New conditions on synchronization of memristor-based neural networks via differential inclusions

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Abstract
In this paper, a general framework named Filippov-framework and some new analytic techniques are presented for dealing with the exponential synchronization of memristor-based neural network system with time-varying delay. Several new sufficient conditions are established to guarantee two different types of exponential synchronization of the coupled networks based on master–slave (drive-response) concept and discontinuous state-feedback controller. In addition, we also provide an estimation of the exponential synchronization rate which depends on the time delay and system parameters. These conditions are good improvement and extension of the existing works on synchronization control of memristive or switching networks. Finally, the validity of our method and theoretical results is demonstrated by concrete examples with numerical simulations.

1. Introduction

In the early 1970s, a new circuit element named memristor (i.e., memory resistor) was firstly predicted by Professor Chua [1]. From the theoretical point of view, memristor possesses the distinctive ability to memorize the passed quantity of electric charge. Due to its unique properties, the memristor is very different from other existing three circuit elements: resistor, inductor and capacitor. However, in the subsequent forty years, memristor did not cause much attention of researchers because it is only an ideal circuit element. In 2008, the first practical memristor device was invented by members of Hewlett-Packard (HP) Lab [2,3]. This new circuit prototype of memristor is based on nanotechnology and exhibits the feature of pinched hysteresis just as the neurons in the human brain have [4,5]. Therefore, by using memristor element, it is more realistic to design neural network model for emulating the human brain. It should be noted that memristive system has its own physical characteristics. Firstly, it possesses memory characteristic. Secondly, it possesses nanometer dimensions. In some classical electronic circuits, suppose we replace the resistors or diodes with memristors, then we can build some new memristive system or circuits. For example, in 2008, Itoh and Chua designed some new nonlinear oscillatory circuits by replacing the “Chua’s diode” in the canonical Chua’s oscillator with a memristor [44]. As shown in Fig. 1, a memristive circuit has been obtained by replacing the Chua’s diode with a flux-controlled memristor. Especially, in the field of artificial neural networks, some memristive neural network circuits can also been builded by replacing the resistors with memristors (see, for example, [17,18,33]). Up to now, considerable efforts have been devoted to study memristor-based neural networks [6–11]. It is noted that the memristor-based neural network model is basically switching dynamical system and its switching rule depends on network’s state. This class of switching dynamical neuron system is usually described by the ordinary (or functional) differential equation with discontinuous right-hand side. However, the additional difficulties will arise since the classical theory of differential equation have been shown to be invalid to deal with discontinuous systems. In this case, the given vector field is no longer continuous and the solutions in the conventional sense might not exist. Fortunately, Filippov proposed a novel method and framework in 1964 [12]. Actually, by constructing Filippov set-valued map named Filippov-regularization, the differential equations with discontinuous right-hand sides could be transformed into differential inclusions. According to the theory of differential inclusion, dynamical behaviors of differential equations possessing discontinuous right-hand sides can be handled under this new framework of
Filippov solution. Recently, the authors in [13–18] investigated the dynamical behaviors of memristor-based neural network via Filippov differential inclusion theory and framework. But the main conditions in many of these literature were not correct. Moreover, there is not much research concerning more complex dynamical behaviors such as periodic oscillation, finite time stability, chaos, bifurcation and synchronization for memristor-based neural networks.

As far as we know, synchronization is a typical collective behavior in nature and it includes many different types such as complete synchronization, anti-synchronization, quasi-synchronization, lag synchronization, projective synchronization, and so on. Actually, synchronization means the dynamics of nodes share the common time-spatial property. So we can understand an unknown dynamical system from the well-known dynamical systems by synchronization. Since the pioneering work of Pecora and Carroll in 1990 [see [19]], the chaotic synchronization has become a hot research topic due to its potential applications in various science and engineering fields such as information science, meteorology, biological systems and secure communications [20–23]. Especially, in the field of neural networks, the problems of synchronization have been extensively and intensively investigated because of their practical significance. There are many excellent results on different types of synchronization for neural networks described by differential equations with continuous or discontinuous right-hand sides [see, for example, [24–31]]. However, the study of synchronization for neural networks possessing discontinuous property is not an easy work. It is worth noting that some complex nonlinear behaviors including chaos and periodic oscillatory usually appear even in a simple network of memristor [see [43–45]], so it is of practical importance and great necessity for us to give a detailed analytical investigation of synchronization control of memristor-based neural networks with the basic oscillator. Moreover, the synchronization control of memristive neural networks plays important roles in many potential applications such as super-dense nonvolatile computer memory and neural synapses [see [46]]. In recent years, based on Filippov-framework, the interest of synchronization issues is shifting to the memristor-based neural networks which possess discontinuous switching jumps with respect to states. For example, the paper [32] concerned the problem of global exponential synchronization for a class of memristor-based Cohen–Grossberg neural networks with mixed delays by designing a novel feedback controller. In [33–35], the authors investigated the synchronization of memristor-based neural networks with time delays by using Lyapunov functional method. In [36,37], the anti-synchronization criteria were obtained for memristive neural networks, respectively. In [38], the complete periodic synchronization control was considered for memristor-based neural networks with time-varying delays. It should be pointed out that, because of abrupt changes at certain instants during the dynamical processes, memristor-based neural networks exhibit some especial state-dependent nonlinear switching or discontinuous behaviors which are different from the discontinuities of discontinuous neuron activation. Hence, the analytical technology for studying synchronization of neural networks with discontinuous activations may not be applicable to deal with the synchronization of memristive neural networks because the discontinuities of these two classes of neural networks are different. In short, there exist more difficulties and challenges in investigating the synchronization of memristor-based neural networks due to the special discontinuous switching features of memristor and the lack of effective analysis methods.

On the basis of the aforementioned discussion, referring to previous works [13–16,33–38], this paper considers a class of memristor-based neural networks with discontinuous switching jumps and time-varying delay which are described by the following differential equations:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}(x_j(t))f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(x_j(t))g_j(x_j(t - \tau(t))) + J_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n,$$  \hfill (1)

where $x_i(t)$ denotes the voltage of the capacitor $C_i$; $J_i$ is the external input to the $i$th neuron; the time-varying delay $\tau(t)$ is a nonnegative continuous function satisfying $0 \leq \tau(t) \leq \tau$ ($\tau \geq 0$ is a constant); $f_j(x_j(t))$ and $g_j(x_j(t - \tau(t)))$ denote feedback functions without and with time-varying delay between the $j$th dimension of the memristor and $x_i(t)$, respectively; $d_i > 0$ denotes the self-inhibition with which the $i$th neuron unit will reset its potential to the resting state in isolations when disconnected from the network; $a_{ij}(x_i(t))$ and $b_{ij}(x_i(t))$ denote memristor-based connection weights, and

$$a_{ij}(x_i(t)) = \frac{W_{ij}}{C_i} \times \text{SGN}_i, \quad b_{ij}(x_i(t)) = \frac{M_{ij}}{C_i} \times \text{SGN}_i,$$

$$\text{SGN}_i = \begin{cases} 1, & \text{if } i \neq j, \\ -1, & \text{if } i = j, \end{cases}$$

in which $W_{ij}$ and $M_{ij}$ represent the memductances of memristors $R_{ij}$ and $F_{ij}$, respectively. $R_{ij}$ denotes the memristor between the feedback function $f_j(x_j(t))$ and $x_i(t)$. $F_{ij}$ represents the memristor between the time-varying delayed feedback function $g_j(x_j(t - \tau(t)))$ and $x_i(t)$. According to the feature of the memristor and the current-voltage characteristic, the memristor-based connection weights $a_{ij}(x_i(t))$ and $b_{ij}(x_i(t))$ of the neural network satisfying the following conditions:

$$a_{ij}(x_i(t)) = \begin{cases} \hat{a}_{ij}, & \text{if } |x_i(t)| \leq Y_i, \\ \tilde{a}_{ij}, & \text{if } |x_i(t)| > Y_i, \end{cases}$$

$$b_{ij}(x_i(t)) = \begin{cases} \hat{b}_{ij}, & \text{if } |x_i(t)| \leq Y_i, \\ \tilde{b}_{ij}, & \text{if } |x_i(t)| > Y_i, \end{cases}$$

for $i, j = 1, 2, \ldots, n$, where the switching jumps $Y_i > 0$, $\hat{a}_{ij}$, $\tilde{a}_{ij}$, $\hat{b}_{ij}$ and $\tilde{b}_{ij}$ are constants.

