Probabilistic linguistic vector-term set and its application in group decision making with multi-granular linguistic information

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A R T I C L E   I N F O

Keywords:
Probabilistic linguistic vector-term set
Multi-granular linguistic information
Multi-attribute group decision making
Personalized hospital selection
Recommender system

A B S T R A C T

With the rapid information explosion and sharing, recommender systems (RS) play an auxiliary role in assisting the Internet users to make decision especially in the e-service platform. Normally, the information in this process is related to opinions and preferences, which are usually expressed through a qualitative way such as linguistic evaluation terms (LETs). However, the LETs may come from different sources such as experts, users, etc., which makes the linguistic evaluation scales (LESs) used in this process probably be different due to their different backgrounds and levels of knowledge. The diversity and flexibility of these LESs determine the quality of information, and further affect the effectiveness of a RS. In this paper, we focus on improving the accuracy of the multi-granular linguistic recommender system by supporting customers to find out the most eligible items according their own preferences. We first propose the probabilistic linguistic vector-term sets (PLVTSs) to promote the application of multi-granular linguistic information. Based on the PLVTSs, we then develop a novel algorithm to tackle multi-attribute group decision making (MAGDM) problems with multiple LESs. Furthermore, the effectiveness of the PLVTSs is validated by an illustration of personalized hospital selection-recommender problem. Finally, we point out some possible research directions regrading to the PLVTSs.

1. Introduction

With the explosive development of the internet and information, it is very common to select an optimal alternative from some available alternatives with respect to a set of attributes, based on the mass related evaluation information [1]. For example, according to physiological and economic condition, a respiratory system disease patient should choose an appropriate hospital for his/her rehabilitation with referring to the advices from doctors and the evaluation information from some other patients [2]. Recommender system (RS), which exploits the past behaviors and the user similarities to provide the personalized recommendations, is a good tool to deal with selection problems. Some authors paid attention to the missing rating prediction for particular users on unknown items [3]. Some other studies focused on developing the measures [4] and recommender methods [5] to improve the recommendation precision. A few of practical applications of RSs with linguistic information can be seen in some papers such as Refs. [5,6].

As for the preference information related to RS, people usually express them by linguistic terms such as “fine” or “slightly good” to evaluate an object. But it is indeed a challenge to accurately simulate the evaluators’ opinions, especially in group decision situations, due to the different evaluators’ knowledge backgrounds and the information extraction methods [7]. The hesitant fuzzy linguistic term set (HFLTS) was proposed [8] and developed [9–11] to improve the flexibility of linguistic preference information within hesitant situations. Probabilistic linguistic term set (PLTS) is an extended expression of HFLTS by adding probability parameter to prevent the loss of original linguistic information [12]. The PLTS is a good tool to represent the evaluator’s hesitancy, but it is still not precisely enough to express the evaluator’s all preference information between these hesitant options as the evaluator may be hesitant not only in the linguistic terms but also in the preference variability corresponding to each term. When such information is used in a group decision making situation, its...
non-precise degree is magnified by the number of evaluators. Therefore, the PLTSs with the change degree of each linguistic term should be much more applicable than the HFLTSs.

Due to the advantage and convenience of linguistic information in describing the preferences of experts, the linguistic RS has been investigated from different points of view such as the methods of capturing the uncertainty of user's preferences [13,14] and detecting the noise by measures [15]. Herreara et al. [16] investigated a fuzzy linguistic RS through the preference relation method on the basis of linguistic hierarchy and similarity measure. After this, a new fuzzy linguistic RS to characterize the users' profiles was presented and applied into the university digital libraries [17,18].

In our opinion, the utilization of linguistic terms can be improved if we consider the sensitivity change parameter, which can ameliorate the effect of linguistic RS. The sensitivity change should be considered because it is different from person to person. Suppose that \( e_1 \) and \( e_2 \) are two evaluators required to make evaluations on an object together. Both they give the linguistic evaluation \( S_2 \). But \( e_1 \) is subtler compared to \( e_2 \), which is implied in the expressions that \( e_1 \) expresses his/her opinion by using a LES with nine terms but \( e_2 \) expresses his/her opinion through a five terms LES. So, the semantics of \( S_2 \) for \( e_1 \) and \( e_2 \) respectively are different even though they give the same linguistic descriptor \( S_2 \). In other words, only using \( S_2 \) cannot describe the evaluators' opinions comprehensively and accurately enough when different LESs are used by a group of experts to deal with a MAGDM. The purpose of this paper is to depict the sensitivity change.

Compared with the existing linguistic approaches for dealing with the MAGDM problems, the innovations and contributions of this paper are summarized as follows:

1. We firstly propose a vector formula of probabilistic hesitant fuzzy LETs, called PLVTSs, which considers not only the score but also the change degree of each LET. It allows the experts to use their own LET sets to express personal opinions flexibly according to their knowledge. We will define the PLVTS as a new tool to express the expert's preferences in a multi-attribute group decision making problem in Section 3.
2. The operations and the operators are the necessary techniques to apply the PLVTSs in a multi-attribute group decision making problem. Then we will define some related concepts and operators for PLVTS after giving the definition of PLVTS in Section 3.
3. We put forward an algorithm to deal with the MAGDM problem. In this algorithm, we construct a value function with some parameters, which can consider both the weight vectors of recommenders and the user's selection weights. This is the specific method to obtain a ranking of all alternatives in a multi-attribute group decision making problem, based on the concept of PLVTS and the related operators. We will interpret the algorithm in Section 4.

The remainder of the paper is arranged as follows: Section 2 briefly reviews some preliminary knowledge of linguistic scales and PLTS. Section 3 defines the concept of PLVT and gives several basic operations to carry out the computing over multi-granular uncertain linguistic information. Section 4 presents an algorithm to accomplish the process of group evaluation and selection. Additionally, an illustrative example about visiting hospital selection in a personalized recommendation system with linguistic evaluation information is introduced in Section 5. We compare the novel expression of multi-granular linguistic information with the traditional method in this section as well. Finally, Section 6 makes a summary and outlook of the paper.

2. Preliminaries

The linguistic variables [19,20] have been widely explored from distinct aspects such as linguistic computations [21] and properties [22,23]. This section briefly reviews the related LESs and the concept of probabilistic linguistic term set (PLTS).

2.1. Linguistic evaluation scale

Based on the ordinal discrete linguistic term set \( S_1 = \{ s_α | α = 1, 2, \cdots, t \} \), Xu [24] redefined the subscript-symmetric discrete linguistic term set:

\[
S_2 = \{ s_α | α = -t, \cdots, -1, 0, 1, \cdots, t \}
\]

where \( t \) is a positive integer, \( s_α \) represents a possible value for a linguistic label and satisfies: (1) \( s_α > s_β \) iff \( α > β \); (2) \( \neg (s_0) = s_β \) such that \( α + β = t \).

To solve the problems of lost and abnormal linguistic information during computations on the original scale, Xu [24] further extended the subscript-symmetric discrete linguistic term set to a continuous form as \( S_2 = \{ s_α | α \in [-q, q] \} \), where \( q (q > t) \) is a sufficiently large positive integer. If \( S_α, S_β \in S_2 \), \( s_α \) is called an original term; otherwise, \( s_α \) is a virtual term. Generally, the original terms are used in evaluation, and the virtual linguistic terms can only appear in operation. To facilitate computing, considering any two linguistic terms \( s_α, s_β, s_γ \in S_3 \), and \( λ \in [0, 1] \), some operational laws are satisfied as follows: (1) \( s_α \odot s_β = s_α \odot s_β \); (2) \( λ s_α = s_α \).

From the distribution of linguistic labels, the LESs \( S_1 \sim S_3 \) are uniform; \( S_1 \) and \( S_3 \) are symmetrical. The unbalanced linguistic information, however, may appear due to the nature of linguistic variables used in the evaluation problems [25,26]. For example, there is an unbalanced LESs defined as [23]:

\[
S_4 = \{ s_α | α = -(t-1), -\frac{2}{3} (t-2), \cdots, 0, \frac{2}{3} (t-2), t-1 \}
\]

where \( t \) is a positive integer and \( t-1 \) is the cardinality value of \( S_4 \). The linguistic terms in \( S_4 \) has the following characteristics: (1) \( s_α > s_β \) iff \( α > β \); (2) \( \neg (s_0) = s_β \), especially, \( \neg (s_0) = s_0 \).

To solve the problems of lost and abnormal linguistic information during computation on the original unbalanced scale, the continuous unbalanced LES can be presented as [24]: \( S_5 = \{ s_α | α \in [-q, q] \} \), where \( q (q > t-1) \) is a sufficiently large positive integer.

The previous LESs and their operational laws provide a good basis and convenient conditions for decision making with linguistic information.
2.2. Probabilistic linguistic term set

Probabilistic linguistic term set (PLTS), combining the linguistic terms with their associated probabilistic, was proposed by Pang et al. [12].

**Definition 2.1.** [12]. Let $S = \{s_0, ..., s_l\}$ be a linguistic term set, a PLTS can be defined as:

$$L(p) = \left\{ L^{(k)}(p^{(k)}) | L^{(k)} \in S, \ p^{(k)} \geq 0, \ k = 1, 2, ..., \#L(p), \ \sum_{k=1}^{\#L(p)} p^{(k)} \leq 1 \right\} \quad (2.3)$$

where $L^{(k)}(p^{(k)})$ is the linguistic term $L^{(k)}$ associated with the probability $p^{(k)}$, and $\#L(p)$ is the number of all different linguistic terms in $L(p)$.

**Example 2.1.** Assume that thirty students are required to evaluate a teacher by using an additional LES with five terms: $S = \{s_0 = \text{very poor}, s_1 = \text{slightly poor}, s_2 = \text{fair}, s_4 = \text{slightly good}, s_5 = \text{very good}\}$. Three students state that the teacher is slightly poor; eighteen students state that the teacher is slightly good; five students state that he is very good; and the others do not express any opinions. Then, we can depict the result as: $S = \{s_1(0.2), s_4(0.6), s_5(0.17)\}$.

