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F-ideals and f-graphs

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\textbf{ABSTRACT}

For a field $K$, a square-free monomial ideal $I$ of $K[x_1, \ldots, x_n]$ is called an $f$-ideal if both its facet complex and Stanley-Reisner complex have the same $f$-vector. In this paper, we introduce a combinatorial concept (LU-set) and use it to characterize an $(n, d)$\textsuperscript{t}h $f$-ideal, whose minimal monomial generating set consists of some monomials of a common degree $d$ from $K[x_1, \ldots, x_n]$. We classify all $(n, 2)$\textsuperscript{t}h $f$-ideals, thus list all $f$-graphs whose edge ideals are exactly the $(n, 2)$\textsuperscript{t}h $f$-ideals. Furthermore, we show that all $f$-graphs are Cohen-Macaulay.

\textbf{1. Introduction}

Throughout the paper, for a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$. Let $\Delta$ be a simplicial complex with vertex set $[n]$. Recall that each element of $\Delta$ is a subset of $[n]$, and is called a face. The maximal faces are called facets of $\Delta$. A vertex cover of $\Delta$ is a subset $A$ of $[n]$, with the property that for every facet $F_i$ there is a vertex $v \in A$ such that $v \in F_i$. A minimal vertex cover of $\Delta$ is a subset $A$ of $[n]$ such that $A$ is a vertex cover, and no proper subset of $A$ is a vertex cover for $\Delta$. Recall that a simplicial complex $\Delta$ is unmixed if all of its minimal vertex covers have the same cardinality. Let $\Delta_i$ be the set of faces of $\Delta$ with dimension $i - 1$, i.e., $\Delta_i = \{F \in \Delta \mid |F| = i\}$. For a simplicial complex $\Delta$ having dimension $d$, its $f$-vector is a $(d + 1)$-tuple, defined as $f(\Delta) = (f_0, f_1, \ldots, f_d)$, where $f_i = |\Delta_{i+1}|$ for $i = 0, \ldots, d$. In general, for a set $A$, let $A_i$ be the set of the subsets of $A$ with cardinality $i$, i.e., $A_i = \{B \mid B \subseteq A, |B| = i\}$.

Throughout, let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ indeterminants over a field $K$. For a monomial ideal $I$ of $S$, let $G(I)$ be the minimal monomial generating set of $I$. If all monomials in $G(I)$ have the same degree $d$, then $I$ is called an $(n, d)$\textsuperscript{t}h ideal. Denote by $sm(S)$ ($sm(I)$, respectively) the set of square-free monomials in $S$ (in $I$, respectively). Clearly, there is a natural bijection between $sm(S)$ and $2^n$, given by

\[\sigma : x_{i_1}x_{i_2}\cdots x_{i_k} \mapsto \{i_1, i_2, \ldots, i_k\}.\]

For other concepts and notations used without mention, one can refer to references [3, 6, 9, 13, 14].

Given a simplicial complex $\Delta$, one can define a Stanley-Reisner ideal $I_\Delta$ and a facet ideal $I(\Delta)$ corresponding to $\Delta$. Conversely, given a square-free monomial ideal $I$, it can be associated with either its facet complex or its Stanley-Reisner complex. Recall that the facet complex of $I$, denoted by $\delta_F(I)$, is generated by the facet set $\sigma(G(I))$, where $\sigma(G(I)) = \{\sigma(g) \mid g \in G(I)\}$. Recall that the Stanley-Reisner complex $\delta_N(I)$ of $I$ (or alternatively, the non-face complex of $I$) is defined by $\delta_N(I) = \{\sigma(g) \mid g \in sm(S) \setminus sm(I)\}$, in which we set $\sigma(1) = \emptyset$. Note that the Stanley-Reisner ideal of $\delta_N(I)$ is $I$. The above correspondences construct a bridge between algebraic properties of ideals and combinatorial properties.
of simplicial complexes. In order to study algebraic properties such as linear resolutions of square-free monomial ideals, one usually takes advantage of the structures of simplicial complexes corresponding to the ideals, see references [5, 7, 10, 15] for further detail.

A square-free monomial ideal \( I \) is called an \( f \)-ideal, if both \( \delta_F(I) \) and \( \delta_N(I) \) have the same \( f \)-vector. Note that the \( f \)-vector of a complex \( \delta_N(I) \) is essential in the computation of the Hilbert series of \( S/I \), and the \( f \)-vector of \( \delta_F(I) \) is generally not easy to calculate. Since the complex \( \delta_F(I) \) corresponds to the ideal \( I \) directly, it is easier to calculate the \( f \)-vector of \( \delta_F(I) \). So, it is easy to calculate the Hilbert series and study other corresponding properties of \( S/I \), whenever \( I \) is an \( f \)-ideal.

Let us give a brief literature review on \( f \)-ideals and related topics. It seems that the original impetus for combining the simplicial complex \( \delta_F(I) \) with \( \delta_N(I) \) comes from Remark 2 of [7], while the formal definition of an \( f \)-ideal first appeared in [1]. The authors of [1] studied the properties of an \((n,2)\)-\( f \)-ideal \( I \), and presented an interesting characterization in terms of unmixedness of ideals together with two conditions on \( n \) and \( |G(I)| \). In [2], the authors gave an analogous characterization for unmixed \((n,d)\)-\( f \)-ideals when \( d \geq 3 \). It is worth mentioning that, only unmixed \((n,d)\)-\( f \)-ideals \( (d \geq 3) \) were characterized in [2], while all \((n,2)\)-\( f \)-ideals were characterized in [1]. The reason may be that all of \((n,d)\)-\( f \)-ideals are not unmixed when \( d \geq 3 \), as Example 5.1 of the present paper shows. In order to describe all \((n,d)\)-\( f \)-ideals for \( d \geq 2 \), we introduce a new combinatorial concept called LU-set, see Theorem 2.3 for a complete characterization of \((n,d)\)-\( f \)-ideals and, Theorem 7.2 for a characterization of \( f \)-ideals in the general case (i.e., the case when the monomials in \( G(I) \) need not have the same degree).

Corresponding to \((n,2)\)-\( f \)-ideal, the authors of [12] introduced and studied \( f \)-graphs. Recall that a graph \( G \) is called an \( f \)-graph, if its edge ideal \( I(G) \) is an \( f \)-ideal. \( f \)-graphs were characterized in [12, Theorem 3.5], while a construction of two classes of \( f \)-graphs was shown in [12, Theorem 3.8].

Note that Theorem 3.5 provides an alternate combinatorial characterization for the \( f \)-ideals of degree 2. Furthermore, the authors also proved that the \( f \)-graphs constructed are Cohen-Macaulay. In the present paper, by giving a classification theorem of \( f \)-graphs (and, of all \((n,2)\)-\( f \)-ideals), we prove that all \( f \)-graphs are pure shellable and hence, Cohen-Macaulay, see Theorem 6.5 and Corollary 6.6 for detail.

