Remarks on an integral functional driven by sub-fractional Brownian motion

Guangjun Shen\textsuperscript{a,b}, Litan Yan\textsuperscript{c,*}

\textsuperscript{a} Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China
\textsuperscript{b} Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China
\textsuperscript{c} Department of Mathematics, Donghua University, Shanghai 201620, PR China

\begin{abstract}
This paper studies the functionals
\begin{align*}
A_1(t, x) &= \int_0^t 1_{(0, \infty)}(x - S_H^t)ds,
A_2(t, x) &= \int_0^t 1_{(0, \infty)}(x - S_H^t)s^{2H-1}ds,
\end{align*}
where \((S_H^t)_{0 \leq t \leq T}\) is a one-dimensional sub-fractional Brownian motion with index \(H \in (0, 1)\). It shows that there exists a constant \(p_H \in (1, 2)\) such that \(p\)-variation of the process \(A_j(t, S_H^t) - \int_0^t \mathcal{L}(s, S_H^s)ds\) \((j = 1, 2)\) is equal to 0 if \(p > p_H\), where \(\mathcal{L}_j, j = 1, 2\), are the local time and weighted local time of \(S_H^t\), respectively. This extends the classical results for Brownian motion.
\end{abstract}

\section{Introduction}

Recently, fractional Brownian motion has become an object of intense study, due to its interesting properties and its applications in various scientific areas including telecommunications, turbulence, image processing and finance. Recall that fractional Brownian motion (fBm in short) with Hurst index \(H \in (0, 1)\) is a mean zero Gaussian process \(B_H = \{B_H^t, t \geq 0\}\) such that
\begin{equation}
E[B_H^t B_H^s] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t-s|^{2H} \right],
\end{equation}
for all \(t, s \geq 0\). For \(H = 1/2\), \(B_H^t\) coincides with the standard Brownian motion \(B\). \(B_H^t\) is neither a semimartingale nor a Markov process unless \(H = 1/2\). The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity and which has stationary increments. On the other hand, based on the sufficient study of fBm, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm were introduced. However, contrast to the extensive studies

\textsuperscript{*} This work was supported by NSFC (10871041) and Key NSF of Anhui Educational Committee (KJ2011A).
\textsuperscript{*} Corresponding author.
\textsuperscript{E-mail addresses: guangjunshen@yahoo.com.cn (G. Shen), litanyan@dhu.edu.cn (L. Yan).}

1226-3192/S - see front matter \textcopyright 2011 The Korean Statistical Society. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.jkss.2010.12.004
on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments.

As an extension of Brownian motion, recently, Bojdecki, Gorostiza, and Talarczyk (2004) have introduced and studied a rather general class of self-similar Gaussian processes. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the sub-fractional Brownian motion. The so-called sub-fractional Brownian motion (sub-fBm in short) with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S^H_t, t \geq 0\}$ with $S^H_0 = 0$ and

$$E \left[ S^H_t S^H_s \right] = t^{2H} + t^{2H} - \frac{1}{2} \left[ (s + t)^{2H} + |t - s|^{2H} \right],$$

for all $s, t \geq 0$. For $H = 1/2$, $S^H$ coincides with the standard Brownian motion $B$. $S^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $S^H$. The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), but it does not have stationary increments. More works for sub-fractional Brownian motion can be found in Bojdecki, Gorostiza, and Talarczyk (2006, 2007) and Tudor (2007, 2009, 2008), Yan and Shen (in press, 2010).

In the study of stochastic area integral for standard Brownian motion $B$, Rogers and Walsh (1990, 1991) were led to analyze the functional of the form

$$A(t, B_t) = \int_0^t 1_{[0, \infty)}(B_t - B_s)ds.$$

In particular, by using the classical Itô formula, essentially the Burkholder–Davis–Gundy inequalities of martingales and the decomposition of the expression

$$\sum_{j=1}^{\infty} |X_{ij/2^n} - X_{ij-(1/2^n)}|^p,$$

with $X_t = A(t, B_t)$, they showed that the process $A(t, B_t)$ is not a semimartingale, and in fact showed that the process

$$A(t, B_t) - \int_0^t \mathcal{S}(s, B_s)dB_s,$$

has finite non-zero $\frac{4}{3}$ variation. Here $\mathcal{S}(t, x)$ is the local time of $B$ at $x$, which is formally defined by $\mathcal{S}(t, x) = \int_0^t \delta(B_s - x)ds$. However, it is difficult to use the method introduced in Rogers and Walsh (1990) to investigate the analogue of sub-fBm.

Recently, Yan, Yang, and Lu (2008) have considered the similar integral functional driven by fractional Brownian motion $B^H$ with Hurst index $H \in (0, 1)$ which arises in the study of integration with respect to fractional local times of fractional Brownian motion (see Yan, Liu, and Yang (2008)).

Inspired by these results, it seems interesting to study the similar integral functional driven by sub-fBm $S^H$, a rather general class of self-similar Gaussian processes which do not have stationary increments.

In this paper, we consider the integral functionals of the form

$$A_1(t, x) = \int_0^t 1_{[0, \infty)}(x - S^H_t)ds,$$

$$A_2(t, x) = \int_0^t 1_{[0, \infty)}(x - S^H_t) s^{2H-1}ds.$$

Note that if $H = \frac{1}{2}$, then $A_1(t, S^H_t) = A_2(t, S^H_t) = A(t, B_t)$.