3. Probabilistic linguistic vector-term set (PLVTS)

In this section, we propose the concept of PLVTS, and then investigate some basic operations of PLVTS.

### 3.1. The concept of PLVTS

According to the utility theory, people have different preferences over different attributes. The unbalanced LESs demonstrate their advantages in this aspect compared to the balanced ones. The strength of preference change can be expressed through different distributions of linguistic labels when linguistic information is used to describe the evaluation values. The smaller the interval between two linguistic labels is, the greater the change of the evaluator’s preferences among these two terms will be, and vice versa. But the existing results of LESs focus on the values of LETs but ignore the change degree of them, which reflects the degree of sensitivity. Therefore, both the linguistic term and its associated change rate should be taken into account when we use LESs to express the evaluators’ preferences.

Besides, we should note that different evaluators may hold different viewpoints when they evaluate the candidate alternatives together. This means that the individuals’ choice of LESs is probably not the same. Specifically, these differences may be reflected in two aspects. Firstly, the cardinalities of different linguistic term sets used by different evaluators may be different. Secondly, the distributions of terms in different linguistic term sets may be different, even if the linguistic term sets contain the same number of terms. For example, suppose that $e_1$ and $e_2$ are two evaluators. Both of them choose five-term linguistic scales to describe their evaluation values. Nonetheless, they may have different attitudes and comprehensions, such as $S_1 = \{s_{-2} = \text{none}, s_{-1} = \text{slight low}, s_0 = \text{medium}, s_1 = \text{high}, s_2 = \text{perfect}\}$ and $S_2 = \{s_{-2} = \text{none}, s_{-2/3} = \text{slight low}, s_0 = \text{medium}, s_{2/3} = \text{high}, s_2 = \text{perfect}\}$. There are two problems which need to deal with: (1) the cardinality of linguistic evaluation sets used by the evaluators; (2) the location of each linguistic term in these LESs. To solve the problem (1), we refer to the result of Ref. [27] that the cardinality of each LES should be not less than 5 but not more than 13. For the problem (2), it attracts our attentions to the step of determining the linguistic information expressions of individual’s opinions. People need to choose an appropriate LES according to his/her own needs. Considering the complexity and diversity of human’s consciousness, we allow the subscript of each term not being derived from the established formulas like Eqs. (2.1) and (2.2). People can flexibly determine the subscripts of terms by arbitrary numbers within a certain range under the precondition of satisfying the cardinality constraint. Because the LETs may be repetitive in the mass linguistic evaluation data, the probability of each LET should not be ignored. We contain the probability in the new formula of LET for facilitating the expression of multi-granular linguistic information in group decision making, which is named as PLVTS. Constructing such as PLVTS involves three steps, i.e., normalizing LESs, calculating the change rate of each normalized term, and establishing the PLVTS.

### 3.1.1. Normalizing linguistic evaluation scales

For the convenience of calculation and comparison, we normalize different LESs in MAGDM into the interval $[0, 1]$. The original linguistic term $s_0$ can be transformed into $s_{\alpha}S$ by the following formula:

$$S \leftrightarrow S_{\alpha} : \ s_0 \rightarrow s_{\alpha}, \ \text{where} \ \alpha = \frac{\alpha - \min_{s_0 \in S} \{\theta\}}{\max_{s_0 \in S} \{\theta\} - \min_{s_0 \in S} \{\theta\}} \quad (3.1)$$

**Example 3.1.** Let $S_1$ be a linguistic term set when $t = 4$ in Eq. (2.2), and $S_2$ be another linguistic term set when $t = 2$ in Eq. (2.1):

$S_1 = \{s_{-3} = \text{none}, s_{4/3} = \text{very low}, s_{1/2} = \text{low}, s_0 = \text{medium}, s_{1/2} = \text{high}, s_{4/3} = \text{very high}, s_3 = \text{perfect}\}$,

$S_2 = \{s_{-2} = \text{none}, s_{-1} = \text{slight low}, s_0 = \text{medium}, s_1 = \text{high}, s_2 = \text{perfect}\}$

After transformed by Eq. (3.1), we can obtain:

$$\leftrightarrow S_1 = \{s_0 = \text{none}, s_{5/18} = \text{very low}, s_{7/12} = \text{low}, s_{1/2} = \text{medium}, s_{7/12} = \text{high}, s_{13/18} = \text{very high}, s_1 = \text{perfect}\}$$,

$$\leftrightarrow S_2 = \{s_0 = \text{none}, s_{1/4} = \text{slightly low}, s_{1/2} = \text{medium}, s_{3/4} = \text{high}, s_1 = \text{perfect}\}$$.
3.1.2. Calculating the change rate of each normalized term

Suppose that $s_{\alpha(t)}$ is the $\tau$th linguistic term in the normalized scale $\leftrightarrow S, \tau = 1, 2, \cdots, L(S)$. In order to obtain the change rate of $s_{\alpha(t)}$, we first calculate the relative change rate of $s_{\alpha(t)}$ by the following formula:

$$r_{s_{\alpha(t)}} = \frac{\langle \alpha(t) \rangle - \langle \alpha(t - 1) \rangle - \langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}{\langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}$$

(3.2)

where $\langle \alpha(t) \rangle$ is the subscript of the $\tau$th term $s_{\alpha(t)}$ in the scale $\leftrightarrow S$. Fig. 1 illustrates the geometric meaning of Eq. (3.2).

If $r_{s_{\alpha(t)}} < 0$, then the change in the left side of $s_{\alpha(t)}$ is relatively slow compared to its right side; if $r_{s_{\alpha(t)}} > 0$, then the change in the left side of $s_{\alpha(t)}$ is relatively quick compared to its right side; if $r_{s_{\alpha(t)}} = 0$, then the speeds of changes in both sides of $s_{\alpha(t)}$ are equivalent. Particularly, we get the relative change rate of the first linguistic term, $r_{s_{\alpha(t)}}$, be equal to $-1$, and the relative change rate of the last linguistic term, $r_{s_{\alpha(t)}}$, be equal to $+\infty$.

It should be noted that the relative change rate of each linguistic term would not be changed in the transformation process.

**Theorem 3.1.** Let $r_{s_{\alpha(t)}}$ be the relative change rate of linguistic term $s_{\alpha(t)}$ in the original linguistic term set $S$. $r_{s_{\alpha(t)}}$ be the relative change rate of the linguistic term $s_{\alpha(t)}$ in the normalized linguistic term set $S \leftrightarrow S$. Then $r_{s_{\alpha(t)}} = r_{s_{\alpha(t)}}$.

**Proof.** The relative change rate of the $\tau$th linguistic term in the original linguistic term set $S$ can be calculated by Eq. (3.2) as:

$$r_{s_{\alpha(t)}} = \frac{\langle \alpha(t) \rangle - \langle \alpha(t - 1) \rangle - \langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}{\langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}$$

When the set $S$ is normalized into $\leftrightarrow S$ through Eq. (3.1), the linguistic term $s_{\alpha(t)}$ is transformed into $s_{\alpha(t)}$. Accordingly, the relative change rate $r_{s_{\alpha(t)}}$ of $s_{\alpha(t)}$ can be calculated by

$$r_{s_{\alpha(t)}} = \frac{\langle \alpha(t) \rangle - \langle \alpha(t - 1) \rangle - \langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}{\langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}$$

By Eq. (3.1), we get

$$r_{s_{\alpha(t)}} = \frac{\langle \alpha(t) \rangle - \langle \alpha(t - 1) \rangle - \langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}{\langle \alpha(t + 1) \rangle - \langle \alpha(t) \rangle}$$

In order to get the change rate of each linguistic term $s_{\alpha(t)}$ in the scale $\leftrightarrow S$, we can take the first linguistic term as a reference and then add all the relative change rates before $s_{\alpha(t)}$. Thus, we can introduce the definition of change rate of the linguistic term $s_{\alpha(t)}$. Fig. 2 illustrates the change rate intuitively.

**Definition 3.1.** The change rate $\leftrightarrow r_{s_{\alpha(t)}}$ of the $\tau$th term $s_{\alpha(t)}$ in the scale $\leftrightarrow S$ can be defined as:

$$\leftrightarrow r_{s_{\alpha(t)}} = r_{s_{\alpha(t)+1}} + r_{s_{\alpha(t)+2}} + \cdots + r_{s_{\alpha(t)}}$$

(3.3)

**Note.** According to the proof of Theorem 3.1, it is easy to conclude that the change rate of each term is not changed through Eq. (3.1).

**Example 3.2.** Let $S = \{s_{-3} = \text{none}, s_{-4/3} = \text{very low}, s_{-1/2} = \text{low}, s_{0} = \text{medium}, s_{1/2} = \text{high}, s_{4/3} = \text{very high}, s_{3} = \text{perfect}\}$. We can get a new LES $\leftrightarrow S$ by Eq. (3.1) as follows:

$$\leftrightarrow S = \{s_{0} = \text{none}, s_{5/18} = \text{very poor}, s_{5/12} = \text{poor}, s_{1/2} = \text{medium}, s_{7/12} = \text{good}, s_{13/18} = \text{very good}, s_{1} = \text{perfect}\}$$
Then, we calculate the change rate of each term in \( \leftrightarrow S \) by Eqs. (3.2) and (3.3) and obtain:

\[
\leftrightarrow r_0 = -1, \quad \leftrightarrow r_{5/18} = 0, \quad \leftrightarrow r_{5/12} = \frac{2}{3}, \quad \leftrightarrow r_{7/12} = \frac{2}{3}, \quad \leftrightarrow r_{7/18} = \frac{4}{15}, \quad \leftrightarrow r_{13/18} = \frac{7}{30}, \quad \leftrightarrow r_1 = +\infty.
\]

Since the normalized linguistic term and its associated change rate should be taken into account simultaneously when we use different LESs to represent the evaluators' assessments, it is appropriate and flexible to propose a vector form of LESs, which can depict the normalized linguistic term and its associated change rate together. The linguistic vector-term set (LVTS) is defined as follows and illustrated in Fig. 3.