In [8], some algorithms are provided for constructing \( f \)-ideals generated by homogeneous square-free monomials of degree \( d \), where \( d \geq 3 \), and more examples of \( f \)-ideals in the general case are constructed. In [11], a simplicial complex \( \Delta \) is said to be an \( f \)-simplicial complex if its facet ideal \( I_F(\Delta) \) is an \( f \)-ideal, and the authors discuss the problem of connectedness of \( f \)-simplicial pure complexes. Moreover, they give a complete characterization of connected and disconnected \( f \)-graphs (i.e., dimension 1 \( f \)-simplicial complexes) and give a classification of all the disconnected \( f \)-graphs.

In the following, we show some results and some immediate observations which are needed in the discussion of next section.

Recall that the degree of a monomial ideal \( I \) is the maximal degree of monomials in \( G(I) \). As a comparison, we have the following definition:

**Definition 1.1.** For a monomial ideal \( I \), the minimal degree of monomials in \( G(I) \) is called the lower degree of \( I \), denoted by \( \text{ldeg}(I) \).

**Lemma 1.2 ([2, Lemma 3.6]).** Let \( I \) be an \((n,d)\)-square-free monomial ideal. Then for each \( 0 \leq i < d-1 \), \( \delta_F(I)_{i+1} \leq \delta_N(I)_{i+1} \) holds. In particular, \( f_i(\delta_F(I)) \leq f_i(\delta_N(I)) \) holds for each \( 0 \leq i < d-1 \).

**Lemma 1.3.** Let \( S = K[x_1, \ldots, x_n] \) and let \( I \) be a square-free monomial ideal of \( S \) with lower degree \( \text{ldeg}(I) = k \). Then \( f_{i-1}(\delta_N(I)) = \binom{n}{i} \) holds for each \( 0 < i < k \). Furthermore,

\[
 f_{i-1}(\delta_F(I)) = f_{i-1}(\delta_N(I)) = \binom{n}{i}
\]

holds for each \( 0 < i < k \), if further \( I \) is an \( f \)-ideal.
Corollary 1.4 ([2, Lemma 3.7]). If $I$ is an $(n, d)^{th}$ $f$-ideal, then

$$f_{i-1}(\delta F(I)) = f_{i-1}(\delta N(I)) = \binom{n}{i}$$

holds for each $0 < i < d$.

This paper is organized as follows: In Section 2, we give a combinatorial characterization of $(n, d)^{th}$ $f$-ideals. In the case $d = 2$, we prove the existence of an $f$-ideal, and give a classification theorem for the set of $(n, 2)^{th}$ $f$-ideals in Sections 3 and 4 respectively. In Section 5, we present a combinatorial proof that $f$-ideals are unmixed when $d = 2$. The Cohen-Macaulayness of $f$-graphs is proved in Section 6. Finally, in Section 7, we give a characterization of $f$-ideals in the general case.

2. LU-sets and $(n, d)^{th}$ $f$-ideals

In order to characterize $f$-ideals, we introduce LU-sets and their corresponding notations.

Let $S = K[x_1, \ldots, x_n]$, and let $A \subseteq sm(S)$. Set $\cup(A) = \{g \in A \mid g_i \geq 1 \leq i \leq n\}$, and $\cap(A) = \{h \mid h = g_i x_i \forall g \in A \text{ and } x_i \}$ with $x_i | g$. Denote inductively $\cup(i) = \cup(\cup^{i-1}(A))$, $\cap(i) = \cap(\cap^{i-1}(A))$. It is easy to see that each of $\cup^{\infty}(A)$ and $\cap^{\infty}(A)$ can be achieved in a finite number of steps.

Definition 2.1. Let $S = K[x_1, \ldots, x_n]$, and let $A \subseteq sm(S)_d$, where $1 < d < n$. A is called an $(n, d)^{th}$ L-set, if $\cap(A) = sm(S)_{d-1}$ holds. Dually, A is called an $(n, d)^{th}$ U-set, if $\cup(A) = sm(S)_{d+1}$ holds. If A is both an $(n, d)^{th}$ L-set and a U-set, then A is called an $(n, d)^{th}$ LU-set, or alternatively, an LU-subset of $sm(S)_d$. For a given pair of numbers $(n, d)$, the smallest number among cardinalities of $(n, d)^{th}$ LU-sets is called the $(n, d)^{th}$ LU-number, and is denoted by $N_{n,d}$.

If there is no confusion, an $(n, d)^{th}$ LU-set would be abbreviated as an LU-set. This applies also to an $(n, d)^{th}$ L-set (U-set, respectively).

Example 2.2. Let $S = K[x_1, x_2, x_3, x_4]$. Consider the following three subsets of $sm(S)_2$:

$A = \{x_1 x_2, x_1 x_3, x_1 x_4\}$, $B = \{x_1 x_2, x_1 x_3, x_2 x_3\}$, $C = \{x_1 x_2, x_3 x_4\}$.

It is direct to check that $A$ is an L-set, $B$ is a U-set, $C$ is an LU-set. Note that $x_2 x_3 x_4 \not\in \cap(A)$, so $A$ is not a U-set. On the other hand, $x_4 \not\in \cap(B)$ implies that $B$ is not an L-set.

With the aid of the bijection $\sigma : sm(S) \to 2^{[n]}$, we can define an (L-subset or U-subset, respectively) LU-subset of $2^{[n]}$. For example, a subset $A$ of $[n]_d$ is called an LU-set, if $\sigma^{-1}(A)$ is an LU-subset of $sm(S)_d$.

The following result classifies $(n, d)^{th}$ $f$-ideals in terms of LU-sets.

Theorem 2.3. Let $S = K[x_1, \ldots, x_n]$, and let $I$ be an $(n, d)^{th}$ square-free monomial ideal of $S$ with the minimal generating set $G(I)$. Then $I$ is an $f$-ideal if and only if, the set $G(I)$ is an LU-set and $|G(I)| = \frac{1}{2} \binom{n}{d}$ holds true.