In order to achieve our main results, the feedback functions $f_j$ and $g_j$ are assumed to satisfy the following basic conditions:

$$(C1) \text{ For any two different } x, y \in \mathbb{R}, \text{ there exist positive constants } L_j, Q_j (j = 1, 2, \ldots, n) \text{ such that }$$

$$|f_j(x) - f_j(y)| \leq L_j|x - y|, \quad |g_j(x) - g_j(y)| \leq Q_j|x - y|.$$
For any \( x \in \mathbb{R} \), there exist positive constants \( M_j, N_j \) (\( j = 1, 2, \ldots, n \)) such that
\[
|f_j(x)| \leq M_j, \quad |g_j(x)| \leq N_j.
\]

The structure of the memristive neural network model is very complex since it consists of too many subsystems. This paper will mainly apply the differential inclusion framework of Filippov in discussing the solution of memristive neural network with switching jumps property (Section 2). By designing novel discontinuous state-feedback controller, this paper gives new testable algebraic criteria to achieve two different types of exponential synchronization control for master-slave system via differential inclusions (Section 3). In order to verify the effectiveness of the proposed approach and results, we present illustrative examples and computer simulations by using MATLAB programming (Section 4). Finally, this paper ends by some conclusions (Section 5). The main contribution and difficulties of this paper can be summarized as follows:

- In handling the complex dynamics of memristive or discontinuous switching neural networks with time-varying delays, we develop a brand new method. In other words, we only use some novel analytic techniques to investigate the synchronization control of master–slave system. These novel analytic techniques of this paper are not needed to construct complex Lyapunov–Krasovskii functional. However, the conventional tool for investigating the synchronization problem of delayed neural networks is the generalized Lyapunov functional approach (see, for example, [33–35]). Unfortunately, the suitable Lyapunov–Krasovskii functional is not easy to be constructed because its structure might be very complex.
- We remove some incorrect assumptions in existing literature such as
\[
\mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell}} f_j(x(t)) + \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x(t)) = \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x(t)),
\]
where \( \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell}} f_j(x(t)) + \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x(t)) \) is a continuous function with respect to state. Some new sufficient conditions are obtained to realize exponential synchronization of memristor-based neural networks. Moreover, these conditions are simple and can be easily verified in practice.
- A more general type of time-varying delay is considered. In this paper, the time-varying delay \( \tau(t) \) is not required to be differentiable with respect to \( t \). We only requested it is continuous and bounded.
- In order to realize the synchronization control of memristor-based neural networks, what kinds of state-feedback controllers should be designed? By using switching state-feedback controller, whether the uncertain differences between the Filippov solutions of the master–slave memristor-based neural networks can be well handled?

**Notations:** \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space and \( \mathbb{B} \) denotes the set of real numbers. For the column vectors \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n \), \( \langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \) denotes the scalar product of \( x \) and \( y \). Let \( x \in \mathbb{R}^n \), \( \| x \| \) denotes any vector norm. Given a set \( E \subset \mathbb{R}^n \), \( \text{conv} [E] \) denotes the closure of the convex hull of \( E \) and \( \text{meas}(\mathbb{S}) \) denotes the Lebesgue measure of set \( E \) in \( \mathbb{R}^n \). If \( x \in \mathbb{R}^n \) and \( \delta > 0 \), \( B(x, \delta) = \{ y \in \mathbb{R}^n : \| y - x \| \leq \delta \} \) represents the ball of \( \delta \) about \( x \). Given a single-valued function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \), by \( \partial V \) we mean the Clarke’s generalized gradient of \( V \).

### 2. Preliminaries

Since the memristor-based connection weights \( a_{j}(x(t)) \) and \( b_{j}(x(t)) \) of system (1) are defined as discontinuous functions and changed according to the state \( x(t) \), the solution of system (1) in the conventional sense has been shown to be invalid. In this case, we need to introduce the Filippov solution with differential inclusion framework to deal with the differential-based neural network described by differential equation with discontinuous right-hand side.

**Definition 1** (see Aubin and Cellina [39]). Let \( E \subset \mathbb{R}^n \), if for every point \( x \) of the set \( E \subset \mathbb{R}^n \) there corresponds a nonempty set \( F(x) \subset \mathbb{R}^n \), then \( x \rightarrow F(x) \) is said to be a set-valued map from \( E \subset \mathbb{R}^n \) to \( \mathbb{R}^n \). A set-valued map \( F \) with nonempty values is called upper semi-continuous (USC) at \( x_0 \in E \), if for any open set \( N \) containing \( F(x_0) \), there exists a neighborhood \( M \) of \( x_0 \) such that \( F(M) \subset N \).

**Definition 2** (see Filippov [12]). For the non-autonomous differential equation \( \frac{dx}{dt} = f(t, x) \), where \( f(t, x) \) is discontinuous in \( x \in \mathbb{R}^n \), let us construct a Filippov set-valued map \( \text{i.e., Filippov regularization} \) \( F : \mathbb{R} \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n} \) defined as follows:
\[
F(t, x) = \bigcap_{\delta > 0} \bigcap_{\text{meas}(\mathbb{S}) = 0} \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x(t)) = \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x(t)),
\]
where \( B(x, \delta) \) is the ball of center \( x \) and radius \( \delta \); \( \mathcal{C} \bigcup \) denotes the closure of the convex hull of \( E \); \( \text{meas}(\mathbb{S}) \) represents the Lebesgue measure of set \( \mathbb{S} \); intersection is taken over all sets \( N \) of Lebesgue measure zero and over all \( \delta > 0 \). A vector-value function \( x(t) \) defined on a non-degenerate interval \( I \subset \mathbb{R} \) is said to be a Filippov solution of this discontinuous differential equation, if it is absolutely continuous on any compact subinterval \( [t_1, t_2] \) of \( I \), and for almost all \( t \in I \), \( x(t) \) satisfies the differential inclusion
\[
\frac{dx(t)}{dt} \in F(t, x).
\]

By applying the above theories of set-valued maps and differential inclusions with Filippov-framework, we can obtain from (1) that
\[
\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} f_j(x_j(t)) + \sum_{j=1}^{n} \mathcal{C} \bigcup_{\ell=1}^{n} \overline{a_{j_{\ell}}} \bigcup_{j_{\ell} = j} g_j(x_j(t-\tau(t))) + f_i, \quad \text{for a.e. } t \geq 0, i = 1, 2, \ldots, n.
\]
where
\[
\mathfrak{C} \{a_{ij}(x_i(t))\} = \begin{cases} 
\hat{a}_{ij}, & |x_i(t)| < Y_i, \\
\hat{a}_{ij}, & |x_i(t)| = Y_i, \\
\tilde{a}_{ij}, & |x_i(t)| > Y_i,
\end{cases}
\mathfrak{C} \{b_{ij}(x_i(t))\} = \begin{cases} 
\hat{b}_{ij}, & |x_i(t)| < Y_i, \\
\hat{b}_{ij}, & |x_i(t)| = Y_i, \\
\tilde{b}_{ij}, & |x_i(t)| > Y_i.
\end{cases}
\]
(3)

\[
\nu_i(t) = \max(\hat{a}_{ij}, \tilde{a}_{ij}), \quad \xi_i(t) = \min(\hat{a}_{ij}, \tilde{a}_{ij}), \\
\nu_i(t) = \max(\hat{b}_{ij}, \tilde{b}_{ij}), \quad \xi_i(t) = \min(\hat{b}_{ij}, \tilde{b}_{ij}), \quad i, j = 1, 2, \ldots, n.
\]

Or, equivalently, there exist measurable functions \(a_{ij}(t)\) and \(b_{ij}(t)\) for a.e. \(t \geq 0\), such that
\[
\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t) - \tau(t)) + f_i(t), \quad \text{for a.e. } t \geq 0, \quad i = 1, 2, \ldots, n.
\]
(4)

In this paper, let the memristor-based neural network model (1) be the master system. We choose a controlled slave system described as follows:
\[
\frac{dy_i(t)}{dt} = -d_i y_i(t) + \sum_{j=1}^{n} a_{ij}(y_j(t))f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}(y_j(t))g_j(y_j(t) - \tau(t)) + f_i(t), \quad \text{for a.e. } t \geq 0, \quad i = 1, 2, \ldots, n.
\]
(5)