**Definition 3.2.** Let \( S_1, S_2, \ldots, S_N \) be a set of distinct LESs, a LVTS can be defined as:

\[
\overrightarrow{S} = \left\{ \overrightarrow{s}(\tau) \mid \tau = 1, 2, \ldots, L(\overrightarrow{S}) \right\} = \left\{ \leftrightarrow a^n_{\tau} \overrightarrow{i} + \leftrightarrow r^n_{\tau} \overrightarrow{j} \mid \tau = 1, 2, \ldots, L(\overrightarrow{S}); \ n = 1, 2, \ldots, N \right\}
\]  

(3.4)

where \( \overrightarrow{s}(\tau) \) is the \( \tau \)th LET in \( \overrightarrow{S}, \tau = 1, 2, \ldots, L(\overrightarrow{S}) \) is the cardinality of \( \overrightarrow{S} \), \( \overrightarrow{i} \) and \( \overrightarrow{j} \) are the unit coordinate vectors, \( \leftrightarrow a^n_{\tau} \) is the subscript of the \( \tau \)th normalized LET \( a^n_{\tau} \) with

\[
\leftrightarrow a^n_{\tau} = \frac{a^n_{\tau} - \min_{\overrightarrow{s} \in \overrightarrow{S}_n} \theta}{\max_{\overrightarrow{s} \in \overrightarrow{S}_n} \theta - \min_{\overrightarrow{s} \in \overrightarrow{S}_n} \theta}
\]  

(3.5)

\( \leftrightarrow r^n_{\tau} \) is the change rate of the \( \tau \)th LET \( a^n_{\tau} \) with

\[
\leftrightarrow r^n_{\tau} = r^n_{\tau(1)} + r^n_{\tau(2)} + \cdots + r^n_{\tau(n)}
\]  

(3.6)

3.1.3. Establishing the PLVS

It is noted that under mass linguistic evaluation information circumstances, the probability of each LET should not be ignored. That is to say, some LETs may be selected by some evaluators, while others may choose different terms. Therefore, different LETs may be selected by different numbers of evaluators. This results in that distinct LETs probably are endowed with different probabilities. Moreover, the probability distributions of different evaluators' linguistic terms descriptions may be different even though they select the same linguistic terms. For example, suppose that there are two groups of evaluators being required to make evaluation for a same object. They may both give a result denoted as \( \overrightarrow{S} = \{s_{-2/3} = \text{slight low}, s_0 = \text{medium}, s_{2/3} = \text{high} \} \). But through the statistical analysis of these two evaluators groups' results, the frequency of \( s_{-2/3} \) in the first group is 60%, but its frequency in the second group is 20%. The similar situation may happen in other linguistic terms of the final expression. This shows that there still are a lot of differences of opinions between the two evaluator groups. Thus, the possibility of each linguistic term should also be considered in expressing multi-granular linguistic information within group decision making problems. Otherwise, the error of group decision making result will be magnified if we neglect this parameter.
Then, we define the concept of PLVTS to represent the comparatively subjective linguistic evaluation information of evaluators.

**Definition 3.3.** Let \( S_1, S_2, \ldots, S_N \) be a set of LESS. A PLVTS can be defined as:

\[
\tilde{S} = \left\{ (s_{i,1}^+, p_{i,1}), (s_{i,2}^-, p_{i,2}), \ldots, (s_{i,N}^+, p_{i,N}) \mid \tau = 1, 2, \ldots, L(\tilde{S}) \right\}
\]

\[
\text{where } s_{i,\tau}^+, \tau \leq L(\tilde{S}), \tilde{i}, \tilde{j}, \leftrightarrow r_{i,\tau}^+ \leftrightarrow r_{i,\tau}^-
\]

are defined as above, \( p_{i,\tau} \) is the probability of the \( \tau \)th LET in \( \tilde{S} \) and satisfies \( 0 \leq p_{i,\tau} \leq 1 \), \( \sum_{\tau=1}^{L(\tilde{S})} p_{i,\tau} \leq 1 \).

For simplicity, \( \left( s_{i,1}^+, p_{i,1} \right) \) is called the probabilistic linguistic vector-term (PLVT).

Note that if \( \sum_{\tau=1}^{L(\tilde{S})} p_{i,\tau} = 1 \), then the distribution on attributes is totally demonstrated by evaluators, i.e., the evaluators have the complete information of probabilistic distribution of all possible linguistic vector terms on this attribute; if \( \sum_{\tau=1}^{L(\tilde{S})} p_{i,\tau} < 1 \), then the probability distribution of the attribute is not totally demonstrated by the evaluators. Especially, \( \sum_{\tau=1}^{L(\tilde{S})} p_{i,\tau} = 0 \) means the attribute is completely unknown. Obviously, how to deal with the ignorance of \( \tilde{S} \) is a very important work for the applications of PLVTSs. Different ways to handle it may produce different results. Nevertheless, as the MAGDM problem discussed in this paper is under mass data background, the linguistic evaluation information we use would have a higher level of completeness. Thus, there is no need to treat the ignorance of linguistic information. We emphatically investigate how to use linguistic evaluation information more efficiently and effectively.

**Example 3.3.** Suppose that there are two evaluators who choose different LESSs \( S_1 \) and \( S_2 \) to express their preferences:

\( \tilde{S}_1 = \{ s_{-3} = \text{none}, s_{-2} = \text{very low}, s_{-1} = \text{low}, s_0 = \text{medium}, s_{1/2} = \text{high}, s_2 = \text{very high}, s_3 = \text{perfect} \} \)

\( \tilde{S}_2 = \{ s_{-4} = \text{extremely poor}, s_{-3} = \text{very poor}, s_{-2} = \text{poor}, s_{-1} = \text{slightly poor}, s_0 = \text{fair}, s_1 = \text{slightly good}, s_2 = \text{good}, s_3 = \text{very good}, s_4 = \text{exremely good} \} \)

Furthermore, assume that the two evaluators’ results are given as: \( l_1 = \{ (s_{-1/2}, 0.31), (s_{1/2}, 0.45) \} \) and \( l_2 = \{ (s_1, 0.55), (s_2, 0.4) \} \).

Then, the normalized PLVTSs given by the group composed by these two evaluators can be given as:

\[
\tilde{S} = \left\{ \left( s_{1/12}^+, p_{1/12} \right), \left( s_{2/12}^+, p_{2/12} \right), \left( s_{3/12}^+, p_{3/12} \right), \left( s_{4/12}^+, p_{4/12} \right) \right\}
\]

\[
= \left\{ \left( s_{1/12}^+, 0.181 \right), \left( s_{2/12}^+, 0.263 \right), \left( s_{3/12}^+, 0.322 \right), \left( s_{4/12}^+, 0.234 \right) \right\}
\]

From **Definition 3.3** and **Example 3.3**, we can see that a PLVT not only takes into account the value of an evaluation but also considers the preference changing and probability. In the definition of PLVT, \( \leftrightarrow r_{\tau,i}^+ \) is the auxiliary or secondary description of \( \leftrightarrow a_{\tau,i}^+ \) is the distribution density interpretation of \( s_{i,\tau}^+ \).

### 3.2. Some operations of PLVTSs

In order to calculate and compare PLVTSs, in this subsection, we define some basic operations of PLVTSs, and then give a method to compare PLVTSs.

**Definition 3.4.** Let \( \left( s_{i,1}^+, p_{i,1} \right) \) and \( \left( s_{i,2}^+, p_{i,2} \right) \) be two PLVTSs whose associated original linguistic term sets are \( S_{n_1} \) and \( S_{n_2} \), respectively, and \( \lambda \) be any real number. We can define the following operations for PLVTSs:

\[
\left( s_{i,1}^+, p_{i,1} \right) + \left( s_{i,2}^+, p_{i,2} \right) = \left( \leftrightarrow a_{i,1}^+ \leftrightarrow a_{i,2}^+ \right) + \left( s_{i,1}^+, \tilde{j}, \tilde{j}, p_{i,1} \right) \quad \text{(1)}
\]

\[
\left( s_{i,1}^+, p_{i,1} \right) - \left( s_{i,2}^+, p_{i,2} \right) = \left( \leftrightarrow a_{i,1}^+ \leftrightarrow a_{i,2}^+ \right) - \left( s_{i,1}^+, \tilde{j}, \tilde{j}, p_{i,1} \right) \quad \text{(2)}
\]

\[
\lambda \cdot \left( s_{i,1}^+, p_{i,1} \right) = \left( \lambda \cdot s_{i,1}^+, \lambda \cdot p_{i,1} \right) \quad \text{(3)}
\]

\[
\left( s_{i,1}^+, p_{i,1} \right) \times \left( s_{i,2}^+, p_{i,2} \right) = \left( \leftrightarrow a_{i,1}^+ \leftrightarrow a_{i,2}^+ \right) \times \left( s_{i,1}^+, \tilde{j}, \tilde{j}, p_{i,1} \right) \quad \text{(4)}
\]

\[
\left( s_{i,1}^+, p_{i,1} \right) \times k \cdot \left( s_{i,2}^+, p_{i,2} \right) = \left( \leftrightarrow a_{i,1}^+ \leftrightarrow a_{i,2}^+ \right) \times \left( s_{i,1}^+, \tilde{j}, \tilde{j}, p_{i,1} \right) \quad \text{(5)}
\]

where \( \tilde{k} \) is the unit coordinate vector of the vertical axis.

