ON CONVERGENCE RATE OF DISTRIBUTED STOCHASTIC
GRADIENT ALGORITHM FOR CONVEX OPTIMIZATION WITH
INEQUALITY CONSTRAINTS

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Abstract. In this paper, we consider an optimization problem, where multiple agents cooperate to minimize the sum of their local individual objective functions subject to a global inequality constraint. We propose a class of distributed stochastic gradient algorithms that solve the problem using only local computation and communication. The implementation of the algorithms removes the need for performing the intermediate projections. For strongly convex optimization, we employ a smoothed constraint incorporation technique to show that the algorithm converges at an expected rate of $O(\ln T / T)$ (where $T$ is the number of iterations) with bounded gradients. For non-strongly convex optimization, we use a reduction technique to establish an $O(1/\sqrt{T})$ convergence rate in expectation. Finally, a numerical example is provided to show the convergence of the proposed algorithms.

Key words. distributed convex optimization, constrained optimization algorithm, stochastic gradient, convergence rate

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1. Introduction. In recent years, the problem of minimizing a sum of local private objective functions that are distributed among a network of multiple interacting nodes has received much attention (see [3, 4, 6, 16, 18, 19, 20, 23] and references therein). Such a problem arises in a variety of real applications such as distributed estimation [2], source localization in sensor networks [24], and smart grid [15, 26]. Most of the existing works build on the average consensus algorithm to design fully distributed algorithms that find the solution of the problem. As uncertainties always exist in communication and environment, stochastic or randomize algorithms also have been discussed for distributed optimization [13, 10].

Following the distributed optimization results without constraints, a constrained multiagent optimization problem is drawing more and more attention. In particular, the objective function is usually a sum of all the nodes’ local objective functions and the constraint is a convex and compact set available to all the nodes. The existing algorithms for solving this problem were involved with a Euclidean projection onto the constraint set at every iteration in order to ensure that the estimates are within the feasible domain; they usually fell into two main categories. The first built on the projected gradient algorithms, mainly based on the assumption that the constraint
set is simple, in the sense that the Euclidean projection step could be easily solved (see, e.g., [1, 5, 12, 13, 17]). To extend the discussion of simple constraint, the works in [22, 25] proposed approximate projected gradient algorithms by allowing the projection step to be solved only approximately at each iteration. The second category of algorithms exploited the particular structure of the constraint set (say, characterized by an inequality constraint) (see, e.g., [7, 14, 15, 21]). In general, these algorithms were primal-dual, and the design of such algorithms usually involved constructing a dual optimal set containing the dual optimal variable; nevertheless this leads to solving a general convex optimization problem. In addition, many problems about constrained multiagent optimization remain to be addressed, including the effect of the stochastic errors on the distributed gradient algorithm.

Convergence rate analysis is also an important issue in the distributed design. Even for consensus problems, how to estimate or improve convergence rate was discussed in many publications (see [9, 11]). Certainly, it is also important to provide convergence rate in the study of distributed optimization. For example, a convergence rate of $O(\ln T/T)$ was obtained for unconstrained optimization, where the individual objective functions are strongly convex and have Lipschitz gradients in [2], and moreover, for constrained optimization, an $O(\ln T/\sqrt{T})$ rate was established for general nonsmooth and convex functions by using a distributed dual averaging algorithm in [5]. In our previous work [21], a distributed algorithm was proposed to drive all the nodes to an optimal point at rate $O(1/T^{1/4})$ for constrained optimization. The algorithm exploited the particular structure of the constraint set, which was described by an inequality and contained in a ball with finite radius. The basic idea of the algorithm was to appropriately penalize the intermediate estimates when they were outside the domain, with the help of a regularized Lagrangian function.

The main objective of this paper is to study a constrained multiagent optimization problem by addressing the following questions: (i) Is there any algorithm that does not require a projection step at every iteration and only needs the noisy samples of the gradients? and (ii) Is it possible to derive the convergence rate under such circumstances? Although the convergence was verified for a class of distributed gradient algorithms in [13], the convergence rate remains unknown for constrained optimization. To this end, we develop a class of distributed gradient algorithms without requiring intermediate projections. Different from the convergence rate results given in [21], we construct a penalty function that incorporates the inequality constraint into the strongly convex objective function without requiring intermediate projections and design an efficient algorithm by adopting the smoothing technique in [8]. Specifically, we can achieve an improvement to an $O(\ln T/T)$ rate. Moreover, we shall also study the convergence rate for the non-strongly convex case with a reduction technique used in online optimization [27].

The technical contribution of the paper can be summarized as follows:

- We propose a class of new stochastic gradient algorithms to study distributed optimization with a global constraint. In contrast to the existing algorithms that require projections at every iteration, the implementation of the proposed algorithm removes the need for intermediate projections; instead, only one projection at the last iteration is needed to get a feasible solution for the entire algorithm.
- We employ the smoothing technique to deal with the case when the objective functions are strongly convex. This is different from the existing works that handle the inequality constraint (see, e.g., [7, 14, 15, 21]), where the algorithms are in general primal-dual. In this paper, we incorporate the in-
equality constraint into the objective function and design a non-primal-dual algorithm. Therefore, the proposed algorithms are relatively simple to implement. In addition, we use the smoothed constraint incorporation technique to provide the explicit convergence rate of the proposed algorithms, and drive all the nodes to the optimal point at an expected rate of $O(\ln T/T)$.

• We study the effect of stochastic errors on the distributed gradient algorithm, where the errors are due to the fact that the nodes only have access to noisy samples of the gradients for the case in which the objective functions are non-strongly convex. To solve the problem, we adopt the idea of a reduction technique in the online convex optimization community (see, e.g., [27]) to derive an expected rate of $O(1/\sqrt{T})$ for the proposed distributed gradient algorithm.

The reminder of the paper is organized as follows. In section 2, we give a detailed description of the constrained multiaagent optimization problem and the related assumptions. In section 3, we propose a class of distributed gradient algorithms and provide its convergence analysis results. In section 4, we illustrate the proposed algorithm with a distributed estimation problem. Finally, we conclude with section 5.

Notation and terminology. Let $\mathbb{R}^d$ be the $d$-dimensional vector space. We use $\langle x, y \rangle$ to denote the standard inner product on $\mathbb{R}^d$ for any $x, y \in \mathbb{R}^d$. We write $\Pi_{\mathcal{X}}[x]$ to denote the Euclidean projection of a vector $x$ onto the set $\mathcal{X}$, i.e., $\Pi_{\mathcal{X}}[x] = \arg \min_{y \in \mathcal{X}} \|x - y\|_2$. We write $\max(a, b)$ to denote the maximum of two real numbers $a$ and $b$ and denote $[\lambda]_+ = \max(\lambda, 0)$. We denote by $\lbrack N \rbrack$ the set of integers $\{1, \ldots, N\}$. We denote the $(i, j)$th element of a matrix $W$ by $[W]_{ij}$. For a convex (possibly nonsmooth) function $f$, its gradient (or subgradient) at a point $y$ is denoted by $\nabla f(y)$, and the following inequality holds for every $x$ in the domain of $f$:

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$ 

In addition, we say $f$ is $\sigma$-strongly convex over the convex set $\mathcal{X}$ if

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\sigma}{2} \|x - y\|^2 \quad \forall x, y \in \mathcal{X}.$$ 

2. Problem setting and assumptions. Consider a time-varying network with $N$ computational nodes (or agents) that is labeled by $V = \{1, 2, \ldots, N\}$. The nodes' connectivity at time $t$ ($t = 1, 2, \ldots$) is described by a directed graph $G(t) = (V, E(t))$, where $E(t)$ is the set of activated links at time $t$; the communication pattern among the nodes is captured by $W(t) \in \mathbb{R}^{N \times N}$, whose elements are defined as follows:

(i) $|W(t)_{ij}| > 0$ for any active links $(j, i) \in E(t)$, including node $i$ itself, that is, $|W(t)_{ii}| > 0$;

(ii) $|W(t)_{ij}| = 0$ for any inactive links at time $t$, as well as those that are not neighbors of node $i$.