Similarly, the memristor-based connection weights \(a_{ij}(y_j(t))\) and \(b_{ij}(y_j(t))\) in slave system (5) can be described as
\[
a_{ij}(y_j(t)) = \begin{cases} 
\hat{a}_{ij}, & |y_j(t)| \leq Y_i, \\
\tilde{a}_{ij}, & |y_j(t)| > Y_i,
\end{cases}
\]
\[
b_{ij}(y_j(t)) = \begin{cases} 
\hat{b}_{ij}, & |y_j(t)| \leq Y_i, \\
\tilde{b}_{ij}, & |y_j(t)| > Y_i.
\end{cases}
\]
(6)

Also, the closure of the convex hull of discontinuous functions \(a_{ij}(y_j(t))\) and \(b_{ij}(y_j(t))\) are given as follows:
\[
\mathfrak{C} \{a_{ij}(y_j(t))\} = \begin{cases} 
\hat{a}_{ij}, & |y_j(t)| \leq Y_i, \\
\tilde{a}_{ij}, & |y_j(t)| > Y_i,
\end{cases}
\mathfrak{C} \{b_{ij}(y_j(t))\} = \begin{cases} 
\hat{b}_{ij}, & |y_j(t)| \leq Y_i, \\
\tilde{b}_{ij}, & |y_j(t)| > Y_i.
\end{cases}
\]
(7)

Let \(C = \mathbb{C}[-r, 0] \times \mathbb{R}^n\) denote the Banach space of continuous functions \(\phi\) mapping the interval \([-r, 0]\) into \(\mathbb{R}^n\) with the norm \(\|\phi\|_C = \sup_{-r \leq s \leq 0} \|\phi(s)\|\). The initial values of master system (1) and slave system (5) are given as \(\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in C[-r, 0, \mathbb{R}^n]\), respectively. Now we define the following synchronization error between the master and the slave system
\[
o(t) = y_i(t) - x_i(t), \quad i = 1, 2, \ldots, n.
\]

In order to reach synchronization of the master–slave system, the state-feedback controller \(u_i(t)\) is defined a discontinuous function given as follows:
\[
u_i(t) = -\kappa_i o_i(t) - \eta_i \text{sign}(o_i(t)),
\]
(9)

where \(\kappa_i\) and \(\eta_i\) are the control gains to be determined. Then, from (4) and (8), we can get the synchronization error system as follows:
\[
\frac{do_i(t)}{dt} = -d_i o_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t)\tilde{f}_j(o_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t)\tilde{g}_j(o_j(t) - \tau(t)) + \sum_{j=1}^{n} (\tilde{a}_{ij}(t) - \alpha_{ij}(t))f_j(y_j(t))
\]
\[
+ \sum_{j=1}^{n} (\tilde{b}_{ij}(t) - \beta_{ij}(t))g_j(y_j(t) - \tau(t)) + u_i(t), \quad \text{for a.e. } t \in [0, +\infty), \quad i = 1, 2, \ldots, n,
\]
(10)

where \(\tilde{f}_j(o_j(t)) = f_j(y_j(t)) - f_j(\xi_j(t))\) and \(\tilde{g}_j(o_j(t) - \tau(t)) = g_j(y_j(t) - \tau(t)) - g_j(\xi_j(t) - \tau(t))\). Substituting the discontinuous state-feedback controller (9) into the synchronization error system (10), we can obtain
\[
\frac{do_i(t)}{dt} = -(d_i + \kappa_i) o_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t)\tilde{f}_j(o_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t)\tilde{g}_j(o_j(t) - \tau(t)) + \sum_{j=1}^{n} (\tilde{a}_{ij}(t) - \alpha_{ij}(t))f_j(y_j(t))
\]
\[
+ \sum_{j=1}^{n} (\tilde{b}_{ij}(t) - \beta_{ij}(t))g_j(y_j(t) - \tau(t)) - \eta_i \text{sign}(o_i(t)), \quad \text{for a.e. } t \in [0, +\infty), \quad i = 1, 2, \ldots, n.
\]
(11)
For the sake of convenience, we denote
\[
\|\phi - \psi\|_1 = \sup_{-r \leq s \leq 0} \sum_{i=1}^n |\phi_i(s) - \psi_i(s)|,
\]
\[
\|\phi - \psi\|_2 = \sup_{-r \leq s \leq 0} \sum_{i=1}^n (\phi_i(s) - \psi_i(s))^2.
\]

**Definition 3.** Under the suitably designed feedback controller, the master system (1) and the slave system (5) with memristor are said to be

- the first-type globally exponentially synchronized if there exist positive constants \(K \geq 1\) and \(\lambda > 0\) such that
  \[
  \sum_{i=1}^n |y_i(t) - x_i(t)| \leq K \|\phi - \psi\|_1 e^{-\lambda t}
  \]
  for all \(t \geq 0\);

- the second-type globally exponentially synchronized if there exist positive constants \(K \geq 1\) and \(\lambda > 0\) such that
  \[
  \sum_{i=1}^n (y_i(t) - x_i(t))^2 \leq K \|\phi - \psi\|_2 e^{-\lambda t}
  \]
  for all \(t \geq 0\).

Here, the constant \(\lambda\) is called the globally exponential synchronization rate.

**Lemma 1** (Chain Rule, Forti et al. [40], Clarke [41]). Suppose that \(V(x) : \mathbb{R}^n \rightarrow \mathbb{R}\) is \(C\)-regular, and \(x(t) : [0, +\infty) \rightarrow \mathbb{R}^n\) is absolutely continuous on any compact subinterval of \([0, +\infty)\). Then, \(x(t)\) and \(V(x(t)) : [0, +\infty) \rightarrow \mathbb{R}\) are differential for almost all \(t \in [0, +\infty)\) and
\[
\frac{dV(x(t))}{dt} = \langle \dot{x}(t), dx(t) \rangle, \quad \forall \dot{x}(t) \in dV(x(t)).
\]

3. Main results

This section is dedicated to investigate the global exponential synchronization of memristor-based neural networks with time-varying delay by introducing discontinuous state-feedback controller (9). By employing Filippov-framework and some new analytic techniques, some new sufficient conditions are derived to guarantee the synchronization of the master–slave network system. Our discussions in this section are different from those of the previous literature.

**Theorem 1.** In addition to the conditions (C1) and (C2), assume further that

(C3) \(\eta_i \geq \sum_{j=1}^n (|\hat{a}_i - \hat{a}_j| M_j + \sum_{j=1}^n |\hat{b}_i - \hat{b}_j| N_j);\)

(C4) \(A + B < D\), where \(D = \min \{1 \leq j \leq n [d_j + \kappa_i]\}, \quad A = \max \{1 \leq j \leq n \left(\sum_{j=1}^n a_{ij}^+ L_j\right), \quad B = \max \{1 \leq j \leq n \left(\sum_{j=1}^n b_{ij}^+ Q_j\right), \quad a_{ij}^+ = \max \{|A_{ij}|, |a_{ij}|\} \quad \text{and} \quad b_{ij}^+ = \max \{|B_{ij}|, |b_{ij}|\}.\)

Then the memristor-based neural networks (1) and (5) can realize the first-type global exponential synchronization under the discontinuous state-feedback controller (9), where the exponential synchronization rate \(\mu^*\) is the unique root of the following algebra equation
\[
\mu = D - A - B e^{\mu^*}.
\]

**Proof.** Choose a non-smooth auxiliary function for the system (11) as follows:
\[
V_i(t) = |o_i(t)|.
\]

It is obvious that \(V_i(t)\) is \(C\)-regular. Moreover, the function \(|\omega(t)|\) is locally Lipschitz continuous in \(\omega(t)\) on \(\mathbb{R}\). By virtue of the definition of Clarke’s generalized gradient of function \(|\omega(t)|\) at \(\omega(t)\), we obtain
\[
\partial(|\omega(t)|) = \begin{cases} \{ -1 \}, & \text{if } \omega(t) < 0, \\ \{ -1, 1 \}, & \text{if } \omega(t) = 0, \\ \{ 1 \}, & \text{if } \omega(t) > 0. \end{cases}
\]