**Theorem 3.2** gives some operational laws of the PLVTSs.

**Theorem 3.2.** Let \( \left( s_{i,1}^+, p_{i,1} \right), \left( s_{i,2}^+, p_{i,2} \right) \) and \( \left( s_{i,3}^+, p_{i,3} \right) \) be three PLVTSs whose associated original linguistic term sets are \( S_{n_1}, S_{n_2} \) and \( S_{n_3} \), respectively, \( \lambda, \lambda_1, \lambda_2 \) be any three real numbers. Then,

\[
\left( s_{i,1}^+, p_{i,1} \right) + \left( s_{i,2}^+, p_{i,2} \right) = \left( s_{i,2}^+, p_{i,2} \right) + \left( s_{i,1}^+, p_{i,1} \right) \quad \text{(1)}
\]
It is easy to prove the result of Theorem 3.2 by using Definition 3.4. Thus, we omit the detailed proof of this theorem.

In order to compare PLVTs, we put forward the score function of a PLVT.

**Definition 3.5.** Let \( (\vec{s}_{t_1}, p_{t_1}) \) be a PLVT, then the score function of \( (\vec{s}_{t_1}, p_{t_1}) \) is defined as:

\[
T \left( \left( \vec{s}_{t_1}, p_{t_1} \right) \right) = \frac{p_{t_1} \cdot R_{\vec{s}_1}^{\alpha(t)}}{|R_{\vec{s}_1}^{\alpha(t)} - 1|} \tag{3.8}
\]

For two PLVTs \( \left( \vec{s}_{t_1}, p_{t_1} \right) \) and \( \left( \vec{s}_{t_2}, p_{t_2} \right) \), if \( T \left( \left( \vec{s}_{t_1}, p_{t_1} \right) \right) > T \left( \left( \vec{s}_{t_2}, p_{t_2} \right) \right) \), then \( \left( \vec{s}_{t_1}, p_{t_1} \right) > \left( \vec{s}_{t_2}, p_{t_2} \right) \); if \( T \left( \left( \vec{s}_{t_1}, p_{t_1} \right) \right) < T \left( \left( \vec{s}_{t_2}, p_{t_2} \right) \right) \), then \( \left( \vec{s}_{t_1}, p_{t_1} \right) < \left( \vec{s}_{t_2}, p_{t_2} \right) \); if \( T \left( \left( \vec{s}_{t_1}, p_{t_1} \right) \right) = T \left( \left( \vec{s}_{t_2}, p_{t_2} \right) \right) \), then these two terms are equivalent and can be denoted as \( \left( \vec{s}_{t_1}, p_{t_1} \right) = \left( \vec{s}_{t_2}, p_{t_2} \right) \).

**Example 3.4.** For the PLVTs \( \vec{s} \) in Example 3.3:

\[
\vec{s} = \left\{ \left( \frac{5}{12} \vec{i} + \frac{2}{3} \vec{j}, 0.181 \right), \left( \frac{7}{12} \vec{i} + \frac{4}{15} \vec{j}, 0.263 \right), \left( \frac{5}{8} \vec{i} - \vec{j}, 0.322 \right), \left( \frac{3}{4} \vec{i} - \vec{j}, 0.234 \right) \right\}
\]

We can calculate the scores of the PLVTs in \( \vec{s} \) by Eq. (3.8) as follows:

\[
T \left( \left( \vec{s}_{i}, p_i \right) \right) = 0.226, \quad T \left( \left( \vec{s}_{j}, p_j \right) \right) = 0.209, \quad T \left( \left( \vec{s}_{k}, p_k \right) \right) = 0.101, \quad T \left( \left( \vec{s}_{l}, p_l \right) \right) = 0.088
\]

Thus, we have \( \left( \vec{s}_{i}, p_i \right) > \left( \vec{s}_{j}, p_j \right) > \left( \vec{s}_{k}, p_k \right) > \left( \vec{s}_{l}, p_l \right) \).

To make a better use of PLVTs in MAGDM, we can develop an aggregation operator for PLVTs.

**Definition 3.6.** Let \( \vec{s} = \left\{ \left( \vec{s}_{t_1}, p_{t_1} \right) \mid r = 1, 2, \ldots, L(\vec{s}) \right\} = \left\{ \left( \left( \vec{i} + r \vec{j}, p_r \right) \mid r = 1, 2, \ldots, L(\vec{s}) \right\} \right\} \) be a PLVT, then the probabilistic linguistic vector-term average (PLVA) operator is defined as:

\[
A \left( \vec{s}, p^* \right) = \left( \vec{s}, p^* \right) = \left( \sum_{t=1}^{L(\vec{s})} p_{t} \cdot R_{\vec{s}_t}^{\alpha(t)} \vec{i} + \frac{1}{L(\vec{s})} \sum_{t=1}^{L(\vec{s})} p_{t} \cdot R_{\vec{s}_t}^{\alpha(t)} \vec{j} \right) \tag{3.9}
\]

It is obvious that the result of the PLVA operator is also a PLVT.

**Definition 3.7.** Let \( \vec{s} = \left\{ \left( \vec{s}_{t_1}, p_{t_1} \right) \mid r = 1, 2, \ldots, L(\vec{s}) \right\} = \left\{ \left( \left( \vec{i} + r \vec{j}, p_r \right) \mid r = 1, 2, \ldots, L(\vec{s}) \right\} \right\} \) be a PLVT. Then, the deviation degree of each PLVT \( \left( \vec{s}_{t_1}, p_{t_1} \right) \) can be defined as:

\[
\sigma_{t} = \left| p_{t} - p^* \right| \times \left( 1 - \frac{\sum_{t=1}^{L(\vec{s})} p_{t} \cdot \vec{s}_{t_1} \cdot \vec{s}_{t_1}^{*}}{\left| \vec{s}_{t_1} \right| \times \left| \vec{s}_{t_1}^{*} \right|} \right) \tag{3.10}
\]

**Example 3.5.** Given a PLVT:

\[
\vec{s} = \left\{ \left( \frac{1}{4} \vec{i} + \vec{j}, 0.129 \right), \left( \frac{3}{8} \vec{i}, 0.232 \right), \left( \frac{1}{2} \vec{i} - \frac{2}{3} \vec{j}, 0.139 \right), \left( \frac{7}{12} \vec{i} + \frac{4}{15} \vec{j}, 0.289 \right), \left( \frac{5}{8} \vec{i}, 0.103 \right) \right\}
\]

Then by Eq. (3.9), we can get the average of the PLVTs:

\[
A \left( \vec{s}, p^* \right) = \left( \vec{s}, p^* \right) = \left( \sum_{t=1}^{L(\vec{s})} p_{t} \cdot R_{\vec{s}_t}^{\alpha(t)} \vec{i} + \frac{1}{L(\vec{s})} \sum_{t=1}^{L(\vec{s})} p_{t} \cdot R_{\vec{s}_t}^{\alpha(t)} \vec{j} \right) = \left( \frac{35}{83} \vec{i} + \frac{11}{97} \vec{j}, 0.178 \right)
\]

Through Eq. (3.10), the deviation degrees of the PLVTs are calculated as:

\[
\sigma_1 = 0.264, \quad \sigma_2 = 0.0019, \quad \sigma_3 = 0.3948, \quad \sigma_4 = 0.0002, \quad \sigma_5 = 0.0012
\]
According to Definition 3.5 and Definition 3.7, for any two PLVTs \((\vec{s}_{(t_1)}, p_{(t_1)})\) and \((\vec{s}_{(t_2)}, p_{(t_2)})\), the comparison can be conducted as follows:

- If \(T((\vec{s}_{(t_1)}, p_{(t_1)})) > T((\vec{s}_{(t_2)}, p_{(t_2)}))\), then \((\vec{s}_{(t_1)}, p_{(t_1)}) > (\vec{s}_{(t_2)}, p_{(t_2)})\);  
- If \(T((\vec{s}_{(t_1)}, p_{(t_1)})) < T((\vec{s}_{(t_2)}, p_{(t_2)}))\), then \((\vec{s}_{(t_1)}, p_{(t_1)}) < (\vec{s}_{(t_2)}, p_{(t_2)})\);
- If \(T((\vec{s}_{(t_1)}, p_{(t_1)})) = T((\vec{s}_{(t_2)}, p_{(t_2)}))\), then \((\vec{s}_{(t_1)}, p_{(t_1)}) = (\vec{s}_{(t_2)}, p_{(t_2)})\).