Proof. For the necessity part, if $I$ is an $(n, d)^{th}$ $f$-ideal, then by definition, $\delta F(I)$ and $\delta N(I)$ have the same $f$-vector. In particular, $\dim(\delta N(I)) = \dim(\delta F(I)) = d - 1$. Since $I$ is the Stanley-Reisner ideal of $\delta N(I)$, $sm(S)_{d+1} \subseteq sm(S)$ holds. Note that $I$ is an $(n, d)^{th}$ ideal, it follows that $G(I)$ is a U-set. Furthermore, by Corollary 1.4, $f_{d-2}(\delta F(I)) = |sm(S)_{d-1}| = \binom{d-1}{2}$. Note that every facet of $\delta F(I)$ has dimension $d - 1$, hence $G(I)$ is an L-set. Finally, it follows from $|\delta F(I)_d| = |\delta N(I)_d|$ that $\binom{n}{d}$ is even, and that $|G(I)| = \frac{1}{2} \binom{n}{d}$ holds true.
Conversely, for the sufficiency part, assume that $G(I)$ is an $(n,d)^{th}$ LU-set and $|G(I)| = \frac{1}{2}\binom{n}{d}$. First, we claim that the simplicial complex $\delta_N(I)$ is identical with the complex $\Delta$ generated by $D = E \cup [n]_{d-1}$, where $E = [n]_d \setminus \sigma(G(I))$. In fact, the Stanley-Reisner ideal $I_\Delta$ of $\Delta$ clearly contains all the monomials in $G(I)$ and thus $I \subseteq I_\Delta$ holds. Note further that $G(I)$ is a $U$-set, it follows that $I = I_\Delta$. On the other hand, since $G(I)$ is an $L$-set, each set in $[n]_{d-1}$ is a face of $\delta_F(I)$. Hence

\[ f_{i-1}(\delta_F(I)) = f_{i-1}(\delta_N(I)) = \binom{n}{i} \]

holds for each $0 < i < d$. Note that $|G(I)| = \frac{1}{2}\binom{n}{d}$, so

\[ f_{d-1}(\delta_F(I)) = f_{d-1}(\delta_N(I)) = \frac{1}{2}\binom{n}{d}. \]

Thus $\delta_F(I)$ and $\delta_N(I)$ have the same $f$-vector, and hence $I$ is an $f$-ideal. \hfill\Box

3. Existence of $(n, 2)^{th}$ $f$-ideals and the LU-number $N_{(n, 2)}$

All graphs in the paper are assumed to be undirected and simple. For a graph $G$, the vertex set of $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$. Let $v$ be a vertex in $V(G)$. The degree of $v$ is the number of edges incident to $v$, denoted by $d(v)$. A clique $C$ of $G$ is a subset of $V(G)$ such that every two distinct vertices in $C$ are adjacent. A maximum clique of $G$ is a clique such that there is no clique of $G$ with more vertices. The clique number of $G$ is the number of vertices in a maximum clique of $G$, and is denoted by $\omega(G)$. Let $G = \langle x_1, \ldots, x_n \rangle$, and let $\tau$ be the bijection sending a subset $A$ of $sm(S)_2$ to a graph $T$ whose vertices are $v_1, \ldots, v_n$, with $v_iv_j \in E(T)$ if and only if $x_ix_j \in A$. In other words, the edge ideal of $T$ is the ideal generated by monomials in $A$. In the following, a question about $f$-ideals will be translated into a corresponding question in graph theory, by taking advantage of the bijection $\tau$.

**Proposition 3.1.** Let $A \subseteq sm(S)_2$. Then

1. $A$ is a $U$-set if and only if $\omega(\tau(A)) \leq 2$ holds, where $\overline{\tau(A)}$ is the complement graph of $\tau(A)$.
2. $A$ is an $L$-set if and only if for each $i \in [n]$, $d_{\overline{\tau(A)}}(v_i) < n - 1$ holds, where $d_{\overline{\tau(A)}}(v_i)$ is the vertex degree of $v_i$ in the graph $\overline{\tau(A)}$.

**Proof.**

1. For the sufficiency part, assume to the contrary that $A$ is not a U-set. Then there exists a subset $\{i,j,t\} \subseteq [n]$ such that none of $x_ix_j, x_ix_t, x_jx_t$ are in $A$, hence $\{v Iv_j, v_i v_t, v_jv_t\} \cap E(\tau(A)) = \emptyset$ holds, thus $\{v_i, v_j, v_t\}$ is a clique in $\overline{\tau(A)}$, contradicting the assumption $\omega(\overline{\tau(A)}) \leq 2$. By reversing this argument, the necessity part follows.

2. It is not hard to see that, $A$ is an L-set if and only if there exists no vertex $v_i \in V(\tau(A))$ with $d_{\tau(A)}(v_i) = 0$, and the latter holds if and only if for each $i \in [n]$, $d_{\overline{\tau(A)}}(v_i) < n - 1$ holds in the graph $\overline{\tau(A)}$. This completes the proof. \hfill\Box

For a subset $B$ of $[n]$ with $1 < |B| < n - 1$, let $\overline{B}$ be the complement of $B$ in $[n]$ and let

\[ W_B = \{x_ix_j \mid i, j \in B \text{ or } i, j \in \overline{B} \} \]

be a subset of $sm(S)_2$. Then $W_B = W_{\overline{B}}$ holds clearly, and $W_B$ is an $(n, 2)^{th}$ LU-set. For an $(n, 2)^{th}$ $f$-ideal $I$, if there exists $B \subseteq [n]$ such that $W_B \subseteq G(I)$, then $I$ is called an $(n, 2)^{th}$ $f$-ideal of $r$ type, where $r = \min(|B|, |\overline{B}|)$. The set of all $f$-ideals of $r$ type is denoted by $W_r$. 


In the following, two symbols, say \( V(n,d) \) and \( U(n,d) \), are needed for clarity. For an \((n,d)\)th \( f \)-ideal \( I \), if there exists an \((n,d)\)th \( f \)-ideals \( A \subseteq G(I) \) such that \(|A| = N_{(n,d)}\) holds, then \( I \) is called an \((n,d)\)th basic \( f \)-ideal. The set of all \((n,d)\)th basic \( f \)-ideals is denoted by \( U(n,d) \). On the other hand, the set of all \((n,d)\)th \( f \)-ideals is denoted by \( V(n,d) \). By Theorem 2.3, in order to study the structure of \( V(n,d) \), it is necessary to investigate \( N_{(n,2)} \). It is easy to see that \( N_{(3,2)} = 2 \) and \( V(3,2) = \emptyset \). We assume \( n \geq 4 \) in the following.

In graph theory, Turán’s theorem is an important result on the number of edges in a \( K_{r+1} \)-free simple graph, where a \( K_{r+1} \)-free graph is a graph which contains no clique \( K_{r+1} \) as a subgraph. Recall that a Turán graph, denoted by \( T_{r,n} \), is a complete \( r \)-partite graph on \( n \) vertices whose parts are of equal or almost equal sizes (that is, \( \lceil \frac{n}{r} \rceil \) or \( \lceil \frac{n}{r} \rceil \)). See [4] for detail.

**Proposition 3.2** (Turán’s Theorem). Let \( G \) be a simple graph with \( n \) vertices, such that \( G \) is \( K_{r+1} \)-free, where \( r \geq 1 \). Then \(|E(G)| \leq |E(T_{r,n})|\), with equality if and only if \( G \cong T_{r,n} \). In particular, while \( r = 2 \), if \( G \) is an \( n \)-vertex triangle-free graph, then \(|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor \), with equality if and only if \( G \cong K_{\lfloor \frac{n}{2} \rfloor,\lceil \frac{n}{2} \rceil} \), where \( K_{\lfloor \frac{n}{2} \rfloor,\lceil \frac{n}{2} \rceil} \) is a complete bipartite graph with two parts containing \( \lfloor \frac{n}{2} \rfloor \) and \( \lceil \frac{n}{2} \rceil \) vertices respectively.