That is to say, for any \(\theta(t) \in \partial(|\omega(t)|)\), we have \(\theta(t) = \text{sign}(\omega(t))\), if \(\omega(t) \neq 0\); while \(\theta(t)\) can be arbitrarily chosen in the interval \([-1, 1]\), if \(\omega(t) = 0\). In particular, we select \(\theta(t) = \text{sign}(\omega(t))\). Obviously, it is easy to see that \(\theta(t)\omega(t) = |\omega(t)|\). According to the chain rule in Lemma 1, calculating the time derivative of \(V_i(t)\) along the right half trajectories of error system (11), we have
\[
\frac{dV_i(t)}{dt} = \frac{d|\omega(t)|}{dt} = \begin{cases} -(d_i + \kappa_i)\omega(t) + \sum_{j=1}^n \alpha_{ij}\psi_j \dot{\theta}(o_j(t)) + \sum_{j=1}^n \beta_{ij}(1 - \psi_j) \dot{\theta}(o_j(t)) + \sum_{j=1}^n (\hat{a}_{ij}(t) - \alpha_{ij})f_j(y_j(t)) + \\
+ \sum_{j=1}^n (\hat{b}_{ij}(t) - \beta_{ij})g_j(y_j(t)) \dot{\theta}(o_j(t)) - \eta_i \text{sign}(\omega(t)) |\omega(t)| \quad \text{if } \omega(t) < 0, \\
+ \sum_{j=1}^n \alpha_{ij}\psi_j \dot{\theta}(o_j(t)) + \sum_{j=1}^n \beta_{ij}(1 - \psi_j) \dot{\theta}(o_j(t)) + \sum_{j=1}^n (\hat{a}_{ij}(t) - \alpha_{ij})f_j(y_j(t)) \quad \text{if } \omega(t) = 0, \\
+ \sum_{j=1}^n (\hat{b}_{ij}(t) - \beta_{ij})g_j(y_j(t)) \dot{\theta}(o_j(t)) - \eta_i \text{sign}(\omega(t)) |\omega(t)| \quad \text{if } \omega(t) > 0. \end{cases}
\]
This, together with (14), leads to

\[ |\omega(t)| = |\omega(0)| e^{-ct} + \int_0^t e^{-ct} \int_\mathbb{R} \phi(y) \psi(t-s) \, ds, \]  

(15)

where

\[ G_i(s) = \left( \sum_{j=1}^n a_{ij}(s) f_j(\omega_j(s)) + \sum_{j=1}^n \beta_{ij}(s) g_j(\omega_j(S-r-s)) + \sum_{j=1}^n (\hat{a}_{ij}(s) - \alpha_{ij}(s)) f_j(\omega_j(S-r-s)) + \sum_{j=1}^n (\hat{\beta}_{ij}(s) - \beta_{ij}(s)) g_j(\omega_j(S-r-s)) \right) \text{sign}(\omega_i(s)) - \eta_i |\text{sign}(\omega_i(s))|. \]

(16)

It is noted that the symbol "\( \int \)" in (15) denotes the Lebesgue integration. Combining with the conditions (C1) and (C2), it follows from (16) that

\[ G_i(s) \leq \sum_{j=1}^n a_{ij} L_j |\omega_j(s)| + \sum_{j=1}^n b_{ij} Q_j |\omega_j(S-r-s)|. \]

(17)

Recalling the condition (C3), we can obtain from (17) that

\[ G_i(s) \leq \sum_{j=1}^n a_{ij} L_j |\omega_j(s)| + \sum_{j=1}^n b_{ij} Q_j |\omega_j(S-r-s)|. \]

(18)

This, together with (15), leads to

\[ |\omega(t)| \leq |\omega(0)| e^{-ct} + \int_0^t e^{-ct} \int_\mathbb{R} \phi(y) \psi(t-s) \, ds \leq |\omega(0)| e^{-ct} + \int_0^t e^{-ct} \int_\mathbb{R} \phi(y) \psi(t-s) \, ds. \]

(19)

Summing up every side of (19), we obtain

\[ \sum_{i=1}^n |\omega_i(t)| \leq \sum_{i=1}^n |\omega_i(0)| e^{-ct} + \int_0^t e^{-ct} \int_\mathbb{R} \phi(y) \psi(t-s) \, ds \leq \sum_{i=1}^n |\omega_i(0)| e^{-ct} + \max_{1 \leq j \leq n} \left( \sum_{i=1}^n a_{ij} L_j \right) \int_0^t e^{-ct} \int_\mathbb{R} \phi(y) \psi(t-s) \, ds. \]

(20)

For the simplification, we set

\[ \omega^{\text{sum}}(t) = \sum_{i=1}^n |\omega_i(t)| = \sum_{i=1}^n |Y_i(t) - X_i(t)|. \]

(21)

It follows from (20) and (21) that

\[ \omega^{\text{sum}}(t) \leq \omega^{\text{sum}}(0) e^{-ct} + A \int_0^t e^{-ct} \omega^{\text{sum}}(s) \, ds + B \int_0^t e^{-ct} \omega^{\text{sum}}(s) \, ds, \]

(22)

where \( A = \max_{1 \leq j \leq n} \left( \sum_{i=1}^n a_{ij} L_j \right) \) and \( B = \max_{1 \leq j \leq n} \left( \sum_{i=1}^n b_{ij} Q_j \right) \). It is clear that \( \omega^{\text{sum}}(S-r-s) \) and \( \omega^{\text{sum}}(S-r) \) are Riemann integrable because of their continuities. It should be pointed out that any Riemann integrable functions are Lebesgue integrable. Therefore, the inequality (22) is still true if we replace the Lebesgue integration in (22) with Riemann integration. For the sake of convenience, the superscript "\( \text{sum} \)" is still used to denote the Riemann integration in the ensuing discussion. Now, we let

\[ \mathcal{W} = \sup_{-r \leq s \leq S} \sum_{i=1}^n |\omega_i(s)| = \| \phi - \psi \|_1. \]
On the interval $[-r, +\infty)$, we define the function $\mathcal{Z}(t)$ as follows:

$$
\mathcal{Z}(t) = \begin{cases} 
W e^{-\mathcal{Z}(t) + A t} & \text{if } t > 0, \\
W & \text{if } -r \leq t \leq 0.
\end{cases}
$$

(23)

Obviously, $\mathcal{Z}(t)$ is a continuous differential function on the interval $[-r, +\infty)$.

In the following, we set

$$
\mathcal{U}(t) = \begin{cases} 
\omega(t) & \text{if } t > 0, \\
W & \text{if } -r \leq t \leq 0.
\end{cases}
$$

(24)

From (22)-(24), it is not difficult to find that the following inequality holds

$$
\mathcal{U}(t) \leq \mathcal{Z}(t), \quad \text{for any } t \geq -r.
$$

(25)

For $t > 0$, by direct calculating the time derivative of $\mathcal{Z}(t)$, we get

$$
\frac{d\mathcal{Z}(t)}{dt} = -D W e^{-\mathcal{Z}(t) - DA t} \int_0^t e^{-\mathcal{Z}(t) - D A s} \omega(s) \, ds + A \omega(t) - DB \int_0^t e^{-\mathcal{Z}(t) - D A s} \omega(s) \, ds
$$

(26)

Set

$$
\mathcal{R}(t) = We^{-\mathcal{Z}(t)}, \quad \text{for } t \geq -r,
$$

(27)

where $\mu^* = \mu^*(D, A, B, \tau)$ represents the unique root of the following algebra equation:

$$
\mu^* = D - A - Be^{\mathcal{Z}(t)}.
$$

In fact, it is easy to see that the function $h(\mu) = \mu - D + A + Be^{\mathcal{Z}(t)}$ is a strictly increasing with respect to $\mu$ on $\mathbb{R}$ due to its derivative $\frac{d h}{d \mu} = 1 + Be^{\mathcal{Z}(t)} > 0$. Meanwhile, we have $h(D - A) = Be^{\mathcal{Z}(t)} > 0$ and $h(0) = A + B - D < 0$ owing to assumption (C4). Thus the algebra equation $h(\mu) = 0$ possesses a unique root $\mu = \mu^*$ on the interval $[0, D - A]$.

Now, let $p > 1$ be a constant. For $-r \leq t \leq 0$, we can derive from (23) and (27) that

$$
\mathcal{Z}(t) = W \mathcal{R}(t) e^{\mathcal{Z}(t)} \leq \mathcal{R}(t) \leq p \mathcal{R}(t).
$$

That is to say, $\mathcal{Z}(t) \leq p \mathcal{R}(t)$, for $-r \leq t \leq 0$.