Our main aim is to study the $p$-variation of the following processes:

$$X_t^{(1)} := A_1(t, S^H_t) - \int_0^t \mathcal{S}_1(s, S^H_s) ds^H,$$

$$X_t^{(2)} := A_2(t, S^H_t) - \int_0^t \mathcal{S}_2(s, S^H_s) ds^H,$$

when $H \geq \frac{1}{2}$, where the stochastic integral is of the Skorohod type and

$$\mathcal{S}_1(t, x) = \int_0^t \delta \left( S^H_t - x \right) ds,$$

$$\mathcal{S}_2(t, x) = \int_0^t \delta \left( S^H_s - x \right) s^{2H-1}ds,$$

are the local time and weighted local time of $S^H$ at $x$, respectively. We find a constant $1 < p_H < 2$ depending only on $H$ such that the $p$-variation of $X_t^{(j)}$ ($j = 1, 2$) is zero if $p > p_H$. The interesting fact is $p_H = \frac{4}{3}$ if $H = \frac{1}{2}$.

This paper is organized as follows. In Section 2, we present some preliminaries and estimates for sub-fBm, in order to study the functionals $A_1(t, S^H_t)$ and $A_2(t, S^H_t)$ given as above. In Section 3, we define the so-called weighted self-intersection local times and consider their derivatives (see Rosen (2005)). In Section 4, we will use the results established in Section 3 to give our main theorems. Some technical estimates are included in the Appendix.
2. Preliminaries for sub-fractional Brownian motion

In this section, we briefly recall some basic definitions and results of sub-fBm.

As we pointed out, sub-fBm $S^H = \{S^H_t, \ t \geq 0\}$ with index $H \in (0, 1)$ is a rather general class of centered Gaussian processes with $S^H_0 = 0$ and

$$E\left[(S^H_t)^2\right] = s^{2H} + \frac{1}{2} (s + t)^{2H} + \frac{1}{2} |t - s|^{2H}, \ \forall s, t \geq 0. \quad (3)$$

The process $S^H$ has the following properties.

(1) **Second moment of increments:** For all $s \leq t$,

$$[(2 - 2^{2H-1}) \wedge 1] (t - s)^{2H} \leq E\left[(S^H_t - S^H_s)^2\right] \leq [(2 - 2^{2H-1}) \vee 1] (t - s)^{2H}. \quad (4)$$

Thus, Kolmogorov’s continuity criterion implies that sub-fractional Brownian motion has a continuous version for each $H$.

(2) **Correlation of increments:** For $0 \leq u < v \leq s < t$, let

$$R_{u,v,s,t} = E\left[(b^H_u - b^H_v)(b^H_s - b^H_t)\right], \quad C_{u,v,s,t} = E\left[(S^H_u - S^H_v)(S^H_s - S^H_t)\right].$$

Then

$$0 < C_{u,v,s,t} \leq R_{u,v,s,t}, \quad \text{if } H > \frac{1}{2}, \quad (5)$$

$$R_{u,v,s,t} < C_{u,v,s,t} \leq 0, \quad \text{if } H < \frac{1}{2}. \quad (6)$$

Similarly, one can show that (5) and (6) hold for $0 \leq u < s < v < t$ and $0 \leq s < u \leq v < t$, respectively.

(3) **Strong local nondeterminism** (see, Yan and Shen (2010)):

The property implies that there exists a constant $\kappa_0 > 0$ depending on $n$ such that (see Berman (1973)) the inequality

$$\text{Var}\left(\sum_{j=2}^n u_j (S^H_{t_j} - S^H_{t_{j-1}})\right) \geq \kappa_0 \sum_{j=2}^n u_j^2 \text{Var}\left(S^H_{t_j} - S^H_{t_{j-1}}\right), \quad (7)$$

holds for $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ and $u_j \in \mathbb{R}, j = 2, 3, \ldots, n$.

As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $S^H$. We refer to Nualart (2006) and Alós, Mazet, and Nualart (2001) for a complete description of stochastic calculus with respect to Gaussian processes, in particular, we refer to Yan and Shen (in press) for stochastic calculus with respect to $S^H$ with $2H \geq 1$. We will use the notation

$$\int_0^T u_s dS^H_s,$$

to express the Skorohod integral of an adapted process $u$.

**Theorem 2.1 (Itô’s Formula (Yan and Shen, in press)).** Let $f \in C^2(\mathbb{R} \times \mathbb{R}_+)$. Suppose that $2H \geq 1$, then we have

$$f(t, S^H_t) = f(0, 0) + \int_0^t \frac{df}{ds}(s, S^H_s) ds + \int_0^t \frac{df}{dx}(s, S^H_s) dS^H_s + (2 - 2^{2H-1}) H \int_0^t \frac{d^2f}{dx^2}(s, S^H_s) s^{2H-1} ds. \quad (8)$$