**Lemma 3.3.** Let \( k \) be a positive integer, and let \( n \geq 4 \). Then the LU-number \( N_{(n,2)} \) is given by the following rules:

\[
N_{(n,2)} = \begin{cases} 
  k^2 - k, & \text{if } n = 2k; \\
  k^2, & \text{if } n = 2k + 1.
\end{cases}
\]

**Proof.** Let \( A \) be an \((n,2)\)th LU-set. By Proposition 3.1, \( \omega(\tau(A)) \leq 2 \) holds. Now consider the following two possible cases:

Case 1: \( n = 2k \). On one hand, it follows from Proposition 3.2 that \( \tau(A) \) has no more than \( k^2 \) edges, which implies that the cardinality of \( A \) is not less than \( \left( \frac{k}{2} \right)^2 - k^2 = k^2 - k \). On the other hand, let \( B \) be a subset of \([n]\) with \( k \) elements. Then it is easy to see that \( W_B \) contains \( \left( \frac{k}{2} \right)^2 + \left( \frac{k}{2} \right) = k^2 - k \) elements, hence \( N_{(n,2)} = k^2 - k \) holds.

Case 2: \( n = 2k + 1 \). By a similar discussion as above, one can see that \( N_{(n,2)} = k^2 \) holds. We omit the detail here.

Note that \( 2 \nmid \left( \frac{n}{2} \right) \) holds whenever \( n = 4k + 2 \) or \( n = 4k + 3 \). Thus, in these cases, \( V(n,2) = \emptyset \). So, it is only necessary to consider the case when \( n = 4k \) or \( n = 4k + 1 \). Now we are ready to settle the existence of \((n,2)\)th \( f \)-ideals:

**Proposition 3.4.** \( V(n,2) \neq \emptyset \) if and only if \( n = 4k \) or \( n = 4k + 1 \) for some positive integer \( k \).

**Proof.** The necessity part is clear. For the sufficiency part, it suffices to show that the \((n,2)\)th LU-number is not greater than \( \frac{1}{2} \left( \frac{n}{2} \right)^2 \) in the two cases respectively. If \( n = 4k \), then by Lemma 3.3, \( N_{(n,2)} = 4k^2 - 2k \) and \( \frac{1}{2} \left( \frac{n}{2} \right)^2 = 4k^2 - k \), so \( N_{(n,2)} < \frac{1}{2} \left( \frac{n}{2} \right)^2 \). If \( n = 4k + 1 \), then \( N_{(n,2)} = 4k^2 + 2k \) and \( \frac{1}{2} \left( \frac{n}{2} \right)^2 = 4k^2 + k \), so we also have \( N_{(n,2)} < \frac{1}{2} \left( \frac{n}{2} \right)^2 \).

**Remark 3.5.** By Proposition 3.2 and Lemma 3.3, the only way to construct an \((n,2)\)th basic \( f \)-ideal is by the following steps. First, decompose the set \([n]\) into a disjoint union of two subsets \( B \) and \( \overline{B} \) uniformly, namely, \( |B| - |\overline{B}| \leq 1 \). Then set \( W_B = \{x \in \{i,j\} | i,j \in B, \text{ or } i,j \in \overline{B}\} \). It follows from Lemma 3.3 and Proposition 3.4 that \( W_B \) is a LU-set with cardinality \( N_{(n,2)} \), which is less than \( \frac{1}{2} \left( \frac{n}{2} \right)^2 \). Let \( D \) be a subset of \( sm(S) \backslash W_B \) with \( \frac{1}{2} \left( \frac{n}{2} \right)^2 - N_{(n,2)} \) monomials. It is easy to see that \( W_B \cup D \) is an \((n,2)\)th LU-set with \( \frac{1}{2} \left( \frac{n}{2} \right)^2 \).
monomials. By Theorem 2.3, the ideal \( I \) generated by \( W_B \cup D \) is a basic \( f \)-ideal. It also follows from the above discussion that \( U(n, 2) = W_{2k} \) when \( n = 4k \) or \( n = 4k + 1 \) for some positive integer \( k \), where \( W_{2k} \) is the set of \( f \)-ideals of \( 2k \) type.

4. Structure of \( V(n, 2) \)

We begin with a counting formula for the cardinality of \( U(n, 2) \), where \( U(n, 2) \) is the set of \((n, 2)^{th}\) basic \( f \)-ideals:

**Proposition 4.1.** If \( k \) is a positive integer, then

\[
|U(n, 2)| = \begin{cases} 
\frac{1}{2} \binom{4k}{2k} \binom{4k^2}{k}, & \text{if } n = 4k; \\
\binom{4k + 1}{2k} \binom{4k^2 + 2k}{k}, & \text{if } n = 4k + 1; \\
0, & \text{otherwise}
\end{cases}
\] (2)

**Proof.** We only prove the case when \( n = 4k \), and the other cases are similar thus their verifications will be omitted. Assume \( I \in U(n, 2) \), where \( n = 4k \). Since \( U(n, 2) = W_{2k} \), there exists a subset \( B \subseteq [n] \) with \( |B| = 2k \), such that \( W_B \subseteq G(I) \) holds. We claim that such a \( W_B \) is unique, i.e., if there exists another \( B_1 \subseteq [n] \) with \( |B_1| = 2k \) such that \( W_{B_1} \subseteq G(I) \), then \( \{B, \overline{B}\} = \{B_1, \overline{B_1}\} \) holds. In fact, note that both \( |G(I)| = \frac{1}{2} \binom{4k}{2} = 4k^2 - k \) and \( |W_B| = 2 \binom{2k}{k} = 4k^2 - 2k \) hold, hence there are at most \( k \) monomials in \( G(I) \setminus W_B \). Now assume to the contrary that \( \{B, \overline{B}\} \neq \{B_1, \overline{B_1}\} \) holds, and further assume without loss of generality that \( \{1, 2\} \subseteq B, 1 \in B_1 \) and \( 2 \notin B_1 \) hold. Note that for each \( j \notin B \), either \( x_j \in W_{B_1} \) or \( x_2x_j \in W_{B_1} \), hence \( W_{B_1} \) contains at least \( 2k \) monomials in \( G(I) \setminus W_B \). Note that \( |G(I) \setminus W_B| \leq k \) hold, a contradiction. The contradiction shows the uniqueness of the set \( W_B \).

In order to count the cardinality of \( U(n, 2) \), we need to first choose a \( 2k \) set \( B \) randomly, then choose \( k \) monomials of sm\((S)_{2k} \setminus W_B \) arbitrarily. Note that \( W_B = W_{\overline{B}} \) holds, thus \( |U(n, 2)| = \frac{1}{2} \binom{4k}{2k} \binom{4k^2}{k} - \frac{1}{2} \binom{4k}{2k} \binom{4k^2}{k} \) also holds. This completes the proof. \( \square \)

In the rest of this section, we will consider a possible decomposition of \( V(n, 2) \) into a disjoint union of the aforementioned \( W_r \).