Each node is endowed with a local private convex cost function; all the nodes in the network collectively minimize the sum of the cost functions, subject to a global convex constraint. The problem is formally given by

$$(2.1) \quad \begin{array}{ll}
\text{minimize} & f(x) = \sum_{i=1}^{N} f^i(x) \\
\text{subject to} & x \in \mathcal{X},
\end{array}$$

where $f^i : \mathbb{R}^d \to \mathbb{R}$ is the convex objective function of node $i$ known only by itself; $\mathcal{X}$ is a compact convex domain that is known to all the nodes, and it is characterized
by an inequality constraint:
\[ \mathcal{X} = \{ x \in \mathbb{R}^d : g(x) \leq 0 \} \subseteq \mathcal{B} = \{ x \in \mathbb{R}^d : \|x\|_2 \leq R \}, \]
where \( g : \mathbb{R}^d \to \mathbb{R} \) is convex, and \( \mathcal{X} \) is compact—specifically, is contained in a Euclidean ball of radius \( R \).

Suppose that each node \( i \) only has access to the noisy samples of its gradient. To be specific, for any point \( x \in \mathcal{X} \), node \( i \) can only obtain the stochastic gradient \( \tilde{\nabla} f^i(x) \) that is generated by
\[ \tilde{\nabla} f^i(x) = \nabla f^i(x) + n^i(x), \quad (2.2) \]
where \( n^i(x) \in \mathbb{R}^d \) is an independent random vector with zero mean and has bounded variance, that is,
\[ \mathbf{E} [n^i(x)] = 0 \quad \text{and} \quad \mathbf{E} [\|n^i(x)\|_2^2] \leq L^2_n. \quad (2.3) \]
The noise \( n^i(x) \) can represent the error due to inexact computation of the gradient, or roundoff error.

In addition, we make the following assumptions on problem (2.1) and the underlying network model.

**Assumption 1.** The graph \( G(t) = (\mathcal{V}, \mathcal{E}(t)) \) and the weight matrix \( W(t) \) satisfy \( (t = 1, 2, \ldots) \):
(a) \( W(t) \) is doubly stochastic.
(b) For all \( i \in [N] \), \( [W(t)]_{ii} \geq \nu \) and \( [W(t)]_{ij} \geq \nu \) if \( (j, i) \in \mathcal{E}(t) \), where \( \nu \) is a positive scalar.
(c) The graph \( (\mathcal{V}, \mathcal{E}(sB + 1) \cup \cdots \cup \mathcal{E}((s + 1)B)) \) is strongly connected for all \( s \geq 0 \) and some positive integer \( B \).

**Assumption 2.** There exists an \( \overline{x} \in \mathcal{X} \) such that \( g(\overline{x}) < 0 \).

**Assumption 3.** For each function \( f^i(x) \) \((i \in [N])\) and function \( g(x) \), we assume that for all \( x \in \mathcal{B} \),
\[ \|\nabla f^i(x)\|_2 \leq L_f \quad \text{and} \quad \|\nabla g(x)\|_2 \leq L_g. \]

**Assumption 4.** Each function \( f^i(x) \) \((i \in [N])\) is \( \sigma \)-strongly convex over the set \( \mathcal{B} \).

**Assumption 5.** For function \( g(x) \), we assume that there exists a \( \rho > 0 \) such that \( \min_{g(x)=0} \|\nabla g(x)\|_2 \geq \rho \).

**Remark 1.** Assumptions 1–3 are standard and widely present in the convergence analysis of consensus-based gradient algorithms that deal with inequality constrained multiagent optimization (see, e.g., \([7, 15]\)). Assumption 4 imposes some strong conditions on the functions \( f^i \), which is standard in the literature on strongly convex optimization (see, e.g., \([2, 17]\)). Assumption 5 is introduced to ensure that the dual optimal variable for the problem in (2.1) is well bounded from above (see section 3.1). In fact, Assumption 5 is satisfied for many constraints, such as a polytope and a positive semidefinite cone.

**Remark 2.** Consider an example with constant edge weights to show how to ensure in a distributed manner that the weight matrix \( W(t) \) satisfies Assumptions 1(a) and 1(b). Specifically, define
\[ [W(t)]_{ij} = \begin{cases} \nu & \text{if } (j, i) \in \mathcal{E}(t), \\ 1 - d_i(t)\nu & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad (2.4) \]
where \( d_i(t) \) is the number of neighbors communicating with node \( i \) at time \( t \). It is easy to show that Assumptions 1(a) and 1(b) are satisfied for \( 0 < \nu \leq \frac{1}{1 + d_i(t)} \).

3. The algorithms and convergence results. In this section, we provide a class of efficient distributed stochastic gradient (DSG) algorithm for solving problem (2.1) and characterize its explicit convergence rate. Building on the developed algorithm, we use a reduction technique to derive the convergence rate for non-strongly convex optimization.

3.1. Distributed stochastic gradient algorithm. To deal with the inequality constraint, we can write problem (2.1) into a min-max optimization problem:

\[
\min_{x \in \mathcal{X}} \max_{\lambda \geq 0} f(x) + \lambda Ng(x).
\]

This is guaranteed by the Slater’s condition (cf. Assumption 2). Denote the optimal solution of the above problem by \((x^*, \lambda^*)\), and it is easy to show that \( \lambda^* \in [0, L_f/\rho] \).

Hence, problem (2.1) can be further written as

\[
\min_{x \in \mathcal{X}} \max_{0 \leq \lambda \leq \theta} f(x) + \lambda Ng(x) = \min_{x \in \mathcal{X}} \sum_{i=1}^N \left( f^i(x) + \theta g(x) \right)_+,
\]

where \( \theta > L_f/\rho \). Note that it is possible, but complicated, to calculate the subgradient of \([g(x)]_+\). We adopt the technique, called the smoothed constraint incorporation technique, given in [8], by adding a smoothing term to the objective function, that is, \( S(\lambda/\theta) \), where \( S(\lambda) = -\lambda \ln \lambda - (1 - \lambda) \ln(1 - \lambda) \) (noting that this idea has been widely used in the conventional convex optimization community [28]). Then the associated problem becomes

\[
\min_{x \in \mathcal{X}} \max_{0 \leq \lambda \leq \theta} f(x) + \lambda Ng(x) + \gamma NS(\lambda/\theta)
\]

\[
= \min_{x \in \mathcal{X}} \sum_{i=1}^N \left( f^i(x) + \gamma \ln \left( 1 + \exp \left( \theta g(x)/\gamma \right) \right) \right).
\]

The smoothing term \( S(\lambda) \) is introduced for the following purposes: (i) it works as a regularization term that prevents the dual variable \( \lambda \) from being too large; (ii) it incorporates the inequality constraint into the objective function, and we can solve the transformed problem instead; and (iii) it avoids the unnecessary complication due to the subgradient of \([g(x)]_+\). We now have a smoothed version of problem (2.1):

\[
(3.1) \quad \min_{x \in \mathcal{X}} F(x) = \sum_{i=1}^N F^i(x),
\]

where \( F^i(x) = f^i(x) + \gamma \ln \left( 1 + \exp \left( \theta g(x)/\gamma \right) \right) \).