Recall that sub-fBm $S^H$ has a local time $\mathcal{L}_1(t, x)$ continuous in $(t, x) \in [0, \infty) \times \mathbb{R}$ which satisfies the occupation formula (see Geman and Horowitz (1980))

$$\int_0^t \phi(S^H_s, s) ds = \int_\mathbb{R} dx \int_0^t \phi(x, s) \mathcal{L}_1(ds, x), \quad (8)$$

for every continuous and bounded function $\phi(x, t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, and such that

$$\mathcal{L}_1(t, x) = \int_0^t \delta(S^H_s - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda (s \in [0, t], |S^H_s - x| < \epsilon), \quad (8)$$

where $\lambda$ denotes Lebesgue measure and $\delta(\cdot)$ is the Dirac delta function. Define the so-called weighted local time $\mathcal{L}_2(t, x)$ of $S^H$ at $x$ as follows

$$\mathcal{L}_2(t, x) = \int_0^t s^{2H-1} \mathcal{L}_1(ds, x) = \int_0^t \lambda(S^H_s - x) s^{2H-1} ds. \quad (8)$$
It follows from (8) that
\[
\int_0^t \phi \left( S_s^H, s \right) s^{2H-1} ds = \int_\mathbb{R} dx \int_0^t \phi(x, s) \mathcal{L}_2(ds, x).
\]

At the end of the section, we give some estimates for the following expressions:
\[
\begin{align*}
\lambda &= \text{Var} (S_t^H - S_s^H), \\
\rho &= \text{Var} (S_t^H - S_{s'}^H), \\
\mu &= \text{Cov} (S_t^H - S_s^H, S_{t'}^H - S_{s'}^H), \\
\mu^* &= \text{Cov} (B_t^H - B_s^H, B_{t'}^H - B_{s'}^H)
\end{align*}
\]
for \(0 < s < t < T, 0 < s' < t' < T\). It is noteworthy that \(|\mu| \leq |\mu^*|\) for \(0 < H < 1\).

According to the property of strong local nondeterminism of sub-fBm (see, Yan and Shen (2010)) and (4), we can obtain the next lemma.

**Lemma 2.1.** There is a constant \(\kappa > 0\) such that the following statements hold:

1. for any \(0 \leq s < s' < t < t' \leq T\),
\[
\lambda, \rho - \mu^2 \geq \kappa ((t-s)^{2H}(t'-t)^{2H} + (t'-s')^{2H}(s'-s)^{2H}),
\]
(9)

2. for any \(0 \leq s < s' < t < t' \leq T\),
\[
\lambda, \rho - \mu^2 \geq \kappa (t-s)^{2H}(t'-s')^{2H},
\]
(10)

3. for any \(0 \leq s < t < s' < t' \leq T\),
\[
\lambda, \rho - \mu^2 \geq \kappa (t-s)^{2H}(t'-s')^{2H}.
\]
(11)

### 3. Weighted self-intersection local times

In order to study the functional \(A_1(t, S_t^H)\) and \(A_2(t, S_t^H)\), in this section we consider the weighted self-intersection local times of sub-fBm, defined as
\[
\begin{align*}
\beta_1(t) &= \int_0^t \int_0^s \delta (S_s^H - S_{s'}^H) s^{2H-1} dr ds, \\
\beta_2(t) &= \int_0^t \int_0^s \delta (S_s^H - B_{s'}^H) (sr)^{2H-1} dr ds.
\end{align*}
\]

Let \(\mathcal{L}_1(t, x) = \int_0^t \delta (S_s^H - x) ds\) and \(\mathcal{L}_2(t, x) = \int_0^t \delta (S_s^H - x) s^{2H-1} ds\) are the local time and weighted local time of \(S_t^H\) at \(x\), respectively. Then we have
\[
\beta_j(t) = \int_0^t \mathcal{L}_j(s, S_s^H) s^{2H-1} ds, \quad j = 1, 2,
\]
which lead to the existence of \(\beta_j, j = 1, 2\) in \(L^2\). Define the functionals \(A_1, A_2 : \mathbb{R}_+ \times \mathbb{R} \to L^2(\Omega, \mathcal{F}, P)\) as follows
\[
\begin{align*}
A_1(t, x) &= \int_0^t 1_{[0, \infty)} (x - S_{s}^H) ds, \\
A_2(t, x) &= \int_0^t 1_{[0, \infty)} (x - S_{s}^H) s^{2H-1} ds.
\end{align*}
\]
(12)

(13)

Then we have
\[
\begin{align*}
A_1(t, x) &= \int_{-\infty}^x \mathcal{L}_1(t, y) dy, \\
A_2(t, x) &= \int_{-\infty}^x \mathcal{L}_2(t, y) dy.
\end{align*}
\]

Thus, the continuity (see, for example, Nualart (2006)) of local time implies that the following proposition holds.

**Proposition 3.1.** (1) The functional \(A_j(t, x)\) \((j = 1, 2)\) is jointly continuous in \((t, x)\). (2) For fixed \(x\), \(A_j(t, x)\) \((j = 1, 2)\) is an increasing Lipschitz continuous function of \(t\). (3) For fixed \(t\), \(A_j(t, x)\) \((j = 1, 2)\) is an increasing \(C^1\) function of \(x\) with
\[
\frac{\partial}{\partial x} A_j(t, x) = \mathcal{L}_j(t, x).
\]
(14)

For \(0 \leq t \leq T\), we define two processes as follows
\[
\begin{align*}
\beta_1'(t) &= \int_0^t \int_0^s \delta' (S_s^H - S_{s'}^H) s^{2H-1} dr ds, \\
\beta_2'(t) &= \int_0^t \int_0^s \delta' (S_s^H - S_{s'}^H) (sr)^{2H-1} dr ds.
\end{align*}
\]