The following example shows that there exist \((n, 2)^{th}\) \( f \)-ideals which are not of \( r \) type for any \( r \).

**Example 4.2.** Let \( S = K[x_1, x_2, x_3, x_4, x_5] \). It is direct to check that

\[ I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_3 \rangle \]

is an \( f \)-ideal, but \( I \) is not of \( r \) type for any \( r \).

Note that if \( T \) is a cycle with 5 vertices, so is \( \overline{T} \). The above example shows that if \( T \) is a cycle with 5 vertices, then the ideal generated by \( \tau^{-1}(\overline{T}) \) is an \( f \)-ideal. Such a class of \( f \)-ideals will be denoted by \( C_5 \), which consists of 12 \( f \)-ideals. In fact, by the proof of the following Theorem 4.5, this is the only class of \( f \)-ideals which is not of any \( r \) type.

The following proposition is easy to check, so the proof is omitted.

**Proposition 4.3.** Let \( 2 \leq r \leq \lfloor n/2 \rfloor \). If \( I \) is an \((n, 2)^{th}\) \( f \)-ideal, then \( I \) is of \( r \) type if and only if \( \tau(G(I)) \) is a bipartite graph with two partite sets which contain \( r \) and \( n - r \) vertices respectively.
By Proposition 3.1 and Proposition 4.3, the following lemma is clear.

**Lemma 4.4.** $I$ is an $(n, 2)^{th}$ $f$-ideal which is not of $r$ type for any $r$, if and only if $H = \tau(G(I))$ satisfies the following four conditions (abbreviated as FC in what follows):

1. For each $i \in [n]$, $d_H(v_i) < n - 1$.
2. $\omega(H) = 2$.
3. $|E(H)| = \frac{1}{2}\binom{n}{2}$.
4. $H$ is not a bipartite graph.

Note that an $(n, 2)^{th}$ square-free monomial ideal $I$ is an $f$-ideal if and only if $H = \tau(G(I))$ satisfies conditions (1), (2) and (3) above.

By considering whether there is a graph satisfying the FC, we find an amazing result as the following theorem shows.

**Theorem 4.5 (Classification Theorem).** Let $V(n, 2)$ be the set of $(n, 2)^{th}$ $f$-ideals for $n \geq 4$. Then $V(5, 2) = W_2 \cup C_5$, and $V(n, 2) = \bigcup_{r=2}^{\lfloor n/2 \rfloor} W_r$ for $n \neq 5$, in which $W_i \cap W_j = \emptyset$ while $i \neq j$.

**Proof.** Note that $V(n, 2) = \bigcup_{r=2}^{\lfloor n/2 \rfloor} W_r$ holds true, if and only if each $f$-ideal is of $r$ type for some $r$; and the latter holds if and only if, there is no graph satisfying the FC. We will show that a graph cannot satisfy condition (3) if it satisfies conditions (2) and (4), except for the case $n = 5$.

Assume that $T$ is a graph satisfying conditions (2) and (4). Note that a graph is bipartite if and only if the graph contains no odd cycle. Since $T$ is not a bipartite graph, there exists at least one odd cycle in $T$. Assume that $D$ is a minimal odd cycle of $T$, with $|V(D)| = 2i + 1$. Note that $\omega(T) = 2$, so $i \geq 2$. Denote by $|E(D)|$ and $|E(T \setminus D)|$ the number of edges of the subgraphs induced on $D$ and $T \setminus D$ respectively, and denote by $|E(D, T \setminus D)|$ the number of edges, each of which has end vertices in $D$ and $T \setminus D$ respectively. It is clear that

$$|E(T)| = |E(D)| + |E(T \setminus D)| + |E(D, T \setminus D)|$$

holds. Note that $|E(D)| = 2i + 1$, since $D$ is a minimal odd cycle. Since there exists no triangles in $T$, it is not hard to see that

$$|E(D, T \setminus D)| \leq (n - 2i - 1)i.$$

We will discuss $|E(T \setminus D)|$ in the following two subcases:

If $n = 2k$ for some positive $k$, then $|V(T \setminus D)| = 2k - 2i - 1$ holds. It follows from Proposition 3.2 that $|E(T \setminus D)| \leq (k - i)(k - i - 1)$ holds, hence

$$|E(T)| = |E(D)| + |E(D, T \setminus D)| + |E(T \setminus D)|$$

$$\leq (2i + 1) + (2k - 2i - 1)i + (k - i)(k - i - 1) = k^2 - k - i^2 + 2i + 1.$$

Note that $\frac{1}{2}\binom{n}{2} = k^2 - k/2$, thus

$$\frac{1}{2}\binom{n}{2} - |E(T)| \geq k/2 + i^2 - 2i - 1 = k/2 + (i - 1)^2 - 2$$

holds. Since $i \geq 2$ and $2k > 2i + 1$, $\frac{1}{2}\binom{n}{2} - |E(T)| > 0$ holds. This shows that there is no graph satisfying FC when $n = 2k$.

If $n = 2k + 1$, then $|V(T \setminus D)| = 2k - 2i$ holds. Again by Proposition 3.2, $|E(T \setminus D)| \leq (k - i)^2$ holds, hence we have

$$|E(T)| = |E(D)| + |E(D, T \setminus D)| + |E(T \setminus D)|$$

$$\leq (2i + 1) + (2k - 2i)i + (k - i)^2 = k^2 - i^2 + 2i + 1.$$
Note that \( \frac{1}{2} \binom{n}{2} = k^2 + k/2 \), thus
\[
\frac{1}{2} \binom{n}{2} - |E(T)| \geq k/2 + i^2 - 2i - 1 = k/2 + (i - 1)^2 - 2
\]
holds true. Then we have \( \frac{1}{2} \binom{n}{2} - |E(T)| \geq 0 \), since \( i \geq 2 \) and \( k \geq i \) hold true by assumption. Note further that the equality \( \frac{1}{2} \binom{n}{2} = |E(T)| \) holds true if and only if \( k = i = 2 \). Thus in this case, there is no graph satisfying FC except \( n = 5 \). Furthermore, the unique exceptions are the f-ideals in \( C_5 \). This completes the proof.

In order to explain the Classification Theorem more precisely, we need the following proposition.

**Proposition 4.6.**

1. If \( n = 4k \) for some positive integer \( k \geq 2 \), then \( W_{2k-i} \neq \emptyset \) if and only if \( 0 \leq i \leq \sqrt{k} \).
2. If \( n = 4k + 1 \), then \( W_{2k-i} \neq \emptyset \) if and only if \( 0 \leq i \leq \frac{\sqrt{1+4k}-1}{2} \).

**Proof.**

1. Note that \( W_{2k-i} \neq \emptyset \) if and only if \( \binom{2k-i}{2} + \binom{2k+i}{2} \leq \frac{1}{2} \binom{4k}{2} \) and \( 2k - i > 1 \) hold. By direct calculation, the latter holds if and only if \( 0 \leq i \leq \sqrt{k} \).