(28)

Next we prove that the inequality (28) still holds for all $t > 0$ by way of contradiction. Otherwise, there exists a real number $t^* \in (0, +\infty)$ such that

$$
\mathcal{Z}(t) \leq p \mathcal{R}(t) \quad \text{for } -r \leq t < t^*, \quad \text{and } \mathcal{Z}(t^*) = p \mathcal{R}(t^*), \quad \frac{d\mathcal{Z}(t)}{dt} \bigg|_{t=t^*} > p \frac{d\mathcal{R}(t)}{dt} \bigg|_{t=t^*}.
$$

(29)

From (26) and (27), we can deduce that

$$
\frac{d\mathcal{Z}(t)}{dt} \bigg|_{t=t^*} = -p(D - A) \mathcal{R}(t^*) + B p \sup_{t^* - n \theta \leq \theta \leq t^*} \mathcal{R}(\theta) = -p(D - A) \mathcal{R}(t^*) + B p \sup_{t^* - n \theta \leq \theta \leq t^*} We^{-\mathcal{Z}(t^*)}
$$

(30)

\begin{align*}
&\leq -(D - A)p \mathcal{R}(t^*) + B p \sup_{t^* - n \theta \leq \theta \leq t^*} We^{-\mathcal{Z}(t^*)} \\
&= -(D - A)p \mathcal{R}(t^*) + B p We^{-\mathcal{Z}(t^*)} e^{\mathcal{Z}(t^*)} \leq -(D - A)p \mathcal{R}(t^*) + B p \mathcal{R}(t^*) e^{\mathcal{Z}(t^*)} \\
&= -(D - A)p \mathcal{R}(t^*) + B p \mathcal{R}(t^*) \mathcal{R}(t^*) = -p \mathcal{Z}(t^*) = p \frac{d\mathcal{R}(t)}{dt} \bigg|_{t=t^*}.
\end{align*}

That is

$$
\frac{d\mathcal{Z}(t)}{dt} \bigg|_{t=t^*} \leq p \frac{d\mathcal{R}(t)}{dt} \bigg|_{t=t^*}.
$$

This contradicts the fact $\frac{d\mathcal{Z}(t)}{dt} \bigg|_{t=t^*} > p \frac{d\mathcal{R}(t)}{dt} \bigg|_{t=t^*}$ given by (29). Therefore, we obtain

$$
\mathcal{Z}(t) \leq p \mathcal{R}(t), \quad \text{for } t \geq 0.
$$

(31)

Then, it follows from (25), (27) and (31) that

$$
\mathcal{U}(t) \leq \mathcal{Z}(t) \leq p \mathcal{R}(t) = p We^{-\mathcal{Z}(t)}, \quad \text{for } t \geq 0.
$$

This yields that

$$
\omega(t) \leq p We^{-\mathcal{Z}(t)} \leq p We^{-n \theta}, \quad \text{for } t \geq 0.
$$

By allowing $p \to 1$, we can get that

$$
\omega(t) \leq \omega(t) \leq p We^{-\mathcal{Z}(t)} \leq p We^{-n \theta} \leq \omega(t) \leq \omega(t), \quad \text{for } t \geq 0.
$$
That is
\[ \sum_{i=1}^{n} |y_i(t) - x_i(t)| \leq \| \phi - \psi \| e^{-\mu t}, \quad \text{for all } t \geq 0. \]

By Definition 3, it follows that the memristor-based neural networks (1) and (5) realize the first-type global exponential synchronization under the discontinuous state-feedback controller (9). This completes the proof.\( \square \)

Remark 1. By using the same method in (17), we can also obtain
\[ G_i(s) \leq \left( \sum_{j=1}^{n} a_{ij} \right) M_j + \left( \sum_{j=1}^{n} b_{ij} \right) N_j \sum_{j=1}^{n} \left| \hat{f}_j(y_j(s)) \right| + \left( \sum_{j=1}^{n} \hat{b}_{ij} \right) N_j - \eta_j \right) \leq \left( \sum_{j=1}^{n} a_{ij} \right) M_j + \left( \sum_{j=1}^{n} b_{ij} \right) N_j \sum_{j=1}^{n} \left| \hat{f}_j(y_j(s)) \right| + \left( \sum_{j=1}^{n} \hat{b}_{ij} \right) N_j - \eta_j \right) \leq \left( 2A + 2B + C + D - \eta_j \right) \leq \left( 2A + 2B + C + D - \eta_j \right) \leq 0. \]

where \( A = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} a_{ij} m_j \right) \), \( B = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} b_{ij} n_j \right) \), \( C = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \hat{a}_{ij} m_j \right) \), \( D = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \hat{b}_{ij} n_j \right) \). When \( 2A + 2B + C + D \leq \eta_j \), it follows that \( G_i(s) \leq 0 \). Therefore, we can obtain from (17) that \( |\omega_i(t)| \leq |\omega_i(0)| e^{-\mu t}. \) That is to say, under the conditions \( 2A + 2B + C + D \leq \eta_j \) and (2), we can also realize the exponential synchronization control of memristor-based neural networks (1) and (5). In this case, the second part \( \eta_j \) of controller (9) plays a key role in solving the exponential synchronization issues.

Meanwhile, by choosing suitable value of \( \kappa \), we can shorten the convergence time of synchronization errors for guaranteed fast response. However, by comparison we can find that the assumption \( 2A + 2B + C + D \leq \eta_j \) is a strong condition which requires that the control gain \( \eta_j \) is relatively larger. It should be pointed out that if the condition \( 2A + 2B + C + D \leq \eta_j \) holds, then the condition (C3) is satisfied. This means that the condition (C3) is relatively weak. This is helpful for us to reduce the control cost due to the relatively smaller value of control gain \( \eta_j \).

Theorem 2. Suppose the conditions (C1)--(C3) are satisfied, assume further that
\[ \text{C5}: A + B + C < 2D, \text{ where} \]
\[ A = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} a_{ij} \right)^2, \quad B = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} b_{ij} \right)^2, \quad C = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \hat{a}_{ij} \right)^2, \quad D = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \hat{b}_{ij} \right)^2, \]

\( D, a_{ij}^2 \) and \( b_{ij}^2 \) have been given in Theorem 1. Then the memristor-based neural networks (1) and (5) can realize the second-type global exponential synchronization under the discontinuous state-feedback controller (9), where the exponential synchronization rate \( \mu \) is the unique root of the following algebra equation
\[ \mu = 2D - A - C - B e^{\mu}. \]

Proof. Multiplying each side of (11) by \( 2\omega_i(t) \), we can obtain following differential equation
\[ \frac{d\omega_i^2(t)}{dt} = -2(d_i + \kappa_\omega) \omega_i^2(t) + 2 \sum_{j=1}^{n} a_{ij} \omega_j(t) \omega_i(t) + 2 \sum_{j=1}^{n} b_{ij} \omega_j(t) \omega_i(t) \]
\[ + 2 \sum_{j=1}^{n} \left( \hat{a}_{ij}(t) \omega_j(t) + \hat{b}_{ij}(t) \omega_j(t) \right) \omega_i(t) - 2 \eta_j |\omega_i(t)|, \quad \text{for a.e. } t \in [0, +\infty). \]

Applying the variation-of-constants formula to system (33), we have
\[ \omega_i^2(t) = \omega_i^2(0) e^{-2(d_i + \kappa_\omega)t} + \int_{0}^{t} e^{-2(d_i + \kappa_\omega)(t-s)} G_i(s) ds. \]

where
\[ \dot{G}_i(s) = 2 \sum_{j=1}^{n} \left( a_{ij}(s) \right) \omega_j(s) \omega_i(s) + 2 \sum_{j=1}^{n} \left( b_{ij}(s) \right) \omega_j(s) \omega_i(s) + 2 \sum_{j=1}^{n} \left( \hat{a}_{ij}(s) \right) \omega_j(s) \omega_i(s) \]
\[ + 2 \sum_{j=1}^{n} \left( \hat{b}_{ij}(s) \right) \omega_j(s) \omega_i(s) - 2 \eta_j |\omega_i(s)|. \]