For \(0 \leq t \leq T\), we define two processes as follows
\[
\begin{align*}
\beta_1'(t) &= \int_0^t \int_0^s \delta' (S_s^H - S_{s'}^H) s^{2H-1} dr ds, \\
\beta_2'(t) &= \int_0^t \int_0^s \delta' (S_s^H - S_{s'}^H) (sr)^{2H-1} dr ds.
\end{align*}
\]
By Itô’s formula for sub-fBm, using \( \frac{d}{dx} 1_{[0,\infty)}(x) = \delta(x) \) and \( \frac{d^2}{dx^2} 1_{[0,\infty)}(x) = \delta'(x) \), for \( H > \frac{1}{2} \) we have

\[
A_1(t, S^H_1) = t + \int_0^t \mathcal{L}_1(s, S^H_1) \, ds^H_1 + (2 - 2^{2H-1}) \, H \beta'_1(t),
\]

(15)

and

\[
A_2(t, S^H_1) = \frac{2H}{2H} + \int_0^t \mathcal{L}_2(s, S^H_1) \, ds^H_1 + (2 - 2^{2H-1}) \, H \beta'_2(t).
\]

(16)

in the setting of distributional, where \( \delta' \) is the distributional derivative of the Dirac delta function, and the stochastic integral \( \int_{0}^{\delta} u, ds^H \) is the Skorohod integral. This motivates the subject matter of this section.

Now, let us prove the existence of the processes \( \beta'_j(t), j = 1, 2 \) in \( L^2 \). For any \( t \in \mathbb{R}_+, \varepsilon > 0 \) and the heat kernel

\[
f_{\varepsilon}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}, \quad x \in \mathbb{R},
\]

define

\[
\beta'_1(t, \varepsilon) := \int_{0}^{t} \int_{0}^{\varepsilon} f_{\varepsilon}(s) (S^H_1 - S^H_1) \, ds^H_1 \, dr \equiv \frac{i}{2\pi} \int_{0}^{t} \int_{0}^{\varepsilon} s^{2H-1} \, ds \int_{\mathbb{R}} \xi e^{i(s^H_1 - s^H_1)\xi} \, e^{-\varepsilon^2 \xi^2} \, d\xi,
\]

\[
\beta'_2(t, \varepsilon) := \int_{0}^{t} \int_{0}^{\varepsilon} f_{\varepsilon}(s) (sr)^{2H-1} \, ds \, dr \equiv \frac{i}{2\pi} \int_{0}^{t} \int_{0}^{\varepsilon} (sr)^{2H-1} \, ds \int_{\mathbb{R}} \xi e^{i(s^H_1 - s^H_1)\xi} \, e^{-\varepsilon^2 \xi^2} \, d\xi.
\]

Denote

\[ T := \{(s, t, s', t') : 0 < s < t < T, 0 < s' < t' < T\}. \]

For any \((s, t, s', t') \in T\), we suppose that \( \zeta_1 = (tt')^{2H-1} \), \( \zeta_2 = (st's')^{2H-1} \) and

\[
\lambda = \text{Var} \left( S^H_1 - S^H_1 \right), \quad \rho = \text{Var} \left( S^H_1 - S^H_1 \right), \quad \mu = \text{Cov} \left( S^H_1 - S^H_1, S^H_1 - S^H_1 \right).
\]

Lemma 3.1. For \( 0 < H < \frac{1}{2} \) and \( j = 1, 2 \), we have

\[
\int_{\mathbb{R}} \frac{\mu \zeta_j}{(\lambda \rho - \mu^2)^\frac{1}{2}} \, ds \, ds' \, dt' < \infty.
\]

(17)

The proof of the lemma will be given in Appendix.

Proposition 3.2. The processes \( \beta'_j(t), j = 1, 2 \) exist in \( L^2 \) if \( 0 < H < \frac{2}{3} \).

Proof.

\[
E \beta'_1(t, \varepsilon) = \frac{i}{2\pi} \int_{0}^{t} \int_{0}^{\varepsilon} s^{2H-1} \, ds \int_{\mathbb{R}} \xi E e^{i(s^H_1 - s^H_1)\xi} \, e^{-\varepsilon^2 \xi^2} \, d\xi = 0,
\]

\[
E \beta'_2(t, \varepsilon) = \frac{i}{2\pi} \int_{0}^{t} \int_{0}^{\varepsilon} (sr)^{2H-1} \, ds \int_{\mathbb{R}} \xi E e^{i(s^H_1 - s^H_1)\xi} \, e^{-\varepsilon^2 \xi^2} \, d\xi = 0.
\]

and, for \( j = 1, 2 \)

\[
E(\beta'_j(t, \varepsilon)^2) = \frac{-1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \xi \, d\xi \, d\eta E \left[ \exp \left( i \xi \left( \zeta_j - \bar{\zeta}_j \right) + i \eta \left( \zeta_j - \bar{\zeta}_j \right) \right) \right] e^{-\xi^2 + \eta^2} \, d\xi \, d\eta
\]

\[
= \frac{-1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \xi \, d\xi \, d\eta \exp \left( \frac{-1}{2} \left( \lambda + \varepsilon \right) \xi^2 - \mu \xi - \frac{1}{2} \left( \rho + \varepsilon \right) \eta^2 \right) \, d\xi \, d\eta
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mu \zeta_j}{(\lambda + \varepsilon) \left( \rho + \varepsilon - \mu^2 \right)^\frac{1}{2}} \, ds \, ds' \, dt'.
\]