Sometimes, we need to collect the opinion subsets into a whole group opinion set. For instance, in order to obtain a sufficient number of opinions from different sources, taking into account efficiency, we do this work via dividing the whole information source into multiple sub-sources, i.e., there are some PLVTs as the opinion interpretations of each subgroup, then we collect all these opinion subsets into a PLVTs as the total opinion expression. The following definition of PLVTs is significant:

**Definition 3.8.** Let \(\vec{S}_1 = \left\{ (\vec{s}_{(t_1)}, p_{(t_1)}) \right\} \) \(t_1 = 1, 2, \ldots, L(\vec{S}_1)\) \(\right\} = \left\{ (\vec{s}_{(t_2)}, p_{(t_2)}) \right\} \) \(t_2 = 1, 2, \ldots, L(\vec{S}_2)\) be two PLVTs, and \(\lambda\) be any real number. Then \(\forall (\vec{s}_{(t_1)}, p) \in \vec{S}_1, (\vec{s}_{(t_2)}, p) \in \vec{S}_2,\)

1. The union of \(\vec{S}_1\) and \(\vec{S}_2\) is defined as:

\[
\vec{S}_1 \cup \vec{S}_2 = \begin{cases} 
\left\{ (\vec{s}_{(t_1)}, \frac{p_{(t_1)}}{L(\vec{S}_1) + L(\vec{S}_2)}), (\vec{s}_{(t_2)}, \frac{p_{(t_2)}}{L(\vec{S}_1) + L(\vec{S}_2)}) \right\}, & \text{if } \vec{s}_{(t_1)} \neq \vec{s}_{(t_2)} \\
\emptyset & \text{if } \vec{s}_{(t_1)} = \vec{s}_{(t_2)} 
\end{cases}
\]

2. The intersection of \(\vec{S}_1\) and \(\vec{S}_2\) is defined as:

\[
\vec{S}_1 \cap \vec{S}_2 = \begin{cases} 
\emptyset & \text{if } \vec{s}_{(t_1)} \neq \vec{s}_{(t_2)} \\
\left\{ (\vec{s}_{(t_2)}, \frac{\min\left(\frac{p_{(t_1)}}{L(\vec{S}_1) + L(\vec{S}_2)}, \frac{p_{(t_2)}}{L(\vec{S}_1) + L(\vec{S}_2)}\right)}{\sum_{t=1}^{t_1} p_{(t)}}) \right\}, & \text{if } \vec{s}_{(t_1)} = \vec{s}_{(t_2)} 
\end{cases}
\]

**Note.** As can be seen by this definition, the probability item \(p_{(t)}\) is a supplementary description for the linguistic vector-item \(s_{(t)}\). For the internal structure of \(s_{(t)}\), the change rate \(\rightarrow r_{\rightarrow \omega(t)}\) is the supplementary description for the LET \(\omega_{\rightarrow \omega(t)}\). Based on this understanding, we define \(\vec{S}_1 \cap \vec{S}_2 = \emptyset\) if \(\vec{s}_{(t_1)} \neq \vec{s}_{(t_2)}\), \(\forall (\vec{s}_{(t_1)}, p) \in \vec{S}_1\) and \(\vec{s}_{(t_2)}, p \in \vec{S}_2\).

**Example 3.6.** Given two normalized PLVTs:

\[
\vec{S}_1 = \left\{ \left(\frac{1}{4}\vec{i} + \vec{j}, 0.25\right), \left(\frac{3}{8}\vec{i}, 0.45\right), \left(\frac{1}{2}\vec{i} - \frac{2}{3}\vec{j}, 0.27\right) \right\}
\]
\[ \hat{s}_2 = \left\{ \left( \frac{3}{8} i, 0.11 \right), \left( \frac{7}{12} i + \frac{4}{15} j, 0.56 \right), \left( \frac{5}{8} i, 0.2 \right), \left( \frac{1}{2} i - \frac{2}{3} j, 0.1 \right) \right\} \]

Then,
\[ \hat{s}_1 \cup \hat{s}_2 = \left\{ \left( \frac{1}{4} i + j, 0.129 \right), \left( \frac{3}{8} i, 0.232 \right), \left( \frac{1}{2} i - \frac{2}{3} j, 0.139 \right), \left( \frac{7}{12} i + \frac{4}{15} j, 0.289 \right), \left( \frac{5}{8} i, 0.103 \right) \right\} \]
\[ \hat{s}_1 \cap \hat{s}_2 = \left\{ \left( \frac{3}{8} i, 0.057 \right), \left( \frac{1}{2} i - \frac{2}{3} j, 0.052 \right) \right\} \]

In order to compare PLVTSs, based on the definition of deviation degree of PLVT, in the following, we define the deviation degree and the variance of a PLVTS to describe which PLVTS is more superior from the concentrative point of view.

**Definition 3.9.** Given a PLVTS \( \hat{s} = \left\{ \left( s_{(\tau)}, p_{(\tau)} \right) \right\} | \tau = 1, 2, \ldots, \mu(\hat{s}) \} \), the deviation degree of \( \hat{s} \) can be defined as:

\[
\sigma(\hat{s}) = \sum_{\tau=1}^{\mu(\hat{s})} \sigma_\tau = \sum_{\tau=1}^{\mu(\hat{s})} \left( |p_{(\tau)} - p^*| \times \left( 1 - \frac{\hat{s}_{(\tau)} \cdot \hat{s}^*}{|\hat{s}_{(\tau)}| \times |\hat{s}^*|} \right) \right)
\]

(3.13)

The deviation degree is a description on the consistency of a PLVTS. For any two PLVTSs \( \hat{s}_1 \) and \( \hat{s}_2 \), if \( \sigma(\hat{s}_1) < \sigma(\hat{s}_2) \), then the concentration of \( \hat{s}_1 \) is higher than \( \hat{s}_2 \); otherwise, if \( \sigma(\hat{s}_1) > \sigma(\hat{s}_2) \), then the concentration of \( \hat{s}_1 \) is lower than \( \hat{s}_2 \); if \( \sigma(\hat{s}_1) = \sigma(\hat{s}_2) \), then these two terms’ concentrations are equivalent.

**Example 3.7.** For the PLVTS in Example 3.5:
\[ \hat{s} = \left\{ \left( \frac{1}{4} i + j, 0.129 \right), \left( \frac{3}{8} i, 0.232 \right), \left( \frac{1}{2} i - \frac{2}{3} j, 0.139 \right), \left( \frac{7}{12} i + \frac{4}{15} j, 0.289 \right), \left( \frac{5}{8} i, 0.103 \right) \right\} \]

we can get the deviation degree of this PLVTS by Eq. (3.13):
\[ \sigma(\hat{s}) = \sum_{\tau=1}^{\mu(\hat{s})} \sigma_\tau = 0.264 + 0.0019 + 0.3948 + 0.0002 + 0.0012 = 0.6621 \]

It is noted that \( (\hat{s}_{(\tau)}, p_{(\tau)}) \) consists of two parts, where \( s_{(\tau)} \) is originated from the evaluation process and \( p_{(\tau)} \) is from the statistical analysis of the realistic mass data. Additionally, the probability \( p_{(\tau)} \) is used to describe the distribution density of \( s_{(\tau)} \).

**Definition 3.10.** Let \( (\hat{s}, p) \) be a PLVT, we can extract the vector item and probability item respectively by the following two functions:

\[
V((\hat{s}, p)) = \hat{s}
\]

(3.14)

\[
P((\hat{s}, p)) = p
\]

(3.15)

**4. Approach to MAGDM with PLVTSs**

In this section, we first describe the MAGDM problem with probabilistic linguistic information, then present an algorithm for conducting the ranking-oriented recommender system.

**4.1. Problem description**

The MAGDM problem with probabilistic linguistic information can be described as follows: Let \( \{e_1, e_2, \ldots, e_K\} \) be a finite set of \( K \) evaluator groups, whose weight vector is \( w = (w_1, w_2, \ldots, w_K)^T \), where \( w_k \geq 0 \) (\( k = 1, 2, \ldots, K \)) and \( \sum_{k=1}^{K} w_k \leq 1 \). Let \( \{A_1, A_2, \ldots, A_l\} \) be a set of \( l \) attributes for the candidate alternatives, whose weight vector is \( u = (u_1, u_2, \ldots, u_l)^T \) with \( u_l \geq 0 \) (\( i = 1, 2, \ldots, l \)) and \( \sum_{i=1}^{l} u_i \leq 1 \). Our purpose is to obtain the final group decision making result based on the linguistic evaluation information for the set of alternatives \( \{X_1, X_2, \ldots, X_M\} \).

It should be noted that the weights of attributes may be completely unknown or partly known due to the complexity and uncertainty of the MAGDM problem itself. The similar situation occurs to the weights of evaluator groups \( \{e_1, e_2, \ldots, e_K\} \). This is an interesting and complex question which deserves in-depth research. Without getting into too much more detail owing to the constraints of space in this paper, we focus on the new mathematic expression of linguistic evaluation information and its application in multi-granular linguistic group decision making. So, the weight vector of attributes used in the following illustration originates from the statistical analysis of evaluation information in a certain medical data collection source. For the weight vector of evaluation groups set, i.e. \( w = (w_1, w_2, \ldots, w_K)^T \), we arrange...
Table 1
The probabilistic linguistic vector-term matrix.

<table>
<thead>
<tr>
<th></th>
<th>A_1</th>
<th>A_2</th>
<th>...</th>
<th>A_l</th>
</tr>
</thead>
<tbody>
<tr>
<td>e_1</td>
<td>(\bar{S}_{11})</td>
<td>(\bar{S}_{12})</td>
<td>...</td>
<td>(\bar{S}_{1l})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>e_k</td>
<td>(\bar{S}_{k1})</td>
<td>(\bar{S}_{k2})</td>
<td>...</td>
<td>(\bar{S}_{kl})</td>
</tr>
</tbody>
</table>

Table 2
New form of the probabilistic linguistic vector-term matrix.

<table>
<thead>
<tr>
<th></th>
<th>A_1</th>
<th>A_2</th>
<th>...</th>
<th>A_l</th>
</tr>
</thead>
<tbody>
<tr>
<td>e_1</td>
<td>(V(\bar{S}_{11}))</td>
<td>(V(\bar{S}_{12}))</td>
<td>...</td>
<td>(V(\bar{S}_{1l}))</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>e_k</td>
<td>(V(\bar{S}_{k1}))</td>
<td>(V(\bar{S}_{k2}))</td>
<td>...</td>
<td>(V(\bar{S}_{kl}))</td>
</tr>
</tbody>
</table>

Note: In this matrix, each item contains two parts, i.e., the LET and the corresponding probability. Subset \(p(\bar{S}_i)\) is a probability density interpretation for the LET subset \(V(\bar{S}_i)\).

it as a special case to interpret the calculation method, which does not present the true meaning in reality. The specific value of it should be obtained by statistical analysis or other methods of data resources according to the actual group divisions.