Note that each \( F^i \) is available to node \( i \) only. We will propose a distributed algorithm (Algorithm 1) to solve the above problem, instead of problem (2.1). This intuition leads to the construction of the algorithm and also the analysis of its convergence performance.

Now we are in a position to propose the algorithm for solving the distributed strongly convex optimization problem, which is shown as follows.
Algorithm 1. Distributed stochastic gradient algorithm.

Input: $x_i^1 = 0$; a step-size sequence $\{\beta_t\}_{t=1}^T$, $\gamma > 0$, and $\theta > L_f/\rho$.

Iteration: $(t = 1, \ldots, T)$
1. $z_i^t = x_i^t - \beta_t \left( \nabla f_i(x_i^t) + n_i(x_i^t) \right) = x_i^t - \beta_t \left( \nabla f_i(x_i^t) + \frac{\exp(\theta g_i(x_i^t)/\gamma)}{1 + \exp(\theta g_i(x_i^t)/\gamma)} \theta \nabla g(x_i^t) \right)$
2. $\hat{x}_i^{t+1} = \frac{1}{N} \sum_{j=1}^N [W(t)]_{ij} z_i^t$
3. $x_i^{t+1} = \Pi_{B} \hat{x}_i^{t+1} = \max \{\|\hat{x}_i^{t+1}\|, R\} \hat{x}_i^{t+1}$

Output: $\bar{x}_T = \Pi_{X} [\hat{x}_T^1]$, where $\bar{x}_T^i = \frac{1}{T} \sum_{t=1}^T x_i^t$.

Remark 3. The essential idea of Algorithm 1 is to replace the projection step with the gradient computation of the inequality function that defines the domain $X$. Instead of projecting the intermediate estimates onto the complex convex domain $X$, Algorithm 1 projects them onto the Euclidean ball that contains the same domain. At the last iteration, only one projection onto the domain $X$ is needed to get a feasible solution. As a result, the implementation of Algorithm 1 removes the need for intermediate projections.

We will seek to establish the convergence of the function value evaluated at $\bar{x}_T^i$ to the optimal value, for every $i \in [N]$, and also characterize the convergence rate of the DSG algorithm.

Denote the average estimate at iteration $t$ by

$$y_t := \frac{1}{N} \sum_{i=1}^N x_i^t. \tag{3.2}$$

The next lemma shows that the average disagreement among all the nodes vanishes at an expected rate $O(\ln T/T)$.

Lemma 3.1. Let Assumptions 1 and 3 hold and the step-size sequence be $\beta_t = \frac{1}{\sigma t}$, $t = 1, \ldots, T$. For all $i \in [N]$ and any number of iterations $T \geq 3$, we have

$$\frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \mathbb{E} \|x_i^t - y_t\|_2 \leq C_1 \frac{\ln T}{T},$$

where $C_1 = 2N \left( \frac{3N}{\sigma (1-c)} \right) + 4 \left( L_f + L_n + \theta L_g \right)$ with $\eta = 1 - \frac{\sigma}{4N^2}$ and $\varsigma = \eta^{1/B}$.

Proof. See Appendix A for the proof.

It is time to present one of the main results of this section, which shows that the DSG algorithm converges also at an expected rate $O(\ln T/T)$.

Theorem 3.2. Under Assumptions 1, 2, 3, 4, and 5, let the step-size sequence be $\beta_t = \frac{1}{\sigma t}$, $t = 1, \ldots, T$, and denote $x^* = \arg \min_{x \in X} f(x)$. If we set $\gamma = \frac{\ln T}{T}$, then, for all $j \in [N]$ and any number of iterations $T \geq 3$,

$$\mathbb{E} \left[ f(\bar{x}_T^j) \right] - f(x^*) \leq \left( 1 + \frac{L_f}{\theta \rho - L_f} \right) C_2 \frac{\ln T}{T},$$

where $C_2 = 2N \sigma \left( \frac{12N}{\sigma (1-c)} + 17 \right) \left( (L_f + \theta L_g)^2 + L_n^2 \right) + N \ln 2$.

The proof of Theorem 3.2 is based on the following three lemmas. Lemma 3.3 provides a basic convergence of the DSG algorithm, while Lemma 3.4 provides the
optimal condition related to the last projection. Lemma 3.5 shows the relation between the function difference in the smoothed version and its original one.

**Lemma 3.3.** Let Assumptions 1, 3, and 4 hold. For all \( j \in [N] \) and \( t \geq 1 \), we have

\[
E[F(x_j^t) - F(x)] \leq \frac{1}{2\beta_t} \left( (1 - \sigma \beta_t) \sum_{i=1}^{N} E[r_i^t(x)] - \sum_{i=1}^{N} E[r_{t+1}^i(x)] \right) + N \left( (L_f + \theta L_g)^2 + L_n^2 \right) \beta_t + (L_f + \theta L_g) \sum_{i=1}^{N} E[\|x_i^t - x_j^t\|_2],
\]

(3.3)

where \( r_i^t(x) = \|x_i^t - x\|^2 \) for all \( x \in \mathcal{B} \) and \( t \geq 1 \).

**Proof.** See Appendix B for the proof.

**Lemma 3.4** (see [8]). Let \( \bar{x} = \arg \min_{g(x) \leq 0} \|x - \bar{x}\|^2 \) with \( g(\bar{x}) > 0 \); then there exists a positive scalar \( \mu \) such that

\[
g(\bar{x}) = 0 \quad \text{and} \quad \hat{x} - \bar{x} = \mu \nabla g(\bar{x}).
\]

**Lemma 3.5.** For all \( x \in \mathcal{X} \), we have

\[
f(x) - f(x^*) \leq F(x) - F(x^*) + N \gamma \ln 2 - N [\theta g(x)]_+.
\]

**Proof.** See Appendix C for the proof.

**Proof of Theorem 3.2.** We divide the proof into two parts. The first part provides a bound on the function difference \( E[F(\bar{x}_j^t) - F(x)] \). The second part derives the conclusion by using the relation between the function difference in the smoothed version and its original one, by using Lemma 3.5.