By Lemma 3.1, for \( 0 < H < \frac{2}{3} \) we can define, in \( L^2 \) space,

\[
\beta'_1(t) = \lim_{\varepsilon \downarrow 0} \beta'_1(t, \varepsilon), \quad \beta'_2(t) = \lim_{\varepsilon \downarrow 0} \beta'_2(t, \varepsilon).
\]
4. $p$-variations

In this section, we give the main result of this paper. We consider $p$-variations of the process

$$X_j^{(i)} := A_j(t, s_j^{(i)}) - \int_0^t \mathcal{L}_j(s, s_j^{(i)}) \, ds_j^{(i)}.$$

(18)

In the remaining part of this paper $C$ will be a generic positive constant depending only on $H, T$ and its values may differ from line to line. The idea used here is essentially due to Rosen (2005). Fix $T > 0$ and let $\tau_n = \{0 = t_0 < t_1 < t_2 < \cdots t_n = T\}$ be a partition of $[0, T]$ such that $| \Delta_n | = \max |t_j - t_{j-1}| \to 0$ as $n$ tends to infinity. For a stochastic process $X = \{X_t; t \geq 0\}$, we denote

$$V_p^0(X; T) := \sum_{j=0}^{n-1} |X_{t_{j+1}} - X_{t_j}|^p,$$

where $p > 0$. Recall that the process $X$ is of bounded $p$-variation if the limit of $V_p^0(X; T)$ exists in $L^1$ as $n$ tends to infinity. We denote this limit by $V_p(X, T)$ and call it $p$-variation of $X$ on $[0, T]$. For any $T_1, T_2 \in [0, T], T_1 < T_2$, we denote

$$\mathcal{D}_1 = \left\{0 < s < s' < t < t' < T_1, T_1 < t, t' < T_2\right\},$$

$$\mathcal{D}_2 = \left\{0 < s < s' < t < t' < T_2, T_1 < t, t' < T_2\right\}.$$

We first consider the case $0 < H < \frac{1}{2}$.

**Lemma 4.1.** For $0 < H < \frac{1}{2}$, and $j = 1, 2$, we have

$$\int_{\mathcal{D}_j} \frac{|\mu| |\xi| dsdt ds'dt'}{(\lambda, \rho - \mu^2)^2} \leq C (T_2 - T_1)^{2-H},$$

(19)

where $\xi = (tt')^{2H-1}$ and $\zeta = (st's't')^{2H-1}$.

This lemma will be proved in Appendix.

**Theorem 4.1.** For all $t \in [0, T]$ and $0 < H < \frac{1}{2}$, we have

$$V_p(\beta_j'; T) = 0, \quad j = 1, 2,$$

if $p > \frac{2}{2-H}$.

**Proof.** For $0 < H < \frac{1}{2}$ and $j = 1, 2$, we have

$$E \left[ \beta_j'(T_2, \epsilon) - (\beta_j'(T_1, \epsilon))^2 \right] = - \frac{1}{(2\pi)^2} \int_{T_1}^{T_2} \int_0^T \int_0^T \int_0^T \xi \eta e^{\frac{\pi}{2H} (\xi^2 - \eta^2)} \, d\xi \, d\eta$$

$$= - \frac{1}{(2\pi)^2} \int_{T_1}^{T_2} \int_0^T \int_0^T \int_0^T \xi \eta e^{\frac{\pi}{2H} (\xi^2 - \eta^2)} \, d\xi \, d\eta$$

$$+ \frac{1}{2\pi} \int_{\mathcal{D}_1} \frac{\mu \xi}{(\lambda, \rho + \epsilon - \mu^2)^2} \, dsdt ds'dt'.$$

Combining this with $\beta_j'(t, \epsilon) \to \beta_j'(t)$ ($j = 1, 2$) in $L^2$ as $\epsilon$ tends to zero, we get by **Lemma 4.1**

$$E[(\beta_j'(T_2) - \beta_j'(T_1))^2] \leq \frac{1}{(2\pi)^2} \int_{\mathcal{D}_1} \frac{|\mu| |\xi| dsdt ds'dt'}{(\lambda, \rho - \mu^2)^2}$$

$$+ \frac{1}{(2\pi)^2} \int_{\mathcal{D}_2} \frac{\xi dsdt ds'dt'}{(\lambda, \rho - \mu^2)^2} \, d\xi \, d\eta$$

$$= \frac{1}{2\pi} \int_{\mathcal{D}_1} \frac{|\mu| |\xi| dsdt ds'dt'}{(\lambda, \rho - \mu^2)^2} + \frac{1}{(2\pi)^2} \int_{\mathcal{D}_2} \frac{\xi \eta}{(\lambda, \rho - \mu^2)^2} \, dsdt ds'dt'$$

$$\leq C (T_2 - T_1)^{2-H} + C (T_2 - T_1)^{2-H} \leq C (T_2 - T_1)^{2-H}.$$
It follows that for $0 < p < 2$ and $j = 1, 2$,
\[
E[V_p^p(\beta_j^j; T)] = \sum_{k=0}^{n-1} (E|\beta_j^j(t_{k+1}) - \beta_j^j(t_k)|^p) \leq \sum_{k=0}^{n-1} (E|\beta_j^j(t_{k+1}) - \beta_j^j(t_k)|^2)^{\frac{p}{2}} \leq C \sum_{k=0}^{n-1} |t_{k+1} - t_k|^{\frac{p(2-3H)}{2}},
\]
which shows that the $p$-variation of the process $\beta_j^j$ ($j = 1, 2$) is zero provided $p > \frac{2}{2-3H}$. This completes the proof. \(\square\)

