Research Article

Bargmann Type Systems for the Generalization of Toda Lattices

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Under a constraint between the potentials and eigenfunctions, the nonlinearization of the Lax pairs associated with the discrete hierarchy of a generalization of the Toda lattice equation is proposed, which leads to a new symplectic map and a class of finite-dimensional Hamiltonian systems. The generating function of the integrals of motion is presented, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

1. Introduction

Differential difference equations have very remarkable applications in modern mathematics and physics; they can model a number of physically interesting phenomena, such as the vibration of particle in lattice [1], the quantum spin chains [2, 3], the Toda lattice [4], the vibration of pulse [5, 6], the nonlinear self-dual network [7], and others. After Toda [8] showed that the Toda lattice was associated with a discretization of the Schrödinger spectral problem, various discrete soliton equations are found, for instance, the discrete nonlinear Schrödinger equation [9], the discrete sine-Gordon equation [10], the discrete KdV equation [11], the discrete mKdV equation [12], and so forth. Recently, the authors have obtained a new discrete hierarchy associated with fourth-order discrete spectral problem, in which a typical member is a generalization of the Toda lattice equation [13].

It has been known that the key to complete integrability of a finite-dimensional Hamiltonian system is the existence of an involutive system of conserved integrals according to the Liouville-Arnold theorem. Many researchers have tried to construct complete integrable Hamiltonian systems. Recently, there are active researches on soliton hierarchies associated with so (3, R) [14]. However, it is a difficult work to search for an involutive system of conserved integrals for a given finite-dimensional Hamiltonian system. An effective method, the nonlinearization of Lax pairs [15, 16], has been developed and applied to various soliton hierarchies associated with 2 × 2 matrix spectral problems to get finite-dimensional completely integrable systems many years ago, such as the nonlinearization of the AKNS hierarchy [15], the coupled KdV hierarchy [17], the discrete Ablowitz-Ladik hierarchy [18], the Heisenberg hierarchy [19], and the Kac-van Moerbeke hierarchy [20]. Subsequently, this method has been generalized to discuss the nonlinearization of Lax pairs and adjoint Lax pairs of soliton hierarchies [21–23]. Moreover, there are attempts to apply the nonlinearization method to the Lax pairs and adjoint Lax pairs of (2 + 1)-dimensional soliton systems, such as the Kadomtsev-Petviashvili equation and the Davey-Stewartson equation, in order to get (1 + 1)-dimensional integrable systems [24]. And it is proved that the binary nonlinearization will be more natural to carry out in the case of higher-order matrix spectral problems [25].

Discrete versions of classical integrable systems have become the focus of common concern in recent years because of their importance. However, the known discrete integrable systems are few compared with the continuous case. In the present paper, the nonlinearization approach is developed and applied to the discrete hierarchy associated with a 4 × 4 discrete eigenvalue problem. Such transformations are adjoint symmetry constraints [26] and a general scheme for doing nonlinearization for lattice soliton hierarchies was presented in [27]. We propose a constraint between the
potentials and eigenfunctions. The nonlinearity of the Lax pairs for the discrete hierarchy leads to a new integrable symplectic map and a class of finite-dimensional integrable Hamiltonian systems.

The outline of this paper is as follows. In Section 2, depending on the spectral problems given in [13], the Bargmann constraint between the potentials and eigenfunctions is introduced, from which a new symplectic map and a class of finite-dimensional Hamiltonian systems are obtained. In Section 3, the generating function approach is used to calculate the involutivity of integrals, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, in Section 4, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

2. A New Symplectic Map

Consider the discrete $4 \times 4$ spectral problem given in [13]

$$E\psi_n = U_n\psi_n, \quad U_n = \begin{pmatrix}
\frac{a_n}{c_n} & 0 & \lambda - b_n & 0 \\
0 & 1 & 0 & c_n \\
0 & 0 & a_n & 0 \\
1 & 0 & 0 & 1
\end{pmatrix},$$

where $\lambda$ is a constant spectral parameter; $E$ is a translation operator defined by $Ef_n = f_{n+1}$. For the sake of convenience, we usually denote $f_{n+k} = E^k f_n$, $f_{n-k} = E^{-k} f_n$. In order to derive the hierarchy of Lattice equations associated with (1), authors of [13] first solve the stationary discrete zero-curvature equation:

$$V_{n+1} U_n - U_{n+1} V_n = 0, \quad V_n = (V_{nij})_{4 \times 4},$$

where the entries of the matrix $V_n$ are Laurent expansions of $\lambda$. Let $\psi_n$ satisfy the spectral problem (1) and its auxiliary problem:

$$\psi_{n+1} = V_{n+1}^{(m)} \psi_n, \quad V_{n+1}^{(m)} = (\lambda^m V_n)_{4 \times 4},$$

then the zero-curvature equation $U_{n+1} \psi_n = V_{n+1}^{(m)} U_n - U_{n+1} V_n^{(m)}$ yields the discrete hierarchy of a generalization of Toda lattices. The first system of evolution equations in this hierarchy is

$$a_n = \frac{1}{2} a_n (b_n - b_{n+1}) + a_{n-1} c_{n-1} - a_{n+1} c_n,$$
$$b_n = a_n^2 - a_n^2 + c_n^2,$$
$$c_n = \frac{1}{2} c_n (b_n - b_{n+2}),$$

which is a generalization of Toda lattice equation.

Let $\lambda_1, \ldots, \lambda_N$ be $N$ distinct nonzero eigenvalues of (1), and the associated eigenfunctions are denoted by

$$q_j^1 = \psi_n^1 (\lambda_j), \quad q_j^2 = \psi_n^2 (\lambda_j),$$
$$p_j^1 = \psi_n^1 (\lambda_j), \quad p_j^2 = \psi_n^2 (\lambda_j),$$

where we denote $q_j^k = q_j^k (n)$ and $p_j^k = p_j^k (n)$ ($k = 1, 2$) for convenience. Then the system associated with (1) can be written in the form

$$E q_j^1 = c_n p_j^2, \quad E q_j^2 = d_j^1 + a_n p_j^2,$$
$$E p_j^1 = \frac{1}{c_n} (a_n p_j^1 + d_j^2 - \lambda_j p_j^1 + b_n p_j^2), \quad E p_j^2 = p_j^1.$$

Now we consider the Bargmann constraint

$$\sum_{j=1}^{N} \nabla \lambda_j = C_n^{(0)},$$

where $G_n^{(0)} = (2a_n, b_n, c_n)^T$ and $\nabla \lambda_j$ is the functional gradient of the eigenvalue $\lambda_j$ with regard to the potentials $a_n$, $b_n$, and $c_n$, that is,

$$\nabla \lambda_j = \left( \begin{array}{c}
\frac{\delta \lambda_j}{\delta a_n} \\
\frac{\delta \lambda_j}{\delta b_n} \\
\frac{\delta \lambda_j}{\delta c_n}
\end{array} \right),$$

Combining (7) and (8), it is easy to see that

$$a_n = \langle p_1^1, p_2^2 \rangle, \quad b_n = \langle p_1^2, p_2^2 \rangle,$$
$$c_n = -\langle p_1^1, p_2^2 \rangle^2 - \langle q_1^2, p_2^2 \rangle^2 + (\Lambda p_2^2, p_2^2) - (p_2^2, p_2^2)^2 \rangle^{1/2},$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$ and $(\cdot, \cdot)$ is the standard inner-product in $\mathbb{R}^N$, $d_q^i = (d_q^1, \ldots, d_q^N)^T$ and $p_q^i = (p_q^1, \ldots, p_q^N)^T$. Substituting (9) into (6), we can get the following system:

$$E q_j^1 = \left( -\langle p_1^1, p_2^2 \rangle^2 - \langle q_1^2, p_2^2 \rangle^2 + (\Lambda p_2^2, p_2^2) - (p_2^2, p_2^2)^2 \rangle \right)^{1/2} p_j^2,$$
$$E q_j^2 = q_j^1 + \langle p_1^1, p_2^2 \rangle^2,$$
$$E p_j^1 = -\left( -\langle p_1^1, p_2^2 \rangle - \langle q_1^2, p_2^2 \rangle + (\Lambda p_2^2, p_2^2) - (p_2^2, p_2^2)^2 \rangle \right)^{-1/2} \times \left( \langle p_1^1, p_2^2 \rangle p_j^1 + q_j^2 - \lambda_j p_j^1 + \langle p_2^2, p_2^2 \rangle p_j^2 \right),$$
$$E p_j^2 = p_j^1.$$
Through tedious calculations one infers

\[
\sum_{j=1}^{N} \sum_{i=1}^{2} d(E_{ij}) \wedge d(E_{ij}) = \sum_{j=1}^{N} \sum_{i=1}^{2} dq_{ij} \wedge dq_{ij},
\]