2. It is similar to (1) to check.

The following refines Theorem 4.5:

**Theorem 4.7.** Let \( n \geq 4 \), and let \( k \) be a positive integer. Then the following equalities hold true:

\[
V(n, 2) = \begin{cases} 
  \bigcup_{0 \leq i \leq \sqrt{k}} W_{2k-i}, & \text{if } n = 4k (k \neq 1); \\
  W_2, & \text{if } n = 4; \\
  \bigcup_{0 \leq i \leq \frac{\sqrt{1+4k}-1}{2}} W_{2k-i}, & \text{if } n = 4k + 1 (k \neq 1); \\
  W_2 \cup C_5, & \text{if } n = 5; \\
  \emptyset, & \text{if } n = 4k + 2 \text{ or } n = 4k + 3.
\end{cases}
\]  

**Remark 4.8.** By Theorem 4.7, one can construct any \((n, 2)^{th}\) f-ideal for \( n \geq 4 \). In the following, we present an algorithm for the construction of an \((n, 2)^{th}\) f-ideal when \( n = 4k (k \neq 1) \). The other cases are similar to construct.

1. Choose a nonempty subset \( B \subseteq [n] \), such that \( |B| = 2k - i \) with \( i \leq \sqrt{k} \);
2. Let \( t = k - i^2 \), and choose a subset \( E_t \subseteq sm(S)_2 \setminus W_B \) such that \( |E_t| = t \);
3. Let \( I \) be the ideal with the minimal generating set \( G(I) = W_B \cup E_t \). It follows from the proof of Proposition 4.6 that \( |G(I)| = \frac{1}{2} \binom{4}{2} \). Since \( W_B \) is an \((n, 2)^{th}\) LU-set, by Theorem 2.3, \( I \) is an \((n, 2)^{th}\) f-ideal.

The proof of the following proposition is similar to Proposition 4.1, so will be omitted.

**Proposition 4.9.** Let \( i, j \in [\lfloor n/2 \rfloor] \). Then the following hold:

1. If \( i \neq j \), then \( W_i \cap W_j = \emptyset \);
2. If \( I \in W_i \), then there exists a unique \( W_B \), such that \( |B| = i \) and \( W_B \subseteq G(I) \).
By Theorem 4.7 and Proposition 4.9, the following proposition is direct to check, so we omit the proof.

**Proposition 4.10.** Let \( n \geq 4 \), and let \( k \) be a positive integer. Then the following formula holds:

\[
|V(n, 2)| = \begin{cases} 
\frac{1}{2} \left( \frac{4k}{2k} \right) \binom{4k^2}{k} + \sum_{1 \leq i \leq \sqrt{k}} \binom{4k - i}{k - i}^2, & \text{if } n = 4k(k \neq 1); \\
12, & \text{if } n = 4; \\
\sum_{0 \leq i \leq \sqrt{k} + k - 1} \binom{4k + 1}{2k - i} \binom{4k^2 + 2k - i}{k - i}^2, & \text{if } n = 4k + 1(k \neq 1); \\
72, & \text{if } n = 5; \\
0, & \text{if } n = 4k + 2 \text{ or } n = 4k + 3. 
\end{cases}
\]  

(4)

The structure of \( V(n, 2) \) is now completely characterized. However, a complete characterization of \( V(n, d) \) for \( d > 2 \) is still open.

5. **Unmixed f-ideals**

It is known that Cohen-Macaulay property is very important in commutative algebra. In [7], Faridi proved that a Cohen-Macaulay complex is an unmixed complex. Recall that an ideal \( I \) is called *unmixed*, if \( \text{codim}(P) = \text{codim}(I) \) holds for all prime ideals \( P \) minimal over \( I \). Comparing the definition of unmixed complex with unmixed ideal, it is not hard to see that a square-free monomial ideal is an unmixed ideal if and only if its facet complex is an unmixed complex. So, it is valuable to study the \( f \)-ideals which are unmixed. Recall also the following famous Unmixed Theorem: If \( I \) is generated by \( r \) elements and \( \text{codim}(I) = r \), then \( I \) is unmixed (see, e.g., [6, Corollary 18.14]).

The following example shows that an \( f \)-ideal need not to be unmixed.

**Example 5.1.** Let \( S = K[x_1, x_2, x_3, x_4, x_5] \), and let

\[ I = \langle x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_2x_3x_4 \rangle. \]

It is not hard to check that \( G(I) \) is an LU-set and \( |G(I)| = 5 = \frac{1}{2}(\frac{5}{2}) \), which satisfies the condition of Theorem 2.3. Hence \( I \) is an \( f \)-ideal. But the standard primary decomposition of \( I \) is \( I = \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_3, x_4, x_5 \rangle \), which shows that \( I \) is not unmixed.

However, when using the formulae of section 3 to consider \((n, 2)^{th} f\)-ideals, we rediscover the following surprising property, which constitutes the main part of [1, Theorem 3.5]. Note that our approach is combinatoric, and is quite different from the proof of [1, Theorem 3.5].

**Proposition 5.2 ([1, Theorem 3.5]).** Let \( I \) be an \((n, 2)^{th} square-free monomial ideal. If \( I \) is an \( f \)-ideal, then \( I \) is unmixed.

**Proof.** Since \( I \) is an \((n, 2)^{th} f\)-ideal, \( \delta_N(I) \) is 1-dimensional. Assume on the contrary that \( I \) is not unmixed. By Corollary 1.11 of [7], \( \delta_N(I) \) is not pure. Hence there exists some \( i \in [n] \), such that \( \{i\} \) is a facet of \( \delta_N(I) \). Assume without loss of generality that \{\( n \)\} is a facet of \( \delta_N(I) \). Then it is easy to see that each of \{\( 1, n \), \( 2, n \), \ldots \}, \{\( n - 1, n \)\} is a facet of \( \delta_P(I) \), hence \( G_1 = \{x_1x_n, x_2x_n, \ldots , x_{n-1}x_n\} \subseteq G(I) \).
Furthermore, since $x_{i_1}x_{i_2}x_{i_3} \in \cup(G(I))$ for each $\{i_1, i_2, i_3\} \subseteq [n-1]$, $G(I)$ contains at least another $(n-1, 2)^{th}$ U-set, denoted by $G_2$. Clearly, $|G(I)| \geq |G_1| + |G_2|$. 

Now that $I$ is an f-ideal, by Proposition 3.4, $n = 4k$ or $n = 4k + 1$ holds for some positive integer $k$. In the following, we will make use of the formula in Lemma 3.3 to estimate the cardinality of $G(I)$ in the two cases respectively.

If $n = 4k$, then $$|G(I)| \geq |G_1| + |G_2| \geq (n - 1) + N_{(n-1,2)} = (4k - 1) + (2k - 1)^2 = 4k^2,$$
and $\frac{1}{2}\binom{2}{2} = 4k^2 - k$, so $|G(I)| > \frac{1}{2}\binom{2}{2}$. In the case, there is a contradiction to Theorem 2.3.