Now, we consider the case $H > \frac{1}{2}$. We first give a lemma which is similar to Lemma 4.1. It will be proved in Appendix.

**Lemma 4.2.** For $\frac{1}{2} \leq H < \frac{2}{3}$ and $j = 1, 2$, we have
\[
\int_{\mathbb{R}^1} \int_{\mathbb{R}^2} \zeta dsdt \cdot ds' dt' \leq C(T_2 - T_1)^{3-3H}, \tag{20}
\]
and
\[
\int_{\mathbb{R}^2} \frac{\mu \zeta_j}{(\lambda, \rho - \mu)^{\frac{1}{2}}} dsdt \cdot ds' dt' \leq C(T_2 - T_1)^{2-H}, \tag{21}
\]
where $\zeta_j = (tt')^{2H-1}$ and $\zeta_2 = (sts't')^{2H-1}$.

One can obtain the following theorem similarly to the proof of Theorem 4.1.

**Theorem 4.2.** For $\frac{1}{2} \leq H < \frac{2}{3}$, we have
\[
V_p(\beta_j^j; T) = 0, \quad j = 1, 2, \quad \text{if } p > \frac{2}{3-3H}.
\]

**Proof.** For $\frac{1}{2} \leq H < \frac{2}{3}$ and $j = 1, 2$, we have
\[
E[(\beta_j^j(T_2) - \beta_j^j(T_1))^2] \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\mu \zeta_j dsdt \cdot ds' dt'}{(\lambda, \rho - \mu)^{\frac{1}{2}}} + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \zeta dsdt \cdot ds' dt' \int_{\mathbb{R}^2} |\xi|e^{-\frac{1}{2}\text{Var}[\xi(s_i^{t'-t_i} + s_i^{t_i-t})]} d\xi d\eta \leq C(T_2 - T_1)^{2-H} + C(T_2 - T_1)^{3-3H} \leq C(T_2 - T_1)^{3-3H}
\]
because $2 - H \geq 3 - 3H$. It follows that for $0 < p < 2$ and $j = 1, 2$,
\[
E[V_p^p(\beta_j^j; T)] = \sum_{k=0}^{n-1} (E|\beta_j^j(t_{k+1}) - \beta_j^j(t_k)|^p) \leq \sum_{k=0}^{n-1} (E|\beta_j^j(t_{k+1}) - \beta_j^j(t_k)|^2)^{\frac{p}{2}} \leq C \sum_{k=0}^{n-1} |t_{k+1} - t_k|^{\frac{p(3-3H)}{2}},
\]
which shows that the $p$-variation of the process $\beta_j^j$ ($j = 1, 2$) is zero provided $p > \frac{2}{3-3H}$. This completes the proof. \(\square\)

Finally, as an application we consider these processes $X^{(1)}$, $X^{(2)}$ given by (18).

**Theorem 4.3.** For any $c > 0$,
\[
\{X^{(1)}_t, 0 \leq t \leq T\} = \{\hat{c} \hat{X}^{(1)}_{t/(c/M)}, 0 \leq t \leq T\}, \quad \{X^{(2)}_t, 0 \leq t \leq T\} = \{\hat{c} \hat{X}^{(2)}_{t/(c/M)}, 0 \leq t \leq T\}
\]
where $\overset{d}{=} \text{means equalities of all finite dimensional distributions. These equalities are some calculus exercises.}$
Acknowledgements

The authors would like to thank the editor and the anonymous earnest referee whose remarks and suggestions greatly improved the presentation of our paper.

Appendix. Some technical estimates

In the Appendix, we will give the proofs of the estimates (17) and (19)–(21).

In order to show these inequalities, we need some preliminaries.

Recall

$$T := \{(s, t, s', t') : 0 < s < t < T, 0 < s' < t' < T\}.$$ 

Without loss of generality, one can assume $t < t'$. For any $(s, t, s', t') \in T$, we denote

$$T_1 = \{0 \leq s < s' < t < t' \leq T\},$$

$$T_2 = \{0 \leq s' < s < t < t' \leq T\},$$

$$T_3 = \{0 \leq s < t < s' < t' \leq T\}.$$

Noting that for all $(s, t, s', t') \in T_2$, by (5) and $\mu^*$ in Hu (2001, P246) we have

$$0 < \mu < \mu^* \leq \kappa(t - s)(t' - s')^{2H - 1}, \quad \frac{1}{2} \leq H < 1. \quad (22)$$