Therefore, (10) determines a symplectic map \( H \) of the Bargmann type:

\[
(Eq^1, Eq^2, Ep^1, Ep^2) = H(q^1, q^2, p^1, p^2).
\]

3. Liouville Integrability

Introducing a matrix \( \mathcal{V}_\lambda \),

\[
\mathcal{V}_\lambda = (\mathcal{V}^{ij}_\lambda)_{4 \times 4} = \left(\begin{array}{cccc}
-Q_\lambda(q^i, p^j) - \frac{\lambda}{2} & Q_\lambda(p^i, p^j) & Q_\lambda(p^i, p^j) + 1 & -Q_\lambda(q^i, p^j) \\
-Q_\lambda(q^i, q^j) - \langle q^i, p^j \rangle & Q_\lambda(q^i, p^j) + \frac{\lambda}{2} & -Q_\lambda(q^i, q^j) - \langle q^i, p^j \rangle & Q_\lambda(q^i, p^j) \\
-Q_\lambda(q^i, q^j) - \langle q^i, p^j \rangle & Q_\lambda(q^i, p^j) + \frac{\lambda}{2} & -Q_\lambda(q^i, q^j) - \langle q^i, p^j \rangle & Q_\lambda(q^i, p^j) \\
-Q_\lambda(q^i, p^j) & Q_\lambda(p^i, p^j) + 1 & -Q_\lambda(q^i, p^j) & Q_\lambda(q^i, p^j)
\end{array}\right),
\]

\[
\mathcal{F}_\lambda^{(2)} = \frac{1}{4} \mathcal{F}_\lambda^{(1)} \lambda^2 + \frac{1}{16} \lambda^4,
\]

We can find that \( \mathcal{V}_\lambda \) and \( \mu I - \mathcal{V}_\lambda \) are two solutions of the stationary discrete zero-curvature equation (2) under the Bargmann constraint (7), where \( \mu \) is a parameter and \( I \) is a 4\( \times \)4 unit matrix. Then we assert that \( \det \mathcal{V}_\lambda \) and \( \det(\mu I - \mathcal{V}_\lambda) \) are independent constants of the discrete variable \( n \). On the other hand,

\[
\det(\mu I - \mathcal{V}_\lambda) = \mu^4 + \mathcal{F}_\lambda^{(1)} \mu^2 + \mathcal{F}_\lambda^{(2)},
\]

Substituting the Laurent expansion of \( Q_\lambda(q^i, p^j), Q_\lambda(p^i, p^j), Q_\lambda(q^i, q^j) \) into (16) we have

\[
\mathcal{F}_\lambda^{(1)} = -\frac{1}{2} \lambda^2 + \sum_{m \geq 1} F_m^{(1)} \lambda^{-m}, \quad \mathcal{F}_\lambda^{(2)} = \sum_{m \geq 1} F_m^{(2)} \lambda^{-m},
\]
In the above equations, the Poisson bracket of two functions is defined as

\[
\{f, g\} = \sum_{i=1}^{N} \sum_{j=1}^{2} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_i} \right)
\]

Then we can prove the following assertions.
Proposition 1. The functions \{F^{(i)}_m \mid i = 1, 2, m \geq 1\} are in involution in pairs; that is,

\[ \{F^{(i)}_m, F^{(j)}_l\} = 0, \quad \forall m, l \geq 1, \quad 1 \leq i, j \leq 2. \quad (20) \]

Proof. Through tedious calculation we can obtain

\[ \{F^{(1)}_{\lambda}, F^{(1)}_{\mu}\} = \{F^{(1)}_{\lambda}, F^{(2)}_{\mu}\} = \{F^{(2)}_{\lambda}, F^{(2)}_{\mu}\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \quad (21) \]

Then we have

\[ \{F^{(1)}_{\lambda}, F^{(1)}_{\mu}\} = \{F^{(1)}_{\lambda}, F^{(2)}_{\mu}\} = \{F^{(2)}_{\lambda}, F^{(2)}_{\mu}\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \quad (22) \]

Then relation (20) follows by comparison of power of \(\lambda^N\) in (22) with (17) taken into account.

Proposition 2. The 2N 1-forms \(dF^{(i)}_j\) \((1 \leq j \leq N, \quad i = 1, 2)\) are linearly independent.