Evaluators with different knowledge backgrounds provide their assessments on attributes by their own linguistic information when they evaluate the attributes together. It is observed that each evaluation group involving different members, and different members may have different viewpoints and perceptions on the objects with respect to different attributes. Therefore, it is appropriate to collect and transform the given linguistic information into PLVTs. All the members' linguistic information in the group \(e_k\) on the attribute \(A_l\) forms a PLVT \(\bar{S}_{ki}\):

\[
\bar{S}_{ki} = \left\{ \left( \bar{S}_{(t_i)}, p_{(t_i)} \right) \mid k = 1, 2, \ldots, K; \quad i = 1, 2, \ldots, I \right\} .
\] (4.1)

After all the linguistic evaluation information of the groups are collected, a probabilistic linguistic vector-term matrix \(\bar{S} = (\bar{S}_{ki})K \times I\) can be constructed to describe the linguistic evaluation information of the groups over different attributes, shown in Table 1.

Without loss of generality, as we assume that all the elements in each \(\bar{S}_{ki}\) are sorted in ascending order of \(\rightarrow a_{ki}\). If the first parts of different PLVTs are the same, then we first integrate them into a PLVT by Eq. (3.9), and sort the PLVTs in ascending order with respect to \(\rightarrow a_{ki}\).

Considering the different meanings and sources of \(d_{(t_i)}\) and \(p_{(t_i)}\) in PLVT, we rewrite the matrix \(\bar{S} = (\bar{S}_{ki})K \times I\) as a new form (See Table 2) according to Eqs. (3.14) and (3.15).

4.2. Algorithm in selection-recommender system with PLVTs

In this subsection, we propose an algorithm for the MAGDM problem in which the weights of evaluators and attributes are all known. The algorithm for MAGDM with PLVTs is given as follows.

Algorithm 4.1. Step 1. Analyze the MAGDM problem and determine the attributes \(\{A_1, A_2, \ldots, A_l\}\) for the candidate alternatives \(\{X_1, X_2, \ldots, X_M\}\) and the evaluation groups \(\{e_1, e_2, \ldots, e_K\}\). Decision makers freely give the weight vectors of \(\{A_1, A_2, \ldots, A_l\}\) and \(\{e_1, e_2, \ldots, e_K\}\) respectively according to their willingness. All the individuals in different evaluation groups express their assessments by additional ES which are suitable and feasible for themselves. Then we can normalize different linguistic scales by Eq. (3.5). The corresponding change rate of each linguistic term is calculated by Eq. (3.6). Therefore, all the linguistic evaluation information of these evaluation groups on attributes is obtained, which forms a matrix \(\bar{S} = (\bar{S}_{ki})K \times I\) presented in Table 2.

Step 2. Through the curve fitting method, we can get a fitting description matrix \(D\) shown in Table 3.

Considering the accuracy of the curve fitting method, we define the degree of fitting precision as the membership degree of the function compared to the reality, denoted as \(\mu(\cdot)\). Thus, we can get a fuzzy probabilistic linguistic evaluation matrix \(E\), shown in Table 4.

Then we calculate the fuzzy linguistic evaluation information of each item in the matrix \(E\) by the following operator:

\[
EI(E) = \left( \int_{\Gamma_{ki}(\alpha)} g_{ki} (\alpha) \, d\alpha, \mu(\Gamma_{ki}(\alpha)) \oplus \mu(g_{ki}(\alpha)) \right)
\] (4.2)

Consider that the change rate of the right endpoint is \(+\infty\), it may lead to some difficult in the subsequent integral operation. Thus, we can assign a large enough number to replace \(+\infty\).

Step 3. For the personality customization, the effects of different positions in the matrix \(E\) are not the same. For example, for a severe patient, when he/she chooses a hospital for medical treatment, the most important evaluation item is the experts' evaluation because they have professional backgrounds on technical attributes. These experts, however, may give little applicable evaluation on price level, transportation and other attributes. As another example, in order to choose a car, an automotive consumer may collect the evaluations from experts for security attribute, the evaluations from the customers for the practical attribute, and the evaluations from good friends for the...
The fitting description matrix.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$...$</th>
<th>$e_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{11}(\alpha), \mu(\Gamma_{11}(\alpha))$</td>
<td>$\Gamma_{12}(\alpha), \mu(\Gamma_{12}(\alpha))$</td>
<td>$...$</td>
<td>$\Gamma_{k1}(\alpha), \mu(\Gamma_{k1}(\alpha))$</td>
</tr>
<tr>
<td>$g_{11}(\alpha), \mu(g_{11}(\alpha))$</td>
<td>$g_{12}(\alpha), \mu(g_{12}(\alpha))$</td>
<td>$...$</td>
<td>$g_{k1}(\alpha), \mu(g_{k1}(\alpha))$</td>
</tr>
</tbody>
</table>

The fuzzy probabilistic linguistic evaluation matrix.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$...$</th>
<th>$e_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{11}(\alpha), \mu(\Gamma_{11}(\alpha))$</td>
<td>$\Gamma_{12}(\alpha), \mu(\Gamma_{12}(\alpha))$</td>
<td>$...$</td>
<td>$\Gamma_{k1}(\alpha), \mu(\Gamma_{k1}(\alpha))$</td>
</tr>
<tr>
<td>$g_{11}(\alpha), \mu(g_{11}(\alpha))$</td>
<td>$g_{12}(\alpha), \mu(g_{12}(\alpha))$</td>
<td>$...$</td>
<td>$g_{k1}(\alpha), \mu(g_{k1}(\alpha))$</td>
</tr>
</tbody>
</table>

The value degree matrix.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$...$</th>
<th>$e_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{11}$</td>
<td>$f_{12}$</td>
<td>$...$</td>
<td>$f_{1k}$</td>
</tr>
<tr>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
</tr>
<tr>
<td>$f_{k1}$</td>
<td>$f_{k2}$</td>
<td>$...$</td>
<td>$f_{kk}$</td>
</tr>
</tbody>
</table>

The weight of an attribute. This step can help users get the comprehensive evaluation on objects. Therefore, the users should assign the selection weight vectors of $\{A_1, A_2, ..., A_k\}$ and $\{e_1, e_2, ..., e_k\}$, which are denoted as $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)^T$ and $\theta = (\theta_1, \theta_2, ..., \theta_k)^T$, respectively (The weight determining methods could be AHP or others, which we do not discuss in this paper). Considering the selection and recommendation two factors, we define a value function to represent the effects of different positions as follows:

$$EP(e_k) = \left(\delta \cdot \frac{1}{1 - \theta_k - \lambda_k}\right)^{\frac{w_i \cdot \mu(e_k)}{1 - \mu(e_k)}}$$

where the parameter $0 < \delta \leq 1$ is a regulated value to make $\delta \cdot \frac{1}{1 - \theta_k - \lambda_k}$ more than 1 but not too large for the $EP(e_k)$ falling in the appropriate range to be researched; then $0 < \psi$ is a magnifying parameter to make $\frac{w_i \cdot \mu(e_k)}{1 - \mu(e_k)}$ large enough to meet the condition that the change rate of $EP(e_k)$ is more than 1, meanwhile it should not be too large for the gaps between different values of $EP(e_k)$ falling in an appropriate range to be researched.

**Remark 1.** For $0 \leq \lambda_k \leq 1$, $0 \leq \theta_k \leq 1$, we have $1 \leq \frac{1}{1 - \theta_k - \lambda_k} \leq +\infty$. Given the index $\frac{w_i \cdot \mu(e_k)}{1 - \mu(e_k)}$, being certain, the bigger the $\frac{1}{1 - \theta_k - \lambda_k}$ is, the faster the change speed of $EP(e_k)$ will be. In addition, the exponential function of base number in this range, i.e., $[1, +\infty)$, is increasing with the increase of $\frac{w_i \cdot \mu(e_k)}{1 - \mu(e_k)}$. It means that $EP(e_k)$ increases as the $\frac{w_i \cdot \mu(e_k)}{1 - \mu(e_k)}$ increases.

**Remark 2.** $\psi$ is used to meet a mathematic axiom that the change rate of the exponential function $\alpha^x (a > 1)$ is much greater than the change rate of a power function $x^a$ when the index $x$ of $\alpha^x (a > 1)$ is large enough.

Using Eq. (4.3) to calculate the elements in the matrix $E$, we can get a value degree matrix $F$, shown in Table 5.

Step 4. Choose the maximum item $f_{k_1i_1}$ of the matrix $F$, and delete the $i_1$th column of the matrix $F$ to get a new matrix $F_1$. Then we choose the maximum item $f_{k_2i_2}$ of $F_1$, and delete the $i_2$ column of the matrix $F_1$ to get a new matrix $F_2$ Keep doing like this again and again till $F_{i-1}$. Finally, we get an order sequence $(k_1i_1, k_2i_2, ..., k_ii_i)$.

Step 5. Aggregate all of the evaluation information of $(k_1i_1, k_2i_2, ..., k_ii_i)$ to get the final evaluation of the MAGDM problem by using the integration operator:

$$EII(E) = \left(\sum_{t=1}^{l} \theta_{k_t} \cdot w_{k_t} \cdot \lambda_{i_t} \cdot U_{i_t} \int_{\Gamma_{k_ti_t}(\alpha)} g_{k_ti_t}(\alpha) d\alpha, \left(\int_{1}^{l} \mu(\Gamma_{k_ti_t}(\alpha)) \otimes \mu(g_{k_ti_t}(\alpha)) \right)^{\theta_{k_t} \cdot \lambda_{i_t} \cdot w_{k_t} \cdot U_{i_t}}\right)$$

Step 6. Select the objects by Eq. (4.4).