Let us first obtain the inequalities (21) and (19). Let

$$A_{j,i} := \int_{\varphi_j} |\mu| \zeta dsdt ds'dt',$$

for $i, j = 1, 2$. We claim that

$$A_{2,i} \leq C(T_2 - T_1)^{2-H}, \quad \frac{1}{2} \leq H < \frac{2}{3}. \quad (23)$$

and

$$A_{j,i} \leq C(T_2 - T_1)^{2-H}, \quad 0 < H < \frac{1}{2}. \quad (24)$$

It follows from (22), (10) and the inequality

$$t' - s' \geq (t' - t) + (s - s') \geq (t' - t) \left( s - s' \right)^{\frac{5}{3}}$$

that for all $(s, t, s', t') \in T_2$ and $\frac{1}{2} \leq H < \frac{2}{3}$,

$$\frac{\mu}{(\lambda, \rho - \mu^2)^\frac{3}{2}} \leq \frac{C(t - s)^{\frac{3}{2}H(t - s)}^{\frac{1}{2} - \frac{1}{2}H(t' - s')^{2H - 1}}}{(t - s)^{\frac{3}{2}H(t' - s')^{2H - 1}}} \leq C(t - s)^{-\frac{1}{2}H(t' - s')^{-\frac{2}{3}H}} \leq C(t - s)^{-\frac{1}{2}H(t' - t)^{-H}(s - s')^{\frac{2}{3}H}},$$

which yields

$$A_{2,i} := \int_{\varphi_2} \mu \zeta dsdt ds'dt' \leq \frac{C}{(\lambda, \rho - \mu^2)^\frac{3}{2}} \leq C \int_{\varphi_2} \frac{\zeta dsdt ds'dt'}{(t - s)^{\frac{3}{2}H(t' - t)^H(s - s')^{\frac{2}{3}H}}} \leq C(T_2 - T_1)^{2-H},$$

for $\frac{1}{2} \leq H < \frac{2}{3}$. This gives the inequality (23) for $i = 1, 2$. To obtain the inequalities (24), we suppose $0 < H < \frac{1}{2}$. For $(s, t, s', t') \in \varphi_1$ by applying the Young inequality (see, for example, Beckenbach and Bellman (1983))

$$\sum_{i=1}^n a_i a_i \geq \prod_{i=1}^n a_i^q,$$

for all $a_i \geq 0, a_i > 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n a_i = 1$ to (9), we have

$$\lambda, \rho - \mu^2 \geq \kappa(t - s)^{\frac{2H}{3}}(t' - t)^{\frac{2H}{3}}(t' - s')^{\frac{2H}{3}}(s' - s)^{\frac{4H}{3}}.$$
Combining this with (10) and the Schwarz inequality $\mu \leq \sqrt{\lambda \rho}$, we have

$$A_{1,1} := \int_{\mathcal{D}} \frac{\mu((tt')^{2H-1}dsdt's'dt')}{(\lambda_1 - \mu_2)^2} \leq \int_{\mathcal{D}} \frac{\mu((ss')^{2H-1}dsdt's'dt')}{(\lambda_1 - \mu_2)^2}$$

$$\leq C \int_{T_1}^{T_2} dt' \int_{T_1}^{t'} (t' - t)^{-H} ds \leq C(T_2 - T_1)^{2-H},$$

and

$$A_{1,2} := \int_{\mathcal{D}} \frac{\mu((ss't')^{2H-1}dsdt's'dt')}{(\lambda_1 - \mu_2)^2} \leq C \int_{\mathcal{D}} \frac{(ss't')^{2H-1}dsdt's'dt'}{(t' - t)^H(t' - s')^H(s' - s)^{2H}}$$

$$\leq C \int_{T_1}^{T_2} dt' \int_{T_1}^{t'} (t' - t)^{-H} ds \int_{0}^{t'} \frac{(ss')^{2H-1}}{(t' - s')^H(s' - s)^{2H}} ds'dt'$$

$$\leq C \int_{T_1}^{T_2} (t')^{2H-1} dt' \int_{T_1}^{t'} t^{2H-1} (t' - t)^{-H} dt \leq C(T_2 - T_1)^{2-H}.$$ Similarly, one can obtain the estimate $A_{2,1} \leq C(T_2 - T_1)^{2-H}, \quad \frac{1}{2} < H < \frac{2}{3}$. These show the inequalities (24) hold. Using the above method, one can obtain the estimate

$$A_{1,1} \leq C(T_2 - T_1)^{2-H}, \quad \frac{1}{2} < H < \frac{2}{3}.$$ Next, we obtain the inequality (20). Suppose $\frac{1}{2} \leq H < \frac{2}{3}$. By the strong local nondeterminism property of the sub-fBm proved by Yan and Shen (2010) and (4), we have

$$\text{Var} \left[ \xi (S^H_t - S^H_s) + \eta (S^H_t - S^H_s) \right] = \text{Var} \left[ \xi (S^H_t - S^H_s) + (\xi + \eta) (S^H_t - S^H_s) + \eta (S^H_t - S^H_s) \right]$$

$$\geq C \left[ \xi^2 \text{Var} (S^H_t - S^H_s) + (\xi + \eta)^2 \text{Var} (S^H_t - S^H_s) + \eta^2 \text{Var} (S^H_t - S^H_s) \right]$$

$$\geq C \left[ \xi^2 (s' - s)^{2H} + (\xi + \eta)^2 (t - s')^{2H} + \eta^2 (t' - t)^{2H} \right],$$