Proof. Assuming that there exist 2N constants \(b^{(i)}_j\), so that

\[ \sum_{j=1}^{N} \left(b^{(1)}_j \frac{\partial F^{(1)}_j}{\partial q^i} + b^{(2)}_j \frac{\partial F^{(2)}_j}{\partial q^i}\right) = 0, \quad i = 1, 2. \quad (23) \]

It is easy to obtain

\[ \frac{\partial F^{(1)}_i}{\partial q^i} \bigg|_{(p^1, p^2, q^i) = 0} = 2\lambda^{j-1} q, \quad \frac{\partial F^{(2)}_i}{\partial q^i} \bigg|_{(p^1, p^2, q^i) = 0} = 0, \quad 1 \leq j \leq N. \quad (24) \]

Then we have

\[ \sum_{j=1}^{N} b^{(1)}_j \lambda_{j-1}^k = 0, \quad 1 \leq k \leq N, \quad (25) \]

which gives rise to \(b^{(1)}_j = 0, \quad 1 \leq j \leq N\), by utilizing the fact that the Vandermonde determinant is not zero. Therefore, (23) is reduced to

\[ \sum_{j=1}^{N} b^{(2)}_j \frac{\partial F^{(2)}_j}{\partial q^i} = 0, \quad i = 1, 2. \quad (26) \]

Take \(P_0 \in \mathbb{R}^{4N}\) with the coordinates \(q^2 = p^1 = 0, q^1 = O(\epsilon), \quad \) and \(p^2 = O(\epsilon)\), where \(\epsilon\) is a small real number. Then, at \(P_0\),

\[ \left. \frac{\partial F^{(1)}_j}{\partial q^i} \right|_{P_0} = \langle \lambda^j q^1, p^2 \rangle p^2 + \langle q^1, p^2 \rangle \lambda^j p^2 - 2 \langle q^1, p^2 \rangle \langle \lambda^{j-1} p^2, p^2 \rangle p^2 = \langle \lambda^j q^1, p^2 \rangle p^2 + \langle q^1, p^2 \rangle \lambda^j p^2 + O(\epsilon^5), \quad (27) \]

and the determinant of the coefficients of the linear system of equations

\[ \sum_{j=1}^{N} b^{(2)}_j \frac{\partial F^{(2)}_j}{\partial q^i} = 0, \quad 1 \leq k \leq N, \quad (28) \]

is

\[
A = \begin{pmatrix}
\langle \lambda^1 q^1, p^2 \rangle p^2_1 + \langle q^1, p^2 \rangle \lambda_1 p^2_1 & \langle \lambda^1 q^1, p^2 \rangle p^2_2 + \langle q^1, p^2 \rangle \lambda_1 p^2_2 & \cdots & \langle \lambda^N q^1, p^2 \rangle p^2_1 + \langle q^1, p^2 \rangle \lambda_N p^2_1 \\
\langle \lambda^1 q^1, p^2 \rangle p^2_2 + \langle q^1, p^2 \rangle \lambda_1 p^2_2 & \langle \lambda^1 q^1, p^2 \rangle p^2_3 + \langle q^1, p^2 \rangle \lambda_1 p^2_3 & \cdots & \langle \lambda^N q^1, p^2 \rangle p^2_2 + \langle q^1, p^2 \rangle \lambda_N p^2_2 \\
\vdots & \vdots & \ddots & \vdots \\
\langle \lambda^1 q^1, p^2 \rangle p^2_N + \langle q^1, p^2 \rangle \lambda_N p^2_N & \langle \lambda^1 q^1, p^2 \rangle p^2_N + \langle q^1, p^2 \rangle \lambda_N p^2_N & \cdots & \langle \lambda^N q^1, p^2 \rangle p^2_N + \langle q^1, p^2 \rangle \lambda_N p^2_N \\
\langle q^1, p^2 \rangle (\lambda_2 - \lambda_1) & \langle q^1, p^2 \rangle (\lambda_2 - \lambda_1) & \cdots & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1) \\
\vdots & \vdots & \ddots & \vdots \\
\langle q^1, p^2 \rangle (\lambda_N - \lambda_1) & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1) & \cdots & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1)
\end{pmatrix} + O(\epsilon^{5N})
\]

\[= \prod_{j=1}^{N} p^2_j \begin{pmatrix}
\langle \lambda^1 q^1, p^2 \rangle & \langle q^1, p^2 \rangle (\lambda_2 - \lambda_1) & \cdots & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1) \\
\vdots & \vdots & \ddots & \vdots \\
\langle q^1, p^2 \rangle (\lambda_N - \lambda_1) & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1) & \cdots & \langle q^1, p^2 \rangle (\lambda_N - \lambda_1)
\end{pmatrix} + O(\epsilon^{5N})
\]
Proposition 4. The symplectic map of the Bargmann type defined by (10) is completely integrable in the Liouville sense.

