Per-Antenna Constant Envelope Precoding and Antenna Subset Selection: A Geometric Approach

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Abstract—Constant envelope (CE) precoding can efficiently control the peak-to-average power ratio (PAPR) and improve the power efficiency of power amplifiers in large-scale antenna array systems. Antenna subset selection (ASS), combined with CE precoding, can further improve power efficiency by using a part of antennas to combine the desired signal. However, due to the inherent nonlinearity, the joint optimization of CE precoding and ASS is very challenging and satisfactory solutions are yet not available. In this paper, we present new methods for CE precoding and ASS optimization from a geometric perspective. First, we show the equivalence between the CE precoder design and a polygon construction problem in the complex plane, thus transforming the algebraic problem into a geometric problem. Aiming to minimize the computational complexity, we further transform the CE precoder design into a triangle construction problem, and propose a novel algorithm to achieve the optimal CE precoder with only linear complexity in the number of used antennas. Then, we investigate the joint optimization of ASS and CE precoding to minimize the total transmit power while satisfying the QoS requirement. Based on the geometric interpretation, we develop an efficient ASS algorithm, which, using only addition and comparison operations, is guaranteed to find the globally optimal solution and provides robustness to channel uncertainty. The complexity of the proposed ASS algorithm is at most quadratic in the number of antennas in the worst case. The optimality and superiority of the proposed geometric methods are demonstrated via numerical results.

Index Terms—Constant envelope precoding, antenna subset selection, large-scale multi-antenna systems.

I. INTRODUCTION

Large-scale multi-antenna cellular communication systems, by employing a large number of antennas at the base station (BS), can significantly improve both spectral efficiency and power efficiency with simple signal processing, e.g., the maximum ratio transmission (MRT) [1], [2]. On the other hand, the hardware implementation of large-scale multi-antenna systems requires highly power-efficient power amplifiers since the number of power amplifiers scales with that of antennas. The power efficiency of a power amplifier is mainly limited by the linear range of the transmitted signal, which, however, may have a high peak-to-average power ratio (PAPR) as a result of adapting channel conditions and modulation schemes. Consequently, the traditional MRT technique requires a large linear range for each amplifier and inevitably leads to low power efficiency [3].

Recently, millimeter wave (mmWave) communication, operating in the band of 30–300 GHz, has been considered as a promising technology for future wireless communication systems