(i) Adding the inequality (3.3) over \( t = 1, \ldots, T \),

\[
\sum_{t=1}^{T} E[F(x_j^t) - F(x)] \leq \sum_{t=1}^{T} \frac{1}{2\beta_t} \left( (1 - \sigma \beta_t) \sum_{i=1}^{N} E[r_i^t(x)] - \sum_{i=1}^{N} E[r_{t+1}^i(x)] \right) + N \left( (L_f + \theta L_g)^2 + L_n^2 \right) \beta_t + (L_f + \theta L_g) \sum_{i=1}^{N} E[\|x_i^t - x_j^t\|_2].
\]

(3.4)

We bound the terms on the right-hand side of the preceding inequality one by one: due to the specific choice of the step size \( \beta_t = \frac{1}{\sigma t} \), the terms \( A_T \) and \( B_T \) can be
bounded as follows:

\[ A_T = \frac{1}{2\beta_1} \sum_{i=1}^{N} \mathbf{E}[r_i^1(x)] - \frac{\sigma}{2} \sum_{i=1}^{N} \mathbf{E}[r_i^1(x)] - \frac{1}{2\beta_T} \sum_{i=1}^{N} \mathbf{E}[r_i^T+1(x)] \]

\[ + \sum_{t=2}^{T} \left( \frac{1}{2\beta_t} - \frac{1}{2\beta_{t-1}} - \frac{\sigma}{2} \right) \sum_{i=1}^{N} \mathbf{E}[r_i^t(x)] \]

\[ = \frac{1}{2\beta_1} \sum_{i=1}^{N} \mathbf{E}[r_i^1(x)] - \frac{\sigma}{2} \sum_{i=1}^{N} \mathbf{E}[r_i^1(x)] - \frac{1}{2\beta_T} \sum_{i=1}^{N} \mathbf{E}[r_i^T+1(x)] \]

\[ = -\frac{1}{2\beta_T} \sum_{i=1}^{N} \mathbf{E}[r_i^T+1(x)] \]

\[ \leq 0 \]

and

\[ B_T = N ((L_f + \theta L_g)^2 + L_n^2) \sum_{t=1}^{T} \mathbf{1}_{t}^2 \]

\[ \leq \frac{2N}{\sigma} ((L_f + \theta L_g)^2 + L_n^2) \ln T, \]

where we have used the inequality (A.8) (see Appendix A). For the term \( C_T \), it follows from Lemma 3.1 that

\[ C_T \leq 2(L_f + \theta L_g) C_1 \ln T \]

\[ \leq \frac{4N}{\sigma} \left( \frac{3N}{\eta^2(1 - \varsigma)} + 4 \right) (L_f + L_n + \theta L_g)^2 \ln T \]

\[ \leq \frac{8N}{\sigma} \left( \frac{3N}{\eta^2(1 - \varsigma)} + 4 \right) ((L_f + \theta L_g)^2 + L_n^2) \ln T. \]

Combining the last three inequalities with (3.4) yields

\[ (3.5) \quad \mathbf{E}[\mathbf{F}(\bar{x}_T^j) - \mathbf{F}(x^*)] \leq \frac{2N}{\sigma} \left( \frac{12N}{\eta^2(1 - \varsigma)} + 17 \right) ((L_f + \theta L_g)^2 + L_n^2) \frac{\ln T}{T}, \]

where we have used the convexity of function \( \mathbf{F} \), that is, \( \mathbf{F}(\bar{x}_T^j) \leq \frac{1}{T} \sum_{t=1}^{T} \mathbf{F}(x_t^j). \)

(ii) We now derive the conclusion by resorting to Lemma 3.5 and the bound (3.5). We distinguish two cases: (a) When \( g(\bar{x}_T^j) \leq 0 \), we have \( \bar{x}_T^j = \Pi_X[x_T^j] = \bar{x}_T^j \), hence, using Lemma 3.5 and setting the first \( x \) to be \( \bar{x}_T^j \) and the second and the third \( x \)'s to be \( \bar{x}_T^j \), respectively, combining with the inequality (3.5), and noting that \( \gamma = \ln T \), we arrive at

\[ \mathbf{E}[f(\bar{x}_T^j)] - f(x^*) \leq \left( \frac{2N}{\sigma} \left( \frac{12N}{\eta^2(1 - \varsigma)} + 17 \right) ((L_f + \theta L_g)^2 + L_n^2) + N \ln 2 \right) \frac{\ln T}{T}, \]

\[ = C_2 \frac{\ln T}{T}. \]

(3.6)

(b) When \( g(\bar{x}_T^j) > 0 \), we have to resort to Lemma 3.4 and Assumption 5:

\[ g(\bar{x}_T^j) = g(\bar{x}_T^j) - g(\bar{x}_T^j) \geq \langle \nabla g(\bar{x}_T^j), \bar{x}_T^j - \bar{x}_T^j \rangle \]

\[ = \| \nabla g(\bar{x}_T^j) \|^2 \| \bar{x}_T^j - \bar{x}_T^j \|^2 \]

\[ \geq \rho \| \bar{x}_T^j - \bar{x}_T^j \|^2, \]
where the equality follows from the fact that \( \hat{x}_T^j - \tilde{x}_T^j \) is in the same direction to \( \nabla g(\tilde{x}_T^j) \) (cf. Lemma 3.4). Moreover, from Assumption 3, we have

\[
f(\hat{x}_T^j) \geq f(\tilde{x}_T^j) + \langle \nabla f(\tilde{x}_T^j), \hat{x}_T^j - \tilde{x}_T^j \rangle 
\geq f(x^*) - NL_f \| \hat{x}_T^j - \tilde{x}_T^j \|_2.
\]

Combining the preceding two inequalities with (3.5), using Lemma 3.5, and taking expectation, we obtain

\[
(N \theta \rho - NL_f) E[\| \hat{x}_T^j - \tilde{x}_T^j \|_2] \leq C_2 \ln T T,
\]

which yields (noting that \( \theta > L_f/\rho \))

\[
E[\| \hat{x}_T^j - \tilde{x}_T^j \|_2] \leq \frac{C_2}{N(\theta \rho - L_f)} \frac{\ln T T}{T}.
\]

This implies that

\[
E[f(\hat{x}_T^j)] - f(x^*) \leq E[f(\tilde{x}_T^j)] - f(x^*) + NL_f E[\| \hat{x}_T^j - \tilde{x}_T^j \|_2] 
\leq C_2 \ln T T + NL_f \frac{E[\| \hat{x}_T^j - \tilde{x}_T^j \|_2]}{T} 
= \left( 1 + \frac{L_f}{\theta \rho - L_f} \right) C_2 \frac{\ln T T}{T},
\]

where we have used Lemma 3.5 and the inequalities (3.5) and (3.7). The conclusion follows by combining the above inequality with (3.6). The proof is complete.

Remark 4. It is worth pointing out that the choice of the step size \( \beta = \frac{1}{\sigma_t} \) is not unique; in fact, the inequality \( A_T \leq 0 \) holds when \( \beta = \frac{1}{\tilde{\sigma}} \) with \( \tilde{\sigma} \leq \sigma \).