Similar to (15), the symbol \( \int_{0}^{t} \) in (34) represents the Lebesgue integration. Moreover, under the conditions (C1)--(C3), by using the element inequality \( 2ab \leq a^2 + b^2 \), we can derive from (35) that
\[ \dot{G}_i(s) \leq 2 \sum_{j=1}^{n} \left| a_{ij}(s) \right| \left| \int_{0}^{s} \omega_j(s) \omega_i(s) \right| + 2 \sum_{j=1}^{n} \left| b_{ij}(s) \right| \left| \omega_j(s) \omega_i(s) \right| + 2 \sum_{j=1}^{n} \left( \hat{a}_{ij}(s) \right) \omega_j(s) \omega_i(s) + 2 \sum_{j=1}^{n} \left( \hat{b}_{ij}(s) \right) \omega_j(s) \omega_i(s) \]
\[ + 2 \sum_{j=1}^{n} \left( \hat{a}_{ij}(s) \right) \omega_j(s) \omega_i(s) + 2 \sum_{j=1}^{n} \left( \hat{b}_{ij}(s) \right) \omega_j(s) \omega_i(s) \]
Obviously, we can obtain from (40) that
\[
\omega^2(t) \leq \omega^2(0) e^{-2\Delta t} + \int_0^t e^{-2\Delta t - \alpha s} \left[ \frac{1}{2} \sum_{i, j = 1}^{n} (a_{ij}^+ + b_{ij}^+) \omega^2(s) + \frac{1}{2} \sum_{i=1}^{n} b_{ij}^2 Q_{ij}^2 \omega^2(s - r(s)) \right] ds.
\]

Making sum on every side of the above inequality (37), we have
\[
\sum_{i=1}^{n} \omega^2(t) \leq \sum_{i=1}^{n} \omega^2(0) e^{-2\Delta t} + \int_0^t e^{-2\Delta t - \alpha s} \left[ \frac{1}{2} \sum_{i, j = 1}^{n} (a_{ij}^+ + b_{ij}^+) \sum_{j=1}^{n} a_{ij}^2 \omega^2(s) + \frac{1}{2} \sum_{i=1}^{n} b_{ij}^2 Q_{ij}^2 \omega^2(s - r(s)) \right] ds.
\]

For the sake of simplification, we set
\[
\omega^\text{sum}(t) = \sum_{i=1}^{n} \omega^2(t) = \sum_{i=1}^{n} (y_i(t) - x_i(t))^2.
\]

Then, from (38) and (39), we can obtain
\[
\omega^\text{sum}(t) \leq \omega^\text{sum}(0) e^{-2\Delta t} + \dot{\mathbf{A}} \int_0^t e^{-2\Delta t - \alpha s} \omega^\text{sum}(s) ds + \mathbf{B} \int_0^t e^{-2\Delta t - \alpha s} \omega^\text{sum}(s - r(s)) ds,
\]
where \( \dot{\mathbf{A}} = \max_{1 \leq i, j \leq n} \left( \sum_{k=1}^{n} a_{ij}^2 \right) \), \( \dot{\mathbf{B}} = \max_{1 \leq i, j \leq n} \left( \sum_{k=1}^{n} b_{ij}^2 \right) \), and \( \mathbf{C} = \max_{1 \leq i \leq n} \left( \sum_{k=1}^{n} b_{ij}^2 \right) \). Similarly, \( \omega^\text{sum}(s) \) and \( \omega^\text{sum}(s - r(s)) \) are Riemann integrable owing to their continuities. And the above inequality (40) is still true if we replace the Lebesgue integration in (41) with Riemann integration. For convenience, we still use “\( \int \)” to denote the Riemann integration. In the following, we let
\[
\mathcal{W}_V = \sup_{-\tau \leq s \leq 0} \sum_{i=1}^{n} \omega^2(s) = \| \phi - y \|_2.
\]

On the interval \([-\tau, +\infty)\), let us define the following function \( \tilde{Z}(t) \)
\[
\tilde{Z}(t) = \begin{cases} 
\mathcal{W}_V e^{-2\Delta t + \dot{\mathbf{A}} t} \int_0^t e^{-2\Delta t - \alpha s} \omega^\text{sum}(s) ds & \text{if } t > 0, \\
\mathcal{W}_V & \text{if } -\tau \leq t \leq 0.
\end{cases}
\]

It is easy to see that \( \tilde{Z}(t) \) is a continuous differential function on the interval \([-\tau, +\infty)\).

Now, let us set
\[
\bar{U}(t) = \begin{cases} 
\omega^\text{sum}(t), & \text{if } t > 0, \\
\mathcal{W}_V, & \text{if } -\tau \leq t \leq 0.
\end{cases}
\]

Obviously, we can obtain from (40)–(42) that
\[
\bar{U}(t) \leq \tilde{Z}(t), \quad \text{for any } t \geq -\tau.
\]

For \( t > 0 \), computing the time derivative of \( \tilde{Z}(t) \), we have
\[
\frac{d\tilde{Z}(t)}{dt} = -2\Delta \tilde{Z}(t) + (\dot{\mathbf{A}} + \mathbf{C}) \omega^\text{sum}(t) + \dot{\mathbf{B}} \omega^\text{sum}(t - r(t)) \leq -2\Delta \tilde{Z}(t) + \dot{\mathbf{A}} \tilde{Z}(t) + \dot{\mathbf{B}} \tilde{Z}(t - r(t)) \leq -2\Delta + \dot{\mathbf{A}} \tilde{Z}(t) + \dot{\mathbf{B}} \sup_{-\tau \leq \theta \leq t} \tilde{Z}(\theta). \]
We denote \( h(\mu) \) by \( h(\mu) = \mu - 2D + A + C + B e^{\mu t} \). Since \( \frac{\partial h(\mu)}{\partial \mu} = 1 + B e^{\mu t} > 0 \), then \( h(\mu) \) is a strictly increasing function with respect to \( \mu \) on \( \mathbb{R} \). Meanwhile, \( h(2D - A - C) = B e^{2D - A - C} > 0 \). And it follows from condition (C5) that \( h(0) = A + B + C - 2D < 0 \). This means that the algebra equation \( h(\mu) = 0 \) possesses a unique root \( \mu = \mu^* \) on the interval \([0, 2D - A - C]\). Set

\[
\mathcal{R}(t) = \lambda(t)e^{-\mu^*t}, \quad \text{for} \quad t \geq -\tau,
\]

where \( \mu^* = \mu^*(D, A, B, C, t) \) denotes the unique root of the following algebra equation

\[
\mu = 2D - A - C - B e^{\mu t}.
\]

Let \( p > 1 \) be a constant. It is not difficult to deduce from (41) and (45) that the following inequality holds for \(-\tau \leq t \leq 0\):

\[
\mathcal{Z}(t) = \mathcal{W}(t)e^{\mu^* t} \leq \mathcal{R}(t) \leq p\mathcal{R}(t).
\]

That is

\[
\mathcal{Z}(t) \leq p\mathcal{R}(t), \quad \text{for} \quad -\tau \leq t \leq 0.
\]

Similar to the proof of Theorem 1 by way of contradiction, we can obtain

\[
\mathcal{Z}(t) \leq p\mathcal{R}(t), \quad \text{for} \quad t > 0.
\]

Hence, we can derive from (43), (45) and (47) that

\[
\dot{\mathcal{U}} \leq \mathcal{Z}(t) \leq p\mathcal{R}(t) = p\lambda(t)e^{-\mu^*t}, \quad \text{for} \quad t \geq 0.
\]

This means that

\[
\omega^\text{sum}(t) \leq p\lambda(t)e^{-\mu^*t}, \quad \text{for} \quad t \geq 0.
\]

Let \( p \to 1 \), we have

\[
\omega^\text{sum}(t) \leq \lambda(t)e^{-\mu^*t}, \quad \text{for} \quad t \geq 0.
\]

That is to say

\[
\sum_{i=1}^{n} (y_i(t) - x_i(t))^2 \leq \|\phi - \psi\|_2 e^{-\mu^*t}, \quad \text{for all} \ t \geq 0.
\]

According to Definition 3, the memristor-based neural networks (1) and (5) achieve the second-type global exponential synchronization under the discontinuous state-feedback controller (9). This completes the proof.

Remark 2. As is well known, the memristor-based neural networks with time delays are basically state-dependent switching delayed dynamical systems which are usually described by delayed differential equations with discontinuous right-hand sides. In order to analyze the synchronization of memristive systems involving time delays, the conventional approach is the Lyapunov functional method (e.g. [32–37]). However, the suitable Lyapunov functional is difficult to be constructed because its structure might be very complex. In this section, we introduce a new method to avoid this difficulty. That is to say, we only use some analytic techniques to investigate the exponential synchronization of memristor-based neural network with time delay under discontinuous state-feedback controller. Furthermore, the new exponential synchronization conditions given by this section are simple and can be easily verified in practice. Clearly, this new method is much different from the previous works and may be applied for analyzing other classes of state-dependent discontinuous dynamical systems with time delays.