If $n = 4k + 1$, then $$|G(I)| \geq |G_1| + |G_2| \geq (n - 1) + N_{(n-1,2)} = 4k + (4k^2 - 2k) = 4k^2 + 2k,$$
and $\frac{1}{2}\binom{2}{2} = 4k^2 + k$, so $|G(I)| > \frac{1}{2}\binom{2}{2}$, another contradiction.

This completes the proof. ☐

It is known that a square-free monomial ideal $I$ is unmixed, if and only if $\delta_N(I)$ is a pure simplicial complex [7]. So, we have the following proposition:

**Proposition 5.3.** Let $S = K[x_1, \ldots, x_n]$. If $I$ is an $(n, d)$th f-ideal, then $I$ is unmixed if and only if $sm(S)_{d} \setminus G(I)$ is an L-set.

**Proof.** Note that if $I$ is an $(n, d)$th f-ideal, then the equation $\delta_N(I)_{d} = \sigma(sm(S)_{d} \setminus G(I))$ holds. By Corollary 1.4, $\sigma(sm(S)_{d-1}) \subseteq \delta_N(I)$. Note that $\delta_N(I)$ is pure if and only if $\delta_N(I)_{d}$ contains all the facets of $\delta_N(I)$, and the latter holds if and only if $sm(S)_{d} \setminus G(I)$ is an L-set. This completes the proof. ☐

6. The Cohen-Macaulayness of f-gra

In [12], a graph $H$ is called an f-graph, if its edge ideal $I(H)$ is an f-ideal. The Cohen-Macaulayness of several classes of f-graphs is studied in [12]. In this section, we will show that all f-graphs are Cohen-Macaulay.

For a graph $H$, there is a natural way to treat it as a simplicial complex: a vertex of $H$ is a face with dimension 0, an edge of $H$ is a face with dimension 1.

**Lemma 6.1.** Let $H$ be an f-graph. Then $\delta_N(I(H)) = \bar{H}$.

**Proof.** It is known that the Stanley-Reisner complex of $I(H)$ is the clique complex of $\bar{H}$. In order to show that the clique complex of $\bar{H}$ equals to $\bar{H}$, it is sufficient to show that there is not any triangle in $\bar{H}$, which is clearly true by part 4 of the present paper. ☐

For some positive integer $r$, the set of f-graphs whose edge ideals are f-ideals in $W_r$ is denoted by $W'_r$, and the set of graphs which are cycles with 5 vertices is denoted still by $C_5$. Note that for an $(n, 2)^{th}$ f-ideal $I$, the graph $\tau(G(I))$ is an f-graph. On the other hand, for any f-graph $H$, we have $H = \tau(G(I(H)))$. Thus Theorem 4.5 can be rephrased as follows:

**Theorem 6.2 (Classification Theorem for f-graphs).** Let $V'(n)$ be the set of f-graphs with $n$ vertices ($n \geq 4$). Then $V'(5) = W'_2 \cup C_5$, and $V'(n) = \bigcup_{i=2}^{\lfloor n/2 \rfloor} W'_i$ for $n \neq 5$, in which $W'_i \cap W'_j = \emptyset$ while $i \neq j$.

Recall that a complex $\Delta$ is called pure shellable, if the facets of $\Delta$ have the same dimension, and there is a shelling order on the facet set $\mathcal{F}(\Delta) : F_1, \cdots, F_m$, i.e., for each pair of $i < j$, there exists a $k < j$,
such that \( |F_j \setminus F_k| = 1 \) and \( F_j \setminus F_k \subseteq F_j \setminus F_l \). A graph \( H \) is called a pure shellable graph if \( \delta_N(I(H)) \) is pure shellable, in which \( \delta_N(I(H)) \) is the Stanley-Reisner complex of the edge ideal \( I(H) \) of \( H \).

In order to show that \( f \)-graphs are pure shellable, we introduce some concepts and notations. The number of vertices and edges in a graph \( H \) are denoted by \( v(H) \) and \( e(H) \) respectively. In particular, if \( U \) is a subset of \( V(H) \), then we use \( e(U) \) to denote the number of edges in the subgraph induced by \( U \). Let \( X \) and \( Y \) be a pair of disjoint subsets of \( V(H) \). Denote by \( E(X, Y) \) the set of edges with one end in \( X \) and the other end in \( Y \), and denote by \( e(X, Y) \) the number of edges in \( E(X, Y) \).

If a simple graph \( H \) has a bipartition \( V(H) = V_1 \cup V_2 \), where \( V_1, V_2 \) are a pair of complete subgraphs, then we call the bipartition a clique bipartition. For an \( f \)-graph \( H \), if \( H \notin C_5 \), then \( H \in W_r \) for some \( r \). It follows from the discussion in Section 4 that such an \( f \)-graph \( H \) has an unique clique bipartition. We call this bipartition a generic bipartition.

**Lemma 6.3.** Let \( H \) be an \( f \)-graph of \( r \) type with the generic bipartition \( V(H) = V_1 \cup V_2 \). Then \( e(V_1, V_2) < \min(v(V_1), v(V_2)) \).

**Proof.** Assume that \( v(H) = n \). Because \( H \) is an \( f \)-graph, hence \( e(H) = \frac{1}{2} \left( \frac{n^2}{2} \right) \). Assume without loss of generality that \( v(V_1) = r \leq v(V_2) = n - r \), then \( e(V_1) = \left( \frac{r}{2} \right) \), \( e(V_2) = \left( \frac{n-r}{2} \right) \). It is direct to calculate that

\[
e(V_1, V_2) = e(H) - e(V_1) - e(V_2) = \frac{1}{2} \left( \frac{n^2}{2} \right) - \left( \frac{r}{2} \right) - \left( \frac{n-r}{2} \right) = -r^2 + nr - \frac{n(n-1)}{4}.
\]

In the following, we will show \( e(V_1, V_2) - r < 0 \). In fact,

\[
e(V_1, V_2) - r = -(r - (n-1)/2)^2 - (n/4 - 1/4).
\]

Note that \( n/4 - 1/4 > 0 \) while \( n \geq 4 \), so \( e(V_1, V_2) - r < 0 \) holds for any \( n \geq 4 \). This completes the proof.

**Lemma 6.4.** Let \( H \) be a simple graph with a bipartition \( V(H) = V_1 \cup V_2 \). If \( e(V_1, V_2) < \min(v(V_1), v(V_2)) \), then \( \overline{H} \) is connected.