Remark 5. We would like to make some comparisons between the proposed algorithm and the existing ones. Based on the work [8], we adapt the algorithm in [8] to the distributed setting. We have designed a consensus mechanism that drives all the nodes’ states to agree on the optimal value and established the convergence results as well. In addition, we have used the idea of reductions to develop a distributed algorithm for non-strongly convex optimization (see Algorithm 2). The work [2] presents a distributed subgradient-push algorithm (with the same convergence rate) for strongly convex optimization, but the problem considered there is unconstrained and the objective functions should have Lipschitz gradients. The authors in [17] present a study of the distributed strongly convex optimization problem in an online setting, where the proposed algorithm involves solving a projection problem at each iteration. In contrast, our proposed algorithm removes the need for intermediate projections and achieves the same convergence rate as that in [17]. Different from our previous work [21], where the convergence rate is \( O(1/T^{1/4}) \), here we show that if the objective functions are further assumed to be strongly convex, then the convergence rate can be improved to \( O(\ln T/T) \) even when the noisy gradients of objective functions are available.

Based on Theorem 3.2, we immediately have the following corollary that shows the convergence rate result for the noise-free version of Algorithm 1, where \( \nabla f_i(x) \) becomes \( \nabla f_i(x) \).
COROLLARY 3.6. Under the conditions of Theorem 3.2 and the assumption that all the nodes have access to the noise-free gradients, we have
\[
f(\tilde{x}_T^j) - f(x^*) \leq \left(1 + \frac{L_f}{\theta_p - L_f}\right) C_2 \ln T,
\]
where \(C_2 = 2N^2 \cdot 12N + 17(L_f + \theta L_g)^2 + N \ln 2\).

3.2. Reduction to non-strongly convex optimization. The previous subsection dealt with strongly convex optimization. In this subsection, we use a reduction technique from the developed algorithm (i.e., Algorithm 1) to derive the convergence rate for non-strongly convex optimization, that is, Assumption 4 does not need to hold.

The reduction technique has been widely used in the study of online optimization (referring to [27]). Its basic idea is to add a controlled amount of strong convexity to each function \(f^i\) at each iteration \(t\), and then apply Algorithm 1 to optimize the new objective function. With the reduction idea, we propose an algorithm as follows.

Algorithm 2. Reduction to non-strongly convex functions.

**Input:** \(f^i, T\), parameter sequence \(\{\sigma_t\}_{t=1}^T\).

**Set** \(f_t^i(x) = f^i(x) + \frac{\alpha}{2} \|x\|_2^2\), for \(t = 1, \ldots, T\).

Apply Algorithm 1 with parameters \(f_t^i, x_t^i = 0\), \(\{\beta_t\}_{t=1}^T\), \(\gamma > 0\), \(\theta > L_f/\rho\), return \(\tilde{x}_T^i\).

The following theorem characterizes the convergence rate for Algorithm 2.

**Theorem 3.7.** Let Assumptions 1, 2, 3, and 5 hold. Let the step-size sequence and the parameter sequence be \(\beta_t = \frac{1}{\sqrt{T}}\) and \(\sigma_t = \frac{1}{\sqrt{T}}\), \(t = 1, \ldots, T\), and denote \(x^* = \arg\min_{x \in X} f(x)\). If we set \(\gamma = \frac{1}{\sqrt{T}}\), we have that for all \(j \in [N]\) and any number of iterations \(T \geq 1\),
\[
E[f(\tilde{x}_T^j)] - f(x^*) \leq \left(1 + \frac{L_f}{\theta_p - L_f}\right) C_3 \frac{1}{\sqrt{T}},
\]
where \(C_3 = 2N^2 \cdot 12N + 17((L_f + R + \theta L_g)^2 + L_n^2) + R^2 + N \ln 2\).

**Proof.** First, we derive the disagreement among the nodes for Algorithm 2. With a slight abuse of notation, we write
\[
\tilde{\nabla}_t^i = \nabla F_t^i(x_t^i) + \n^i(x_t^i)
\]
\[
= \nabla f_t^i(x_t^i) + \frac{\exp(\theta g(x_t^i)/\gamma)}{1 + \exp(\theta g(x_t^i)/\gamma)} \theta \nabla g(x_t^i) + \n^i(x_t^i)
\]
\[
= \nabla f_t^i(x_t^i) + \sigma_t x_t^i + \frac{\exp(\theta g(x_t^i)/\gamma)}{1 + \exp(\theta g(x_t^i)/\gamma)} \theta \nabla g(x_t^i) + \n^i(x_t^i),
\]
where \(F_t^i(x) = f_t^i(x) + \gamma \ln (1 + \exp (\theta g(x)/\gamma))\). Hence we have the bound
\[
E[\|\tilde{\nabla}_t^i\|_2] \leq L_f + \sigma_t \|x_t^i\|_2 + \theta L_g + L_n
\]
\[
\leq L_f + R + \theta L_g + L_n
\]
because of the inequality (A.6) and $\sigma_t \leq 1$. On the other hand, we have the following bound for the step-size sequence $\{\beta_t\}_{t=1}^T$:

$$
\sum_{t=1}^T \beta_t = 1 + \sum_{t=2}^T \frac{1}{\sqrt{t}} \leq 1 + \int_1^T \frac{1}{\sqrt{u}} \, du = 2\sqrt{T} - 1 \leq 2\sqrt{T}.
$$

Building on the proof of Lemma 3.1 and using the preceding inequalities, we can easily get that

$$
\sum_{t=1}^T \beta_t \leq 1 + \frac{1}{\sqrt{T}} \leq 2 \sqrt{T},
$$

where $C_4 = 2N\left(\frac{3N}{\theta^2(1-\varsigma)} + 4\right)(L_f + R + \theta L_g + L_n)$. Second, based on the proof of Lemma 3.3, it follows that

$$
\sum_{t=1}^T \mathbb{E}[F_t(x_j^t) - F_t(x)] \leq \sum_{t=1}^T \frac{1}{2\beta_t} \left(1 - \sigma_t \beta_t \right) \sum_{i=1}^N \mathbb{E}[r_i^t(x)] - \frac{1}{2\beta_T} \sum_{i=1}^N \mathbb{E}[r_i^{T+1}(x)]
$$

where the last two terms can be easily bounded by using the estimates (3.10) and (3.11), respectively, and we hence turn our attention to bound the first term:

$$
A'_T = \frac{1}{2\beta_1} \sum_{i=1}^N \mathbb{E}[r_i^1(x)] - \frac{\sigma_1}{2} \sum_{i=1}^N \mathbb{E}[r_i^1(x)] - \frac{1}{2\beta_T} \sum_{i=1}^N \mathbb{E}[r_i^{T+1}(x)]
$$

where in the first inequality we have used the fact that $\beta_1 = \sigma_1 = 1$, and in the last inequality we have used the inequality that $\sqrt{t} - \sqrt{t-1} - \frac{1}{\sqrt{t}} < 0$ for all $t \geq 2$. This yields

$$
\frac{1}{T} \sum_{t=1}^T \mathbb{E}[F_t(x_j^t) - F_t(x)] \leq 2N \left(\frac{12N}{\eta^2(1-\varsigma)} + 17\right) ((L_f + R + \theta L_g)^2 + L_n^2) \frac{1}{\sqrt{T}}.
$$

Hence, we have

$$
F_t(x_j^t) = F(x_j^t) + \frac{\sigma_t}{2} \|x_j^t\|_2^2 \quad \text{and} \quad F_t(x) = F(x) + \frac{\sigma_t}{2} \|x\|_2^2,
$$
and the bound of the left-hand side of (3.13) can be further lowered by
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[f_t(x_t^i) - F_t(x)] \geq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[F(x_t^i) - F(x)] - \frac{1}{T} \sum_{t=1}^{T} \frac{\sigma_t}{2} R^2 \\
&\geq \mathbb{E}[F(\bar{x}_T^i) - F(x)] - \frac{R^2}{\sqrt{T}}.
\]
(3.14)

Thus, the conclusion can be obtained by following an argument similar to that of the proof of Theorem 3.2.