Step 7. End.
Note. The algorithm interpreted above is based on the assumption that the number of data in $\tilde{S}_k$ is very large. When the number of terms is not very large, the number of data in $\tilde{S}_k$ is also not very large. In such a case, we can calculate them through a reduced discrete case of Algorithm 4.1. The specific formula is as follows:

$$
\left( \tilde{S}_{\alpha}, p_{\alpha} \right)_{\tilde{k}} = \left( \left( \tilde{S}_{\alpha(\tau)}, p_{\alpha(\tau)} \right)_{\tilde{k}} \right), \forall \left( \tilde{S}_{\alpha(\tau)}, p_{\alpha(\tau)} \right)_{\tilde{k}} \in \tilde{S}_k \text{ and } \alpha_1 = \alpha_2 = \cdots = \alpha_{\max(\tau)}
$$

(4.5)

where $\left( \tilde{S}_{\alpha}, p_{\alpha} \right)_{\tilde{k}} \in \tilde{S}_k$. We can integrate all the PLVTs $\left( \tilde{S}_{\alpha}, p_{\alpha} \right)_{\tilde{k}}$ in $\tilde{S}_k$ to a PLVT $\tilde{s}_k$ by Eq. (3.8). Then the following steps are the same as those in Algorithm 4.1. The method that deals with $\tilde{s}_k$ in Algorithm 4.1 can be seen as the generalization of Eq. (4.5).

Next, we demonstrate the utility and the effect of PLVTs, through using Algorithm 4.1 to evaluate and select the most suitable hospital for a respiratory disease patient.

5. Application in personal hospital selection-recommender system

In recent decades, climatic deterioration and the multiple phenomenon of respiratory diseases make people pay attention to the relation between them. The associations between socioeconomic, environment and health provide a research topic that has a bearing on a long-term development. High morbidity and wide prevalence in respiratory diseases lead to more people going to the hospital for medical treatment. Therefore, the management, configuration and application of medical resources are more important than before. Data collection of recent years has established a database resources to promote hospital service quality and utilization of medical recourses. Based on this, structure personalized hospital recommendation system through the multi-attribute decision making to help users compare different resources such as different hospitals to meet their own needs is just one significant application aspect of the medical data background [22,28]. For example, the mild respiratory disease patients may emphasize on the geographical attribute among several properties of hospitals when he/she chooses the hospitals. But the severe patients such as the patients with lung cancer would pay more attention to the technical attribute, not the geographical one. So, we should consider the weight parameters. In the following, we use the example concerning the hospital recommendation to illustrate the proposed approach.

Step 1. Assume that there is a respiratory disease patient who wants to select a hospital between two alternatives ($H_1$: the first hospital; $H_2$: the second hospital) for medical treatment. In order to get an objective evaluation on candidate hospitals, the patient consults the opinions of the following three groups:

- $e_1$: The medical experts group (the number of samples in this group is $K_1$);
- $e_2$: The patients group with experience in clinic (the number of samples in this group is $K_2$);
- $e_3$: The third party (an organization whose members do not belong to the first two groups. The number of samples in this group is $K_3$).

The recommendation weight vector of $\{e_1, e_2, \ldots, e_k\}$ is $w = (0.45, 0.35, 0.2)^T$, which is obtained by the method introduced in Subsection 4.1. Furthermore, we assume that this patient assigns the weight vector of the evaluator groups $\{e_1, e_2, \ldots, e_k\}$ as $\theta = (0.4, 0.4, 0.2)^T$.

Suppose the three groups evaluate the hospitals in terms of the following attributes: $A_1$ : Medical equipment condition; $A_2$ : Health care service quality; $A_3$ : Price level; $A_4$ : Convenience transportation.

Similar to $w$, the recommender weight vector of these attributes is $\mathbf{w} = (0.33, 0.27, 0.25, 0.15)^T$, which is obtained through the approach introduced in Subsection 4.1. Then, we assume that this patient arranges the selection weight vector of $\{A_1, A_2, \ldots, A_4\}$ as $\lambda = (0.3, 0.3, 0.3, 0.1)^T$. Then, the probabilistic linguistic vector-term matrices provided by the groups on two hospitals over different attributes are shown in Tables 6 and 7, respectively.

Note. (1) Because of limited space, Tables 6 and 7 only show parts of data to illustrate the method. (2) The probabilities in Tables 6 and 7 are not necessary to be normalized.

Step 2. Through the curve fitting method, we can get a fuzzy probabilistic linguistic evaluation matrix $E$ (see Table 8). Notice that there are many methods for curve fitting. Here we select the well-known polynomial approximation as it is convenient for calculation.

Let the change rate of the right endpoint be equal to 100. We can calculate the linguistic evaluation information of each item in the matrix $E$ by Eq. (4.2). The results are set out in Table 9.

Step 3. Using Eq. (4.3) and let $\delta = \psi = 1$, then we can get the value degree matrix $F$ (see Table 10).

Step 4. Choose the maximum item of the matrix $F$. Obviously, it is $f_{11}$. We delete the first column of matrix $F$ to get a new matrix:

$$
F_1 = \begin{bmatrix}
2.8 \times 10^{67} & 1.34 \\
2.88 \times 10^{52} & 1.12 \\
1.60 & 2.62
\end{bmatrix}
$$

Then we choose the maximum item $f_{24}$ of $F_1$, and delete the forth column of the matrix $F_1$ to get:

$$
F_2 = \begin{bmatrix}
2.8 \times 10^{67} & 1.34 \\
2.88 \times 10^{52} & 1.12 \\
1.60 & 2.62
\end{bmatrix}
$$
Table 6
The probabilistic linguistic vector-term matrix of the first hospital.

<table>
<thead>
<tr>
<th></th>
<th>A1 (0.33)</th>
<th>A2 (0.27)</th>
<th>A3 (0.25)</th>
<th>A4 (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1(0.45)) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
</tr>
</tbody>
</table>

Table 7
The probabilistic linguistic vector-term matrix of the second hospital.

<table>
<thead>
<tr>
<th></th>
<th>A1 (0.33)</th>
<th>A2 (0.27)</th>
<th>A3 (0.25)</th>
<th>A4 (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1(0.45)) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.28: 0.44, 0.38)</td>
</tr>
</tbody>
</table>

Table 8
The fuzzy probabilistic linguistic evaluation matrix.

<table>
<thead>
<tr>
<th></th>
<th>A1 (0.33)</th>
<th>A2 (0.27)</th>
<th>A3 (0.25)</th>
<th>A4 (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1(0.45)) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
<td>(\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2}) (\frac{1}{2}: \frac{5}{8}: \frac{1}{2})</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
</tr>
<tr>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
<td>(\frac{1}{2} \cdot 0.18: 0.34, 0.38)</td>
</tr>
</tbody>
</table>

\[ \text{e}_1(0.2) \]
\[ \text{e}_2(0.35) \]
\[ \text{e}_3(0.22) \]
Table 9
The result of the matrix $E$ calculated by Eq. (4.2).

<table>
<thead>
<tr>
<th>$e_1$ (0.45)</th>
<th>$e_2$ (0.35)</th>
<th>$e_3$ (0.2)</th>
<th>$e_4$ (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5113.02, 1)</td>
<td>(3454.19, 0.9735)</td>
<td>(5041.46, 0.9999)</td>
<td></td>
</tr>
<tr>
<td>(5076.11, 0.9999)</td>
<td>(5308.30, 0.9999)</td>
<td>(5282.01, 0.9999)</td>
<td>(5584.67, 0.9533)</td>
</tr>
<tr>
<td>(4752.12, 0.9999)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10
The value degree matrix.

<table>
<thead>
<tr>
<th>$e_1$ (0.45)</th>
<th>$e_2$ (0.35)</th>
<th>$e_3$ (0.2)</th>
<th>$e_4$ (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+\infty$</td>
<td>$1.72$</td>
<td>$5.42 \times 10^{17}$</td>
<td>$2.8 \times 10^{37}$</td>
</tr>
<tr>
<td>$1.12$</td>
<td>$1.60$</td>
<td>$2.62$</td>
<td>$9.24 \times 10^{11}$</td>
</tr>
</tbody>
</table>

Table 11
The value degree matrix.

<table>
<thead>
<tr>
<th>$e_1$ (0.45)</th>
<th>$e_2$ (0.35)</th>
<th>$e_3$ (0.2)</th>
<th>$e_4$ (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+\infty$</td>
<td>$1214.88$</td>
<td>$659.93$</td>
<td>$2.3$</td>
</tr>
<tr>
<td>$944.91$</td>
<td>$7.55$</td>
<td>$674.93$</td>
<td>$0.89$</td>
</tr>
<tr>
<td>$15.58$</td>
<td>$99.97$</td>
<td>$+\infty$</td>
<td>$+\infty$</td>
</tr>
</tbody>
</table>

Table 12
The derivative of the fuzzy probabilistic linguistic evaluation matrix $E$.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^\varepsilon_{11}$ ($a_{11}$)</td>
<td>$\Gamma^\varepsilon_{12}$ ($a_{12}$)</td>
<td>$\Gamma^\varepsilon_{13}$ ($a_{13}$)</td>
<td>$\Gamma^\varepsilon_{14}$ ($a_{14}$)</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\Gamma^\varepsilon_{k1}$ ($a_{k1}$)</td>
<td>$\Gamma^\varepsilon_{k2}$ ($a_{k2}$)</td>
<td>$\Gamma^\varepsilon_{k3}$ ($a_{k3}$)</td>
<td>$\Gamma^\varepsilon_{k4}$ ($a_{k4}$)</td>
</tr>
</tbody>
</table>

Keeping doing this, we can get the third maximum item as $f_{12}$ with

$$F_3 = \begin{pmatrix} / & 1.34 & / \\ / & 1.12 & / \\ / & 2.62 & / \end{pmatrix}$$

and the fourth maximum item is $f_{33}$. Therefore, we get a selection order sequence $\{11, 24, 12, 33\}$.

Step 5. Integrate all the evaluation information according to the sequence $\{11, 24, 12, 33\}$ to get the final evaluation of the first hospital by Eq. (4.4). Thus, we have $ECC(E) = (243.431, 0.99923)$.