**Proof.** Assume that \( v(H) = n \), and assume without loss of generality that \( v(V_1) = k \leq n/2 \). Let \( V_1 = \{x_1, \ldots, x_k\} \) and \( V_2 = \{y_1, \ldots, y_{n-k}\} \) be the vertices. Because \( e(V_1, V_2) < k \), there is vertex \( x_j \in V_1 \) such that \( x_jy_1, \ldots, x_jy_{n-k} \) are not edges of \( E(V_1, V_2) \) (and consequently, not in \( H \)). So \( x_jy_1, \ldots, x_jy_{n-k} \) are edges of \( \overline{H} \). Similarly, there is a vertex \( y_l \) such that \( x_1y_l, \ldots, x_ky_l \) are not in \( H \), and thus in \( \overline{H} \). It is clear that the edges \( x_jy_1, \ldots, x_jy_{n-k}, x_1y_l, \ldots, x_{j-1}y_l, x_{j+1}y_l, \ldots, x_ky_l \) form a spanning tree of \( \overline{H} \). Thus, \( \overline{H} \) is connected.

**Theorem 6.5.** All \( f \)-graphs are pure shellable.

**Proof.** Let \( H \) be an \( f \)-graph. By Lemma 6.1, it is sufficient to show that the complex \( \overline{H} \) is pure shellable. Since all the facets of \( \overline{H} \) have dimension 1, \( \overline{H} \) is pure. In order to show that \( \overline{H} \) is shellable, it is clearly sufficient to show that \( \overline{H} \) is connected. In the following, consider the two subcases:

1. \( H \in C_5 \). It is clear that \( \overline{H} \) is connected.

2. \( H \) is an \( f \)-graph of \( r \) type for some positive integer \( r \). Assume that the generic bipartition of \( H \) is \( V(H) = V_1 \cup V_2 \). By Lemma 6.3, \( e(V_1, V_2) < \min(v(V_1), v(V_2)) \). It is clear that \( \overline{H} \) is connected by Lemma 6.4.

Since the graph \( \overline{H} \) is connected in any case, the complex \( \overline{H} \) is pure shellable. Hence the graph \( H \) is pure shellable.

\[\square\]
It is well known that every pure shellable graph is Cohen-Macaulay. So we have the following corollary.

**Corollary 6.6.** All $f$-graphs are Cohen-Macaulay.

It is shown by [9, Lemma 9.1.10] that every Cohen-Macaulay graph is unmixed. So, Proposition 5.2 also follows from Corollary 6.6.

Recall that a graph $H$ is a Gorenstein graph if $K[x_1, \ldots, x_n]/I(H)$ is a Gorenstein ring. It is well known that Gorenstein graphs are a special class of Cohen-Macaulay graphs. Recall from [13, 6.2.16] that for a graph $H$, if $H$ is connected and has no triangles, then $H$ is Gorenstein if and only if $H$ is a cycle. We have the following proposition.

**Proposition 6.7.** Let $H$ be an $f$-graph. Then $H$ is Gorenstein if and only if $H \in C_5$.

**Proof.** Note that the complement of a cycle with 5 vertices is still a cycle with 5 vertices, so the sufficiency part is clear. For the necessity part, by Proposition 6.2, the complement of $f$-graph $H$, which is not in $C_5$, is a bipartite graph. Consider the number of edges, $H$ is clearly not a cycle. Hence $H$ is not Gorenstein.

Recall that a Cohen-Macaulay graph $H$ is said to be saturated, if $q = g(g+1)/2$ holds. By the above discussion, for an $f$-graph $H$, it is saturated if and only if it has 4 vertices, and by Proposition 6.2, the latter holds if and only if it is isomorphic to the following graph:

Let $R = k[x_1, \ldots, x_n]$. Recall from [13, 6.2.22] that for a Cohen-Macaulay graph $H$ with $n$ vertices, it is saturated if and only if $R/I(H)$ has a 2-linear resolution. By the above discussion, the following proposition clearly holds:

**Proposition 6.8.** Let $H$ be an $f$-graph. Then $R/I(H)$ has a 2-linear resolution if and only if $H$ is isomorphic to the line graph in Figure 1.

### 7. $F$-ideals in general case

For a square-free monomial ideal $I$, denote $G(I) = \cup_{i=1}^{k} G_{d_i}$, in which $G_{d_i}$ consists of monomial generators of degree $d_i$. The following lemma is a simple fact, thus the verification will be omitted:

**Lemma 7.1.** If $I$ is a square-free monomial ideal, then

1. $\sigma (\cap_{i=1}^{k} (G(I))) = \delta_F(I) \cap \delta_N(I);$ 

2. $\sigma (\cup_{i=1}^{k} (G(I))) \cap \delta_F(I) = \emptyset$ and $\sigma (\cup_{i=1}^{k} (G(I))) \cap \delta_N(I) = \emptyset$.

The following result characterizes the $f$-ideals in general case.

![Figure 1. Line graph with 4 vertices.](image-url)
Theorem 7.2. Let I be a square-free monomial ideal of $S = K[x_1, \ldots, x_n]$, with the minimal generating set $G(I) = \bigcup_{i=1}^{k} G_d$. Then I is an f-ideal if and only if
\[
|G| = \frac{1}{2} \left( \binom{n}{l} - |\bigcup_{d_i \in I} (\cap_l^{d_i - l}(G_d))| - |\bigcup_{d_i \in I} (\cup_l^{d_i - d_i}(G_d))| \right)
\]
holds for each $l \in [n]$.

Proof. For each $l \in [n]$, denote by $A_l$ the faces in $(\delta_{X'}(I) \setminus \delta_X(I)) \cap [n]$. Note that $sm(S)_l$ is a disjoint union of four parts:
\[
sm(S)_l = G_l \cup \bigcup_{d_i > l} (\cap_l^{d_i - l}(G_d)) \cup \bigcup_{d_i < l} (\cup_l^{d_i - d_i}(G_d)) \cup \sigma^{-1}(A_l).
\]
By Lemma 7.1, $f_{l-1}(\delta_X(I)) = |G_l| + |\bigcup_{d_i > l} (\cap_l^{d_i - l}(G_d))| + |\sigma^{-1}(A_l)|$. Thus I is an f-ideal if and only if $f_{l-1}(\delta_X(I)) = f_{l-1}(\delta_X(I))$ holds for each $l > 0$ and the latter holds if and only if $|G_l| = |\sigma^{-1}(A_l)| = \frac{1}{2}\binom{n}{l} - |\bigcup_{d_i > l} (\cap_l^{d_i - l}(G_d))| - |\bigcup_{d_i < l} (\cup_l^{d_i - d_i}(G_d))|$ holds for each $l$.

Even though the abstract properties of f-ideals in general case are characterized, it is still not easy to show an example of an f-ideal with generating elements not on the same degree. The following result is easy to see by Theorem 7.2.

Corollary 7.3. Let I be an f-ideal of $S = K[x_1, \ldots, x_n]$. For an integer $l \in [n]$ with $l < \deg(I)$, $sm(S)_l \subseteq \cap_{l}^{\infty}(G(I))$ holds true. On the other hand, if $l > \deg(I)$, then $sm(S)_l \subseteq \cup_{l}^{\infty}(G(I))$ holds.

It is easy to see that Theorem 2.3 is a special case of Theorem 7.2.

For further results on the $(n, d)^{th}$ f-ideals and on the general case, see [8].

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References