Remark 6. Under appropriate conditions, our convergence rate is a factor of $1/T^{1/4}$ better than that attained in [21] (i.e., $O(1/T^{1/4})$). It is also worth noting that for general nonsmooth and convex functions, the best achievable rate for centralized subgradient methods is $O(1/\sqrt{T})$.

Similarly, we have the following corollary that shows the convergence rate result for the noise-free version of Algorithm 2.

Corollary 3.8. Under the conditions of Theorem 3.7 and the assumption that all the nodes have access to the noise-free gradients, we have
\[
\mathbb{E}[f(\bar{x}_T^i)] - f(x^*) \leq \left(1 + \frac{L_f}{\theta \rho} - L_f\right) C_3 \frac{1}{\sqrt{T}},
\]
where $C_3 = 2N\left(\frac{12N}{\eta^2(1-\epsilon)} + 17\right)(L_f + R + \theta L_g)^2 + R^2 + N \ln 2$.

4. Simulation results. In this section, we demonstrate some simulations of the DSG algorithm. To be specific, we consider a distributed estimation problem of the following form:
\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{N} w^i \|x - \nu^i\|^2 \\
\text{subject to} & \quad x \in \mathcal{X},
\end{align*}
\]
where $w^i \in \mathbb{R}$ and $\nu^i \in \mathbb{R}^d$ are deterministic parameters known only to node $i$. Note that this is a well-known problem in distributed estimation (see [2] and references therein).

We illustrate the proposed algorithm with a specific problem instance where $g(x) = \|x\|_1 - 1$ (i.e., $\mathcal{X} = \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}$), with $w^i$ and $\nu^i$ generated from the interval (1, 2) and a unit norm distribution, respectively. It is easy to see that $\mathcal{X} \subseteq \mathcal{B} = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$. We give some specific choices of the parameters that are used in simulations. First, note that each $f_i$ is $2w^i$-strongly convex, hence we set the step size as $\beta_i = \frac{1}{2T}$, where $\sigma = \min_{i \in [N]} 2w^i$. Second, we set $\rho = 1$, based on the following argument: from Assumption 5 we can choose $\rho = \min_{g(x) = 0} \|g(x)\|_2$, where $g(x) = \|x\|_1 - 1$; the minimum can be achieved by setting $x = e_i$ and choosing the associated subgradient as that of $e_i$, where $e_i$ is the standard basis vector in $\mathbb{R}^d$. A reasonable estimate for $L_f$ is that $L_f = \max_{i \in [N]} 2w^i(1 + \|\nu^i\|_2)$. Hence, we set $\theta = L_f/\rho = L_f$.

The noises $n^i(x_t^i)$ are random variables generated independent and identically distributed from the normal distribution $\mathcal{N}(0, \sigma I_{d \times d})$, where $\sigma$ is some positive constant that will be specified in what follows. In all cases, we use the setting of the network size $N = 60$, the dimension of the estimate $d = 2$, and the initial estimates $x_0 = 0$ for all $i$. Note that the simulation results are based on the average of 100 realizations.
Next, let us investigate the convergence performance of the DSG algorithm. We first implement the proposed algorithm over a random graph, which is generated as in [9] (see Figure 1). Figure 2 provides a plot of the relative function error (on the log-scale), $\frac{|f(\tilde{x}_i^T) - f(x^*)|}{|f(x^*)|}$, versus the number of iterations $T$ for two randomly selected nodes, together with the corresponding noise-free case for comparison. Moreover, we have shown the sample mean and standard deviation of $\frac{|f(\tilde{x}_i^T) - f(x^*)|}{|f(x^*)|}$ for $T$ in multiplies of 20 in Figure 2, where the error bars show the mean plus and minus one standard deviation, respectively. Note that the parameter $\sigma$ is chosen as $\sigma = 0.15$. We have illustrated the same quantities for a ring graph in Figure 3.
Fig. 3. Plot of the relative function error versus the number of iterations $T$ for a ring graph.

Fig. 4. Plot of the disagreement versus the iteration $t$.

Figures 2 and 3 show that the estimates of the nodes converge in both cases, and the error decays faster for the random graph than that of the ring graph. This is consistent with common sense, because the random graph (shown in Figure 1) clearly has a better connectivity than that of the ring graph. A similar phenomenon can also be seen in Figure 4, which shows that the disagreement, $\sum_{i=1}^{N} \|\mathbf{x}_i^t - \mathbf{y}_i^t\|_2$, evolves over the iteration $t$. We can see from Figure 4 that the disagreement diminishes at a faster rate for the random graph than that of the ring graph. Note that we have shown the sample mean and standard deviation of the disagreement for $T$ in multiples of 100 in Figure 4.
5. Conclusion. In this paper, we studied the constrained multiagent optimization problem. We proposed a class of DSG algorithms without intermediate projections. By virtue of the smoothed constraint incorporation technique, we established an $O(\ln T/T)$ convergence rate for strongly convex optimization. For the case when the individual objective functions are non-strongly convex, we established an optimal $O(1/\sqrt{T})$ convergence rate based on the idea of the reduction method. There are several interesting questions that remain to be explored. For instance, one possible future research direction is to apply the proposed algorithm to solve the distributed supervised learning problem. Also, it would be of interest to relax the assumption requiring all the objective functions to be strongly convex.

Appendix A. Proof of Lemma 3.1.

**Lemma A.1** (see [16]). Under Assumption 1, for all $i, j$ and all $t \geq s$, we have

$$|\langle W(t : s) \rangle_{ij} - N^{-1}| \leq \eta^t \sum_{\pi} \tau^{\pi} \left( \sum_{t=s+1}^{\infty} \eta^{t-s} \right)^{-2} \leq \eta^{-2} \frac{t-s+1}{s+1}.$$

where $W(t : s)$ is a transition matrix defined by $W(t : s) = W(t)W(t-1)\cdots W(s+1)W(s)$ with $W(t : t) = W(t)$.