Remark 3. In [33,34,36,37], the authors utilized the theory of differential inclusion to study the synchronization and anti-synchronization problems of memristive neural networks based on the main condition as follows:

\[
\begin{align*}
\text{co} \left[ \left[ a_{ji}, a_{ij} \right] f_j(x_i(t)) \right] \pm \text{co} \left[ \overline{a}_{ji}, \overline{a}_{ij} \right] f_j(y_i(t)) \leq \text{co} \left[ \left[ a_{ji}, a_{ij} \right], \left[ \overline{a}_{ji}, \overline{a}_{ij} \right] \right] \left( f_i(x_i(t)) \pm f_j(y_i(t)) \right).
\end{align*}
\]
Remark 4. To the authors' knowledge, in the existing literature (e.g., [42]), there are some results on stability analysis of classical neural slave network system with memristor. It is noted that our results are also applicable to the common neural network system without memristor. Discontinuous factors. In this case, the analytic techniques of [42] are invalid for handling switching or discontinuous dynamical neuron neural networks via new analytic techniques. A further issue worth addressing concerns the possibility of systems. Therefore, our method is novel. In addition, only a few papers have studied the synchronization problems of memristor-based systems. Based on the theory of differential inclusions given by Filippov, we have extended this method to switching or discontinuous systems. Therefore, our method is novel. In addition, only a few papers have studied the synchronization problems of memristor-based neural networks via new analytic techniques. A further issue worth addressing concerns the possibility of finding better tools to investigate the synchronization problems of memristor-based neural networks system or other discontinuous dynamical systems.

Remark 5. The main difference between the two types of globally exponential synchronization is the norm of the vector. However, there also exists some other differences between the two kinds of synchronization. From the proofs of Theorems 1 and 2, one of important differences is the estimated rate of exponential synchronization. That is, the two kinds of estimated rates of exponential synchronization are determined by different algebra equations. As is well known, in the design of neural network circuit for synchronization, it is often desired that the trajectories of the error states converge to zero in different exponential rate for guaranteeing fast response. Therefore, it is necessary and significant to provide the different of estimated rates for exponential synchronization.

4. Numerical examples

In this section, two numerical examples are given to show the effectiveness of our method and theoretical results obtained above.

Example 1. Consider a two-dimensional memristor-based neural network system as follows:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -d_1 x_1(t) + \sum_{j=1}^{2} a_{1j}(x_1(t)) f_j(x_j(t)) + \sum_{j=1}^{2} b_{1j}(x_1(t)) g_j(x_j(t-\tau(t))) + f_1, \\
\frac{dx_2(t)}{dt} &= -d_2 x_2(t) + \sum_{j=1}^{2} a_{2j}(x_2(t)) f_j(x_j(t)) + \sum_{j=1}^{2} b_{2j}(x_2(t)) g_j(x_j(t-\tau(t))) + f_2,
\end{align*}
\]

where

\[
\begin{align*}
da_{11}(x_1(t)) &= \begin{cases} 
-0.2, & |x_1(t)| \leq 0.5, \\
0.2, & |x_1(t)| > 0.5,
\end{cases} & b_{11}(x_1(t)) &= \begin{cases} 
-0.1, & |x_1(t)| \leq 0.5, \\
0.2, & |x_1(t)| > 0.5,
\end{cases} \\
da_{12}(x_1(t)) &= \begin{cases} 
2, & |x_1(t)| \leq 0.5, \\
3, & |x_1(t)| > 0.5,
\end{cases} & b_{12}(x_1(t)) &= \begin{cases} 
2, & |x_1(t)| \leq 0.5, \\
2.5, & |x_1(t)| > 0.5,
\end{cases} \\
da_{21}(x_2(t)) &= \begin{cases} 
3, & |x_2(t)| \leq 0.5, \\
2, & |x_2(t)| > 0.5,
\end{cases} & b_{21}(x_2(t)) &= \begin{cases} 
3, & |x_2(t)| \leq 0.5, \\
2.5, & |x_2(t)| > 0.5,
\end{cases} \\
da_{22}(x_2(t)) &= \begin{cases} 
-0.1, & |x_2(t)| \leq 0.5, \\
0.1, & |x_2(t)| > 0.5,
\end{cases} & b_{22}(x_2(t)) &= \begin{cases} 
-0.1, & |x_2(t)| \leq 0.5, \\
0.2, & |x_2(t)| > 0.5.
\end{cases}
\]

And \(d_1 = d_2 = 0.6, f_1 = f_2 = 0, \tau(t) = 1\), taking the feedback functions as follows:

Unfortunately, this condition is not correct. Yang et al. have pointed out this mistake in the literature [32] and the interested readers can consult [32] to get more detailed explanations. In this paper, we have dropped this wrong assumption. On the other hand, since the memristor-based conventional controllers such as linear state-feedback controllers \(u_i(t) = -\epsilon_i (y_i(t) \pm x_i(t))\) and \(u_i(t) = -\sum_{j=1}^{m} \epsilon_j (y_j(t) \pm x_j(t))\). Therefore, we introduce a discontinuous state-feedback controller (9) which plays an important role to realize exponential synchronization control of master–slave network system with memristor.
\[ f_j(\theta) = g_j(\theta) = \cos(\theta), \quad j = 1, 2. \]

Obviously, the feedback functions satisfy the conditions (C1) and (C2) with \( L_j = Q_j = M_j = N_j = 1(j = 1, 2). \) Figs. 2 and 3 show the chaotic-like trajectory of memristive system (48) with initial value \( \phi(t) = (6, 1)^T, \ t \in [-1, 0]. \)

Now we choose the control gains \( k_1 = k_2 = 8 \) and \( \eta_1 = \eta_2 = 3. \) It can be easily verified that

\[ 3 = \eta_1 > \sum_{j=1}^{2} |\tilde{a}_{ij} - \tilde{a}_{ij}| M_j + \sum_{j=1}^{2} |\tilde{b}_{ij} - \tilde{b}_{ij}| N_j = 2.2. \]
the second-type global exponential synchronization with the corresponding slave system under the discontinuous state-feedback controller (9).

Example 2. Consider the following three-dimensional memristor-based neural network system:

\[
A = \max_{1 \leq j \leq 2} \left( \sum_{i=1}^{2} a_{ij} L_j \right) = 3.2, \quad B = \max_{1 \leq j \leq 2} \left( \sum_{i=1}^{2} b_{ij}^+ Q_j \right) = 3.2, \quad D = \min_{1 \leq i \leq 2} \left| d_i + \kappa_i \right| = 8.6.
\]

So we get \(A + B < D = 8.6\) which shows that the condition (C4) is satisfied. According to Theorem 1, we conclude that the master system (48) can realize the first-type global exponential synchronization with the corresponding slave system under the discontinuous state-feedback controller (9).

On the other hand, we have

\[
A' = \max_{1 \leq j \leq 2} \left( \sum_{i=1}^{2} a_{ij}^+ L_j \right) = B' = \max_{1 \leq j \leq 2} \left( \sum_{i=1}^{2} b_{ij}^+ Q_j \right) = 3.2, \quad C' = \max_{1 \leq i \leq 2} \left( \sum_{j=1}^{2} \left( a_{ij}^+ + b_{ij}^+ \right) \right) = 6.3.
\]

It can be seen that \(12.7 = A' + B' + C' < 2D = 17.2\) and (C5) is also satisfied. By Theorem 2, we know that the master system (48) can also realize the second-type global exponential synchronization with the corresponding slave system under the discontinuous state-feedback controller (9).

For numerical simulations, we take the initial condition of master system as \(x(t) = (6, 1)^T\) for \(t \in [-1, 0] \text{ and } y(t) = (-10, 8)^T\) for \(t \in [-1, 0].\) Figs. 4 and 5 show that the synchronization between the master system (48) and its corresponding slave system can also be achieved under the discontinuous state-feedback controller (9). These numerical simulations verify the theoretical results perfectly.