for $(s, t', s', t') \in \mathcal{D}_1$. It follows that

$$I_1 := \int_{\mathcal{D}_1} \xi dsdt'ds'dt' \int_{\mathcal{D}_2} |\xi| \exp \left\{ -\frac{C}{2} \left[ \xi^2 (s' - s)^{2H} + (\xi + \eta)^2 (t - s')^{2H} + \eta^2 (t' - t)^{2H} \right] \right\} d\xi d\eta$$

$$\leq C \int_{\mathcal{D}_1} \int_{\mathcal{D}_2} |\xi| |\eta| d\xi d\eta \int_{\mathcal{D}_3} \exp \left\{ -\frac{C}{2} \left[ \xi^2 r_2^{2H} + (\xi + \eta)^2 r_3^{2H} + \eta^2 r_4^{2H} \right] \right\} dr_1 dr_2 dr_3 dr_4,$$

where $\mathcal{D}_3 = \{ T_1 \leq \sum_{j=1}^{3} r_j \leq T_2, T_1 \leq \sum_{j=1}^{4} r_j \leq T_2, 0 < r_1, r_2, r_3, r_4 < T \}$. Noting that

$$\int_{T_1 - r_1 - r_2 - r_3}^{T_2 - r_1 - r_2 - r_3} e^{-\frac{C}{2} |\xi|^2 r_2^{2H}} dr_4 \leq (T_2 - T_1)^a \left( \int_{0}^{T} e^{-\frac{C}{2} |\xi|^2 r_2^{2H}} dr_4 \right)^{1-a}$$

$$\leq C(T_2 - T_1)^a \frac{1}{1 + |\eta|^{\frac{1}{2H}}},$$

by Hölder inequality with parameters $\frac{1}{a}$ and $\frac{1}{1-a} (0 < a < 1)$, then we obtain

$$\int_{\mathcal{D}_3} \exp \left\{ -\frac{C}{2} \left[ \xi^2 r_2^{2H} + (\xi + \eta)^2 r_3^{2H} + \eta^2 r_4^{2H} \right] \right\} dr_1 dr_2 dr_3 dr_4$$

$$\leq C(T_2 - T_1)^{1+a} \frac{1}{1 + |\eta|^{\frac{1}{2H}}} \int_{0}^{T} \int_{0}^{T} \exp \left\{ -\frac{C}{2} \left( \xi^2 r_2^{2H} + (\xi + \eta)^2 r_3^{2H} \right) \right\} dr_2 dr_3$$

$$\leq C(T_2 - T_1)^{1+a} \frac{1}{1 + |\eta|^{\frac{1}{2H}}} \frac{1}{1 + |\xi|^{\frac{1}{2H}}} \frac{1}{1 + |\xi + \eta|^{\frac{1}{H}}},$$

where we have used a basic inequality

$$\int_{0}^{1} e^{-\xi^2 x^2} dx \leq \frac{C}{1 + |\xi|^{\frac{1}{2H}}}.$$
for all \( x \in \mathbb{R} \). Thus, as \( a < 2 - 3H \), i.e., when \( \frac{1-a}{H} - 1 + \frac{1}{H} - 1 > 1 \), we obtain

\[
I_1 \leq C(T_2 - T_1)^{1+a} \int_{\mathbb{R}^2} \frac{|\xi \eta|}{1 + |\eta|^{\frac{1-a}{H}}} \frac{1}{1 + |\xi|^{\frac{1}{H}}} \frac{1}{1 + |\xi + \eta|^{\frac{1}{H}}} \, d\xi \, d\eta \\
\leq C(T_2 - T_1)^{1+a} \leq C(T_2 - T_1)^{3-3H}.
\]

which shows the inequality (20) holds for all \( \frac{1}{2} \leq H < \frac{2}{3} \).

Finally, let us prove the estimates (17). Because of the above proof, we just need to prove the estimate

\[
\int_{\mathbb{T}_3} \frac{\mu}{(\lambda, \rho - \mu^2)^{\frac{1}{2}}} < \infty
\]

for \( j = 1, 2 \) and \( 0 < H < \frac{3}{4} \). Therefore by using the estimate for \( \mu^* \) in Hu (2001, P248) imply that

\[
|\mu| < |\mu^*| \leq C(s' - t)^{2\alpha(1-H)}(t - s)(t' - s')^{2\beta(1-H)+1}
\]

for \( \alpha > 0, \beta > 0 \) and \( \alpha + 2\beta = 1 \). Combining this with (11), we get

\[
\frac{|\mu|}{(\lambda, \rho - \mu^2)^{\frac{1}{2}}} \leq C(s' - t)^{2\alpha(1-H)}(t - s)^{2\beta(1-H)+1} (t' - s')^{2\beta(1-H)+1 - 3H}.
\]

Taking \( 0 < \beta < \frac{2 - 3H}{2 - 2H} \) leads to \( 2\beta(H - 1) + 1 - 3H > -1 \) and \( 2\alpha(H - 1) > -1 \) and so that when \( 0 < H < \frac{2}{3} \), we have

\[
\left| \int_{\mathbb{T}_3} \frac{\mu}{(\lambda, \rho - \mu^2)^{\frac{1}{2}}} \, ds \, dz \, dt' \right| < \infty.
\]

Thus, we have proved (17).

References