**Proof of Lemma 3.1.** To simplify the notation, we denote for all $i$ and $t$,

$$\tilde{\nabla}_i = \nabla F_i(x_i) + n_i(x_i).$$

The average estimate $y_t$ evolves as follows:

\begin{align}
y_t &= \frac{1}{N} \sum_{i=1}^{N} \Pi_B[\bar{x}_i] \\
&= \frac{1}{N} \sum_{i=1}^{N} (\bar{x}_i + \bar{p}_t) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} [W(t-1)]_{ij} z_{i-1} + \frac{1}{N} \sum_{i=1}^{N} \bar{p}_t \\
&= \frac{1}{N} \sum_{i=1}^{N} z_{i-1} + \frac{1}{N} \sum_{i=1}^{N} \bar{p}_t,
\end{align}

where $\bar{p}_t = \Pi_B[\bar{x}_i] - \bar{x}_i$, and the first equality follows from the fact that $x_i = R_{\max(\|x_i\|,R)} \Pi_B[\bar{x}_i]$, and in the last equality we have used the relation

\begin{equation}
\sum_{i=1}^{N} \sum_{j=1}^{N} [W(t-1)]_{ij} \phi^j = \sum_{j=1}^{N} \left( \sum_{i=1}^{N} [W(t-1)]_{ij} \right) \phi^j = \sum_{i=1}^{N} \phi^i
\end{equation}

because $W(t-1)$ is doubly stochastic. Substituting the update formula for $z_{i-1}$ into (A.1), we have

\begin{align}
y_t &= \frac{1}{N} \sum_{i=1}^{N} (x_{i-1} - \beta_{t-1} \tilde{\nabla}_{i-1}) + \frac{1}{N} \sum_{i=1}^{N} \bar{p}_t \\
&= y_{t-1} - \beta_{t-1} \frac{1}{N} \sum_{i=1}^{N} \tilde{\nabla}_{i-1} + \frac{1}{N} \sum_{i=1}^{N} \bar{p}_t;
\end{align}
applying the above equation recursively, we obtain

(A.3) \[ y_t = y_1 - \sum_{s=1}^{t-1} \beta_s \frac{1}{N} \sum_{i=1}^{N} \tilde{\nabla}_s^i + \sum_{s=1}^{t-1} \frac{1}{N} \sum_{i=1}^{N} p_{s+1}^i. \]

Similarly, the estimate of node \( i \) evolves as follows:

(A.4) \[ x_i^t = \sum_{j=1}^{N} [W(t-1:1)]_{ij} x_j^1 - \sum_{s=1}^{t-1} \beta_s \sum_{j=1}^{N} [W(t-1:s)]_{ij} \tilde{\nabla}_s^j + \sum_{s=1}^{t-2} \sum_{j=1}^{N} [W(t-1:s+1)]_{ij} p_{s+1}^j + p_t^i. \]

Combining the inequalities (A.3) and (A.4), we arrive at

(A.5) \[ \| x_i^t - y_t \|_2 \leq \sum_{j=1}^{N} [W(t-1:1)]_{ij} x_j^1 - y_1 \|_2 \]

\[ + \sum_{s=1}^{t-1} \beta_s \sum_{j=1}^{N} [W(t-1:s)]_{ij} - N^{-1} \| \tilde{\nabla}_s^j \|_2 \]

\[ + \sum_{s=1}^{t-2} \sum_{j=1}^{N} [W(t-1:s+1)]_{ij} - N^{-1} \| p_{s+1}^j \|_2 \]

\[ + \| p_t^i \|_2 + \frac{1}{N} \sum_{i=1}^{N} p_t^i \|_2, \]

where the first term on the right-hand side is zero, because \( x_j^1 = 0 \) for all \( j \in [N] \) and \( y_1 = 0 \). We are left to bound the norm of the error due to the projection

\[ \| p_t^i \|_2 = \| \Pi_B \left[ x_i^t \right] - \sum_{j=1}^{N} [W(t-1)]_{ij} x_j^1 \left( x_{t-1}^j - \beta_{t-1} \tilde{\nabla}_{t-1}^j \right) \|_2 \]

\[ \leq \| \Pi_B \left[ x_i^t \right] - \sum_{j=1}^{N} [W(t-1)]_{ij} x_j^1 \|_2 \]

\[ + \beta_{t-1} \sum_{j=1}^{N} [W(t-1)]_{ij} \| \tilde{\nabla}_{t-1}^j \|_2. \]

Note that \( x_i^1 \in \mathcal{B} \) holds for all \( i \) and \( t \geq 1 \). Hence, we can bound the first term by using the nonexpansiveness of the Euclidean projection:

\[ \| \Pi_B \left[ x_i^t \right] - \sum_{j=1}^{N} [W(t-1)]_{ij} x_j^1 \|_2 \leq \| x_i^t - \sum_{j=1}^{N} [W(t-1)]_{ij} x_j^1 \|_2 \]

\[ \leq \beta_{t-1} \sum_{j=1}^{N} [W(t-1)]_{ij} \| \tilde{\nabla}_{t-1}^j \|_2. \]
This yields the final bound on $\|\mathbf{p}_i^t\|_2$ in expectation, that is,

$$
\mathbb{E} \left[ \|\mathbf{p}_i^t\|_2 \right] \leq 2 \beta_{t-1} \sum_{j=1}^{N} \|\mathbf{W}(t-1)\|_{ij} \mathbb{E} \left[ \|\tilde{\nabla}_j^{t-1}\|_2 \right]
$$

$$
\leq 2 (L_f + L_n + \theta L_g) \beta_{t-1},
$$
due to the double stochasticity of the weight matrix and the following bound on the stochastic gradient $\|\tilde{\nabla}_i^t\|_2$ by Assumption 3 and the assumption on the stochastic noise (2.3):

$$
(A.6) \quad \mathbb{E} \left[ \|\tilde{\nabla}_i^t\|_2 \right] \leq \|\nabla f(\mathbf{x}_i^t)\|_2 + \theta \|\nabla g(\mathbf{x}_i^t)\|_2
$$

$$
\leq L_f + \sqrt{\mathbb{E} \left[ \|\nabla g(\mathbf{x}_i^t)\|_2^2 \right]} + \theta L_g
$$

$$
\leq L_f + L_n + \theta L_g,
$$
where in the first inequality we have used Jensen’s inequality. Combining the above relation with (A.5) and using Lemma A.1, we have

$$
\mathbb{E} \left[ \|\mathbf{x}_i^t - y_i^t\|_2 \right] \leq N \eta^{-2} (L_f + L_n + \theta L_g) \sum_{s=1}^{t-1} \beta_s \zeta^{t-s}
$$

$$
+ 2 N \eta^{-2} (L_f + L_n + \theta L_g) \sum_{s=1}^{t-2} \beta_s \zeta^{t-s-1}
$$

$$
+ 4 (L_f + L_n + \theta L_g) \beta_{t-1},
$$
where we have used the convexity of the norm function, that is, $\| \frac{1}{N} \sum_{i=1}^{N} \mathbf{p}_i^t \|_2 \leq \frac{1}{N} \sum_{i=1}^{N} \|\mathbf{p}_i^t\|_2$. By noting that $\zeta < 1$, we can get a more compact bound:

$$
(A.7) \quad \mathbb{E} \left[ \|\mathbf{x}_i^t - y_i^t\|_2 \right] \leq 3 N \eta^{-2} (L_f + L_n + \theta L_g) \sum_{s=1}^{t-1} \beta_s \zeta^{t-s-1}
$$

$$
+ 4 (L_f + L_n + \theta L_g) \beta_{t-1}.
$$
Summing (A.7) over $t = 1, \ldots, T$ yields

$$
\sum_{t=1}^{T} \mathbb{E} \left[ \|\mathbf{x}_i^t - y_i^t\|_2 \right] = \sum_{t=2}^{T} \mathbb{E} \left[ \|\mathbf{x}_i^t - y_i^t\|_2 \right]
$$

$$
\leq 3 N \eta^{-2} (L_f + L_n + \theta L_g) \sum_{s=1}^{t-1} \beta_s \zeta^{t-s-1}
$$

$$
+ 4 (L_f + L_n + \theta L_g) \sum_{t=2}^{T} \beta_{t-1}.
$$

The term $\Delta_T$ can be bounded as follows:

$$
\Delta_T \leq \sum_{t=1}^{T-1} \beta_t \left( \sum_{s=0}^{T-2} \zeta^s \right) \leq \sum_{t=1}^{T-1} \beta_t \left( \sum_{s=0}^{\infty} \zeta^s \right) \leq \frac{1}{1 - \zeta} \sum_{t=1}^{T-1} \beta_t.
$$
A direct consequence of the step-size sequence is that for all $T \geq 3$,

$$
\sum_{t=1}^{T} \beta_t = \frac{1}{\sigma} \left( 1 + \sum_{t=2}^{T} \frac{1}{t} \right) \leq \frac{1}{\sigma} \left( 1 + \int_{1}^{T} \frac{1}{u} \, du \right) \leq \frac{2}{\sigma} \ln T.
$$

This, combined with the last two inequalities, gives our final estimate.

\textbf{Appendix B. Proof of Lemma 3.3.} We follow the standard analysis to derive the general evolution of $r_t^i(x)$:

$$
r_{t+1}^i(x) = \|x_{t+1}^i - x\|^2
= \|\Pi_B[x_{t+1}^i] - x\|^2
\leq \|x_{t+1}^i - x\|^2
\leq \sum_{j=1}^{N} [W(t)]_{ij}\|z_j^i - x\|^2,
$$

where the first inequality follows from the nonexpansiveness of the Euclidean projection and the last inequality from the convexity of the norm square function and the double stochasticity of $W(t)$. For the term $\|z_j^i - x\|^2$, we have the following estimate, according to the update formula for $z_j^i$:

$$
\|z_j^i - x\|^2 = \|x_j^i - \beta_t \tilde{\nabla}_t^i - x\|^2
= r_t^i(x) + \beta_t^2 \|\tilde{\nabla}_t^i\|^2 + 2\beta_t \langle x - x_j^i, \tilde{\nabla}_t^i \rangle.
$$

Substituting the preceding inequality into (B.1) and then summing over all $i$ yields

$$
\sum_{i=1}^{N} \sum_{j=1}^{N} [W(t)]_{ij} \left( r_t^i(x) + \beta_t^2 \|\tilde{\nabla}_t^i\|^2 + 2\beta_t \langle x - x_j^i, \tilde{\nabla}_t^i \rangle \right)
= \sum_{j=1}^{N} \left( \sum_{i=1}^{N} [W(t)]_{ij} \right) \left( r_t^j(x) + \beta_t^2 \|\tilde{\nabla}_t^j\|^2 + 2\beta_t \langle x - x_j^i, \tilde{\nabla}_t^j \rangle \right)
= \sum_{j=1}^{N} \left( r_t^j(x) + \beta_t^2 \|\tilde{\nabla}_t^j\|^2 + 2\beta_t \langle x - x_j^i, \tilde{\nabla}_t^j \rangle \right)
$$

with the same manipulation as in (A.2). The term $\|\tilde{\nabla}_t^i\|^2$ can be bounded as follows:

$$
\|\tilde{\nabla}_t^i\|^2 = \|\nabla F_i(x_t^i) + n_t^i(x_t^i)\|^2 \leq 2\|\nabla F_i(x_t^i)\|^2 + 2\|n_t^i(x_t^i)\|^2.
$$

This leads to

$$
\mathbb{E}[\|\tilde{\nabla}_t^i\|^2] \leq 2(L_f + \theta L_g)^2 + 2L_n^2.
$$
Combining the inequalities (B.2) and (B.3) gives

\begin{equation}
\sum_{i=1}^{N} E \left[ \langle x_i' - x, \nabla F(x_i') \rangle \right] = \sum_{i=1}^{N} E \left[ \langle x_i' - x, \nabla F'(x_i) + n_i' \rangle \right] \\
= \sum_{i=1}^{N} E \left[ \langle x_i' - x, \tilde{v}_i' \rangle \right] \\
\leq \frac{1}{2\beta_t} \left( \sum_{i=1}^{N} E \left[ r_i'(x) \right] - \sum_{i=1}^{N} E \left[ r_{i+1}'(x) \right] \right) \\
+ N \left( (L_f + \theta L_g)^2 + L_n^2 \right) \beta_t,
\end{equation}

where the second equality follows from \( E \left[ \langle x_i' - x, n_i'(x_i') \rangle \right] = 0 \), by using the independence of \( x_i' \) and \( n_i'(x_i') \) and the relation \( E \left[ n_i'(x_i') \right] = 0 \). It remains to provide a lower bound on the left-hand side of the preceding inequality, which is given as follows:

\begin{equation}
\sum_{i=1}^{N} \left( F'(x_i') - F'(x) + \frac{\sigma}{2} \| x_i' - x \|_2^2 \right) \\
= \sum_{i=1}^{N} \left( F'(x_i') - F'(x) + F'(x_i') - F'(x_i') + \frac{\sigma}{2} \| x_i' - x \|_2^2 \right) \\
\geq F(x_i') - F(x) + \frac{\sigma}{2} \sum_{i=1}^{N} r_i'(x) + \sum_{i=1}^{N} \langle \nabla F'(x_i'), x_i' - x_i' \rangle \\
\geq F(x_i') - F(x) + \frac{\sigma}{2} \sum_{i=1}^{N} r_i'(x) - (L_f + \theta L_g) \sum_{i=1}^{N} \| x_i' - x_i' \|_2
\end{equation}

because each \( F' \) is \( \sigma \)-strongly convex, that is, \( F'(x) \geq F'(x_i') + \langle x - x_i', \nabla F'(x_i') \rangle + \frac{\sigma}{2} \| x_i' - x \|_2^2 \) (since each \( f_i' \) is \( \sigma \)-strongly convex, by Assumption 4). This, combined with the estimate (B.4), gives the desired result.

**Appendix C. Proof of Lemma 3.5.** To simplify the notation, we denote \( g_r(x) = \gamma \ln (1 + \exp (\theta g(x)/\gamma)) \). It is easy to show that \( g_r(x^*) \leq N \gamma \ln 2 \), due to the fact that \( g(x^*) \leq 0 \). For any \( x \in \mathcal{X} \), we can show that

\[ g_r(x) \geq [\theta g(x)]_+ \]

based on the following argument: if \( g(x) \leq 0 \), we have \( g_r(x) > \gamma \ln 1 = 0 = [\theta g(x)]_+ \); otherwise we have \( g_r(x) \geq \gamma \ln (1 + \exp (\theta g(x)/\gamma)) > \theta g(x) = [\theta g(x)]_+ \), according to the inequality that \( \ln (1 + \exp(a)) > a \) for all \( a \geq 0 \). Hence, using the fact that \( F(x) = f(x) + N g_r(x) \), it is easy to show that

\[ f(x) - f(x^*) = F(x) - F(x^*) + N g_r(x^*) - N g_r(x) \]
\[ \leq F(x) - F(x^*) + N \gamma \ln 2 - N[\theta g(x)]_+ \]

This gives the desired result.

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ON CONVERGENCE RATE OF DSG ALGORITHM

REFERENCES