Example 2. Consider the following three-dimensional memristor-based neural network system:

\[
\frac{dx_i(t)}{dt} = -d_x x_i(t) + \sum_{j=1}^{3} a_{ij}(x_j(t)) f_j(x_j(t)) + \sum_{j=1}^{3} b_{ij}(x_i(t)) g_j(x_j(t-r(t))) + J_i, \quad i = 1, 2, 3,
\]

where

\[
a_{11}(x_1(t)) = \begin{cases} -0.1, & |x_1(t)| \leq 0.5, \\ 0.1, & |x_1(t)| > 0.5, \end{cases} \quad b_{11}(x_1(t)) = \begin{cases} 0.1, & |x_1(t)| \leq 0.5, \\ -0.2, & |x_1(t)| > 0.5, \end{cases}
\]

\[
a_{12}(x_1(t)) = \begin{cases} 1, & |x_1(t)| \leq 0.5, \\ 2, & |x_1(t)| > 0.5, \end{cases} \quad b_{12}(x_1(t)) = \begin{cases} 2, & |x_1(t)| \leq 0.5, \\ 3, & |x_1(t)| > 0.5, \end{cases}
\]

\[
a_{21}(x_2(t)) = \begin{cases} 2.5, & |x_2(t)| \leq 0.5, \\ 2, & |x_2(t)| > 0.5, \end{cases} \quad b_{21}(x_2(t)) = \begin{cases} 2.5, & |x_2(t)| \leq 0.5, \\ 1.5, & |x_2(t)| > 0.5, \end{cases}
\]

\[
a_{22}(x_2(t)) = \begin{cases} -0.2, & |x_2(t)| \leq 0.5, \\ 0.1, & |x_2(t)| > 0.5, \end{cases} \quad b_{22}(x_2(t)) = \begin{cases} 0.1, & |x_2(t)| \leq 0.5, \\ -0.1, & |x_2(t)| > 0.5, \end{cases}
\]

\[
a_{31}(x_3(t)) = \begin{cases} 0.2, & |x_3(t)| \leq 0.5, \\ -0.2, & |x_3(t)| > 0.5, \end{cases} \quad b_{31}(x_3(t)) = \begin{cases} -0.2, & |x_3(t)| \leq 0.5, \\ 0.3, & |x_3(t)| > 0.5, \end{cases}
\]

\[
a_{32}(x_3(t)) = \begin{cases} 2, & |x_3(t)| \leq 0.5, \\ 3, & |x_3(t)| > 0.5, \end{cases} \quad b_{32}(x_3(t)) = b_{31}(x_3(t)) = a_{31}(x_3(t)) = 0.
\]
Clearly, the above feedback functions satisfy the conditions (C1) and (C2) with \( \eta \) given by Yang et al.\[32\] plays a key role in dealing with the uncertain differences between the Filippov solutions of the master memristive or switching neural networks can be investigated under the framework of Filippov’s solution. In order to realize the synchronization hysteresis. Because of the switching jumps property of memristor, this kind of dynamical neuron system is described by the delayed differential equation with discontinuous right-hand side. According to the theories of set-valued maps and differential inclusions, the dynamical behaviors of memristive or switching neural networks can be investigated under the framework of Filippov’s solution. In order to realize the synchronization of memristor-based neural networks, we have added a discontinuous state-feedback controller to the slave system. Such a new controller given by Yang et al.\[32\] plays a key role in dealing with the uncertain differences between the Filippov solutions of the master–slave systems. Under this discontinuous state-feedback controller, we only have used some new analytic techniques to discuss the first-type and second-type

\[ a_{13}(x_1(t)) = b_{13}(x_1(t)) = a_{23}(x_2(t)) = b_{23}(x_2(t)) = 0. \]

In Example 2, take \( d_1 = d_2 = d_3 = 0.7, J_1 = J_2 = J_3 = 0 \) and \( r(t) \equiv 1 \). Let the feedback functions turn to be

\[ f_j(\theta) = g_j(\theta) = \sin(\theta), \quad j = 1, 2, 3. \]

Clearly, the above feedback functions satisfy the conditions (C1) and (C2) with \( L_j = Q_j = M_j = N_j = 1(j = 1, 2, 3) \). Fig. 6 shows the chaotic-like trajectory of memristive system \((49)\) with initial value \( \phi(t) = (1, -4, -7)^T \) for \( t \in [-1, 0] \). Select the control gains \( k_1 = k_2 = k_3 = 10 \) and \( \eta_1 = \eta_2 = \eta_3 = 3 \). Similar to Example 1, it is straightforward to check that

\[
3 = \eta_1 > 3 \sum_{j=1}^{3} (\tilde{a}_{1j} - \tilde{a}_{1j} | M_j + \sum_{j=1}^{3} (\tilde{b}_{1j} - \tilde{b}_{1j}) | N_j) = 2.5,
\]

\[
3 = \eta_2 > 3 \sum_{j=1}^{3} (\tilde{a}_{2j} - \tilde{a}_{2j} | M_j + \sum_{j=1}^{3} (\tilde{b}_{2j} - \tilde{b}_{2j}) | N_j) = 2,
\]

\[
3 = \eta_3 > 3 \sum_{j=1}^{3} (\tilde{a}_{3j} - \tilde{a}_{3j} | M_j + \sum_{j=1}^{3} (\tilde{b}_{3j} - \tilde{b}_{3j}) | N_j) = 1.9.
\]

This shows that the condition (C3) is satisfied.

By simple computing, we get

\[
A = \max_{1 \leq j \leq 1} \left( \sum_{i=1}^{3} a_{ij} L_j \right) = 5.2, \quad B = \max_{1 \leq j \leq 3} \left( \sum_{i=1}^{3} b_{ij} Q_j \right) = 3.1, \quad C = \min_{1 \leq j \leq 1} (d_i + k_i) = 10.7.
\]

It is easy to see that \( 8.3 = A + B < D = 10.7 \). By now we have checked that all the conditions in Theorem 1 hold. So the first-type global exponential synchronization between the master system \((49)\) and its corresponding slave system can be realized under the discontinuous state-feedback controller \((9)\).

In addition, we can easily calculate that

\[
\mathcal{A} = \max_{1 \leq j \leq 1} \left( \sum_{i=1}^{3} a_{ij}^+ L_j \right) = 5.2, \quad \mathcal{B} = \max_{1 \leq j \leq 3} \left( \sum_{i=1}^{3} b_{ij}^+ Q_j \right) = 3.1, \quad \mathcal{C} = \max_{1 \leq j \leq 1} \left( \sum_{i=1}^{3} (a_{ij}^+ + b_{ij}^+) \right) = 5.3.
\]

Obviously, \( 13.6 = \mathcal{A} + \mathcal{B} + \mathcal{C} < 2D = 21.4 \). Therefore, all the conditions required in Theorem 2 are satisfied. Then, under the discontinuous state-feedback controller \((9)\), the second-type global exponential synchronization between the master system \((49)\) and its corresponding slave system can also be achieved. To simulate the obtained results, the initial values of master system \((49)\) and its corresponding slave system are set to be \( \phi(t) = (1, -4, -7)^T \) for \( t \in [-1, 0] \) and \( \mathcal{W}(t) = (5, -7, -2)^T \) for \( t \in [-1, 0] \). Fig. 7 presents the time evolution of every variable of coupled master system \((49)\) and the corresponding slave system which also illustrates the master–slave system can realize synchronization. Fig. 8 depicts the trajectory of every error state approaches to zero quickly as time goes. The above numerical simulations are in accordance with our results.

5. Conclusions

This paper has introduced a general class of memristor-based neural networks with time-varying delay to characterize the feature of pinched hysteresis. Because of the switching jumps property of memristor, this kind of dynamical neuron system is described by the delayed differential equation with discontinuous right-hand side. According to the theories of set-valued maps and differential inclusions, the dynamical behaviors of memristive or switching neural networks can be investigated under the framework of Filippov’s solution. In order to realize the synchronization control of memristor-based neural networks, we have added a discontinuous state-feedback controller to the slave system. Such a new controller given by Yang et al.\[32\] plays a key role in dealing with the uncertain differences between the Filippov solutions of the master–slave systems. Under this discontinuous state-feedback controller, we only have used some new analytic techniques to discuss the first-type and second-type...
global exponential synchronization of memristor-based neural network with time-varying delay. This new method is different from the conventional Lyapunov functional approach. Therefore, the obtained results in this paper are novel and effectively complement or improve the previously results. Besides, we have also given the estimated rate of exponential synchronization which are determined by some algebra equations involving delays and parameters of memristive system. In short, our outcomes possess an important instructional significance in the design and applications of synchronized neural networks circuits with memristor. It should be pointed out that our method and results are also applicable to the continuous neural network system without switching jumps.

References


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