Step 6. In analogous, we can get $ECC(E_2) = (149.27, 1)$ as the final evaluation information of the second hospital. We illustrate the material calculation in the appendixes.

Step 7. Since the final ranking of the evaluations about $H_1$ and $H_2$ is $243.431 > 149.27$, we should recommend the first hospital.

Thus, we have selected the suitable hospital for the respiratory disease patient. In the following, we analyze the result from the patient and the product-provider, and evaluate the ranking performance by comparing the result considering the changing rate with the ones neglecting the changing rate:

5.1. The effectiveness of the selecting weight vector parameters

Eq. (4.3) considers the users' personal view. If we do not take this aspect into account, the value degree matrix will be obtained, shown as Table 11.

Comparing Tables 10 and 11, we can find that the use of selecting weight vectors, with the fixed regulated parameter $\delta$ and the magnifying parameter $\varphi$ defined in Eq. (4.3), enlarges the gaps between the items in value degree matrix. This likes a magnifying glass to make people easy to make a choice in hesitancy of small distinction between alternatives. It verifies Remark 2 of Eq. (4.3).

5.2. Significance from the point of product-provider

Algorithm 4.1. aims to provide support for consumer to make selection among different alternatives. Additionally, the fuzzy probabilistic linguistic evaluation matrix $E$ given as Table 3 also has some practical guiding significance for the product-provider (such as hospital) supposing that the membership degree of each item in the matrix $E$ has attained an acceptable level. We calculate the maximum point of each function $g_{k1}(\alpha)$ and denote it as $\alpha_{k1}^*$, and $\alpha_{k2}^*$, $\ldots$, $\alpha_{k4}^*$ denote as $\Gamma^\varepsilon_{k1}(\alpha_{k1}^*)$, $\Gamma^\varepsilon_{k2}(\alpha_{k2}^*)$, $\ldots$, $\Gamma^\varepsilon_{k4}(\alpha_{k4}^*)$. Thus, a derivative matrix $F = (\psi_{ki})_{KL} = (\Gamma^\varepsilon_{ki}(\alpha_{ki}^*))_{KL}$ is constructed (see Table 12).
Table 13
The derivative matrix.

<table>
<thead>
<tr>
<th></th>
<th>A₁ (0.33)</th>
<th>A₂ (0.27)</th>
<th>A₃ (0.25)</th>
<th>A₄ (0.15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁ (0.45)</td>
<td>43.01</td>
<td>46.13</td>
<td>179.22</td>
<td>169.64</td>
</tr>
<tr>
<td>e₂ (0.35)</td>
<td>62.07</td>
<td>59.8</td>
<td>493.3</td>
<td>4.88</td>
</tr>
<tr>
<td>e₃ (0.2)</td>
<td>8.01</td>
<td>56.9</td>
<td>99.41</td>
<td>8.05</td>
</tr>
</tbody>
</table>

Table 14
The hesitant fuzzy matrix.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>A₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
</tr>
<tr>
<td>e₂</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
</tr>
<tr>
<td>e₃</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
<td>{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} }</td>
</tr>
</tbody>
</table>

Table 15
The probabilistic linguistic matrix.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>A₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>{ \frac{3}{8} (0.18), \frac{1}{2} (0.34), \frac{3}{4} (0.38) }</td>
<td>{ \frac{3}{8} (0.13), \frac{5}{12} (0.32), \frac{3}{8} (0.18) }</td>
<td>{ \frac{3}{8} (0.05), \frac{1}{2} (0.22), \frac{3}{8} (0.15) }</td>
<td>{ \frac{3}{8} (0.16), \frac{5}{12} (0.08), \frac{3}{8} (0.27) }</td>
</tr>
<tr>
<td>e₂</td>
<td>{ \frac{5}{12} (0.11), \frac{3}{4} (0.35), \frac{1}{2} (0.25) }</td>
<td>{ \frac{5}{12} (0.21), \frac{1}{2} (0.07), \frac{5}{12} (0.32) }</td>
<td>{ \frac{5}{12} (0.24), \frac{5}{12} (0.32), \frac{5}{12} (0.25) }</td>
<td>{ \frac{5}{12} (0.16), \frac{5}{12} (0.15), \frac{5}{12} (0.23) }</td>
</tr>
<tr>
<td>e₃</td>
<td>{ \frac{5}{12} (0.2), \frac{1}{2} (0.16), \frac{3}{4} (0.4) }</td>
<td>{ \frac{5}{12} (0.2), \frac{13}{24} (0.35), \frac{1}{2} (0.27) }</td>
<td>{ \frac{5}{12} (0.31), \frac{1}{2} (0.12), \frac{5}{12} (0.27) }</td>
<td>{ \frac{5}{12} (0.22), \frac{1}{2} (0.15), \frac{7}{12} (0.36) }</td>
</tr>
</tbody>
</table>

Note. There are several points which are selected as the maximum points of \( g_{k_{i}}(\alpha) \), then we can integrate the derivatives of these points as \( \sum_{\tau = 1}^{\max(\tau)} I_{\alpha}(\alpha_{k_{i}}) \), where \( \tau \) is the number of the maximum points of \( g_{k_{i}}(\alpha) \). In addition, if the values in Table 11 are very big or very small, then we can balance them by a positive parameter such as \( n \times I_{\alpha}(\alpha_{k_{i}}) \), where \( n \) is a reasonable large positive real number.

After establishing the derivative matrix \( F = (\psi_{k_{i}})_{K_{i}} \), it is easy to rank all the elements in \( F \) and get a descending sequence \((k_{1}, k_{2}, \ldots, k_{e}, k_{d}, \ldots, k_{e_{i}})\) with \( \psi_{k_{1}, i} > \psi_{k_{2}, i} > \cdots > \psi_{k_{d}, i} \). Note that \( \psi_{k_{i}} = I_{\alpha}(\alpha_{k_{i}}) \) denotes the maximum change rate of the fitting curve of linguistic terms at the point \( \alpha_{k_{i}} \). Thus, the sequence implies the sensitivities of the evaluators’ linguistic assessments on these attributes. For example, if \( \psi_{k_{3}} \) is the top of this sequence, then it means that the change rate of the linguistic terms given by the second evaluation group \( e_{2} \) on the attribute \( A_{3} \) is the maximum one. That is to say, the second group is very sensitive toward the third attribute \( A_{3} \). Thus, the product-provider should improve the quality of his/her product on the third attribute towards the group \( e_{2} \). This shows great significance in individual product customization.

Furthermore, if we calculate the sum of orders in the sequence in each column and arrange the results in descending order, then we can choose the attribute which needs to be adjusted with the smallest value.

Example. We calculate the derivative matrix of Table 8 as shown in Table 13.

It is obvious that \( \psi_{23} > \psi_{13} > \psi_{14} > \psi_{33} > \psi_{21} > \psi_{22} > \psi_{32} > \psi_{12} > \psi_{11} > \psi_{34} > \psi_{31} > \psi_{24} \). The sum of orders in each column are 25, 21, 7 and 25, respectively. Then the order of adjustment for attributes should be \( A_{5}, A_{3}, A_{1}, A_{4} \), which means that the manager of the hospital should start to adjust the attributes from the third one according their actual situations if he/she wants to improve the reputation of the hospital.

5.3. Comparative analysis

5.3.1. Comparing to the hesitant linguistic evaluation information without changing rate and probability

If we deal with the illustration by the traditional hesitant fuzzy sets, the linguistic evaluation information of the hospital can be denoted as a hesitant fuzzy matrix shown as Table 14.

Then, we can get the score of the first hospital as 6.42. Similarly, we can obtain that the score of the second hospital is 23.06. According to this final result, we should choose the second hospital. This result is distinct from the one considering the changing rate and probability. It implies that Algorithm 4.1 is significant.

5.3.2. Comparing to PLTS without changing rate

If we deal with the illustration by the probabilistic linguistic term sets, the linguistic evaluation information of the hospital can be denoted as a probabilistic linguistic matrix shown as Table 15.
6. Conclusions

The PLVTS proposed in this paper is helpful to deal with multi-granular multi-attribute group decision making problems with uncertain linguistic information. In this paper, we have first introduced the concept of PLVTS and given their basic operations. This expression improves the accuracy of multi-granular linguistic information in the MAGDM problems. Then an algorithm has been developed to aid MAGDM with multiple LESSs. The advantage of this approach is that the influence factors considered are more comprehensive, which makes the description or simulation of the selection-recommendation system more profound and subtle. The method proposed in this paper not only can deal with the small amount of data, but also can deal with the mass data. It is an improvement compared to the existing method used in multi-granular decision making problems. Additionally, the approach introduced in this paper can be used by the service supplier to analyze the specific situation and help them make project management and improve service to customs. It is closely connected with the daily lives of people.

Moreover, there are some potential directions for further investigation. Firstly, as preferences are a powerful tool to deal with the group decision making problems with linguistic information, different individuals may prefer to express their preferences by linguistic information with distinct scales in realistic situations. In such a case, it is reasonable to use PLVTS to represent the individuals’ preferences by taking into account both the distinct linguistic evaluation values and their associated change rates of preferences. Preferences with PLVTSs are an interesting research direction in qualitative group decision making. Secondly, different probabilities and statistics indicators could be used in the model and algorithm proposed in this paper. Thirdly, it is also interesting to use other fuzzy numbers, such as intuitionistic fuzzy numbers, interval fuzzy numbers, etc., to represent the fitting accuracy. We will focus on these issues in the future.

Acknowledgements

The authors thank the Associate Editor and the anonymous reviewers for their helpful comments and suggestions, which have led to an improved version of this paper. The work was supported by the National Natural Science Foundation of China (Nos. 61273209, 71571123, 71532007), and the Central University Basic Scientific Research Business Expenses Project (No. skgt201501).

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