Proposition 3. The systems defined as follows are completely integrable in the Liouville sense:

\[
\begin{align*}
\frac{\partial q}{\partial t} &= \frac{\partial F_m^{(1)}}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial F_m^{(1)}}{\partial q}, \quad m \geq 1, \quad i = 1, 2.
\end{align*}
\] (32)

Then we obtain \( b_j^{(2)} = 0, 1 \leq j \leq N \). The proof is complete.

Combining Propositions 1 and 2, we have immediately the following conclusions.

Proposition 5. Let \( q^i(t) \) and \( p^i(t) \) (1 \( \leq i \leq 3 \)) be a solution of (33); define

\[
\begin{align*}
(q^1(n,t), q^2(n,t), p^1(n,t), p^2(n,t)) &= H^n \left( q^1(t), q^2(t), p^1(t), p^2(t) \right).
\end{align*}
\] (35)

Then

\[
\begin{align*}
a_n &= \langle p^1, p^2 \rangle, \quad b_n = \langle p^2, p^2 \rangle, \\
c_n &= \left( -\langle p^1, p^2 \rangle^2 - \langle q^1, p^2 \rangle + \langle \Lambda p^2, p^2 \rangle - \langle p^2, p^2 \rangle \right)^{1/2},
\end{align*}
\] (36)

and solve the lattice equation (4).
Proof. It is easy to see that (35) is equivalent to (12), that is, (10) with \((q'(0, t), p'(0, t)) = (q'(t), p'(t)).\) Using (33), (36), and (10), a direct calculation shows that

\[
a_{ij} = \frac{1}{2} \left( p_1^1 p_2^2 \right) \left( \left( p_1^1 p_1^1 \right) + \left( p_2^2 p_2^2 \right) \right) + \left( p_2^1 p_2^2 \right) \left( \left( p_1^1 p_1^2 \right) + \left( p_2^2 p_2^2 \right) \right) - \left( \Lambda p_1^1 p_2^2 \right)
\]

\[
= \frac{1}{2} \left( (E^{-1} c_n - c_n E) \right) \left( 2 \left( p_1^1 p_2^2 \right) \right) + \frac{1}{2} a_n (1 - E) \left( p_2^2 p_2^2 \right),
\]

\[
b_{ij} = \left( p_2^1 p_2^2 \right) \left( p_2^1 p_2^2 \right) + 2 \left( \Lambda a_2^1, p_2^2 \right) - \left( p_2^2 p_2^2 \right)
\]

\[
= \frac{1}{2} \left( (E^{-1} - 1) a_n \right) \left( 2 \left( p_1^1 p_2^2 \right) \right) + \left( E^2 - 1 \right)
\]

\[
\times \left(-a_n \left( p_1^1 p_2^2 \right) - \left( q_2^1 p_2^1 \right) + \Lambda p_2^2, p_2^2 \right) - \left( p_2^2 p_2^2 \right),
\]

\[
c_{ij} = \frac{1}{2} \left( - p_2^2 p_2^2 \right) \left( p_2^2 p_2^2 \right) + \left( q_2^1 p_2^1 \right) - \Lambda p_2^2, p_2^2 \right) - \left( p_2^2 p_2^2 \right) - \left( p_2^2 p_2^2 \right) - \left( q_2^1 p_2^1 \right) \right)
\]

\[
= \frac{1}{2} c_n \left( 1 - E^2 \right) \left( p_2^2 p_2^2 \right).
\]

(37)

Therefore, we have

\[
\frac{\partial}{\partial t} (a_n, b_n, c_n)^T = J_n \sum_{j=1}^N \nabla \lambda_j = J_n G_n^{(0)},
\]

(38)

where

\[
J_n = \begin{pmatrix}
\frac{1}{2} \left( E^{-1} c_n - c_n E \right) & \frac{1}{2} a_n (1 - E) & 0 \\
\frac{1}{2} \left( E^{-1} - 1 \right) a_n & 0 & \frac{1}{2} \left( E^2 - 1 \right) c_n \\
0 & \frac{1}{2} c_n \left( 1 - E^2 \right) & 0
\end{pmatrix}.
\]

(39)

Then (38) is equivalent to the generalization of Toda lattice equation (4). This proves Proposition 5.

\[\square\]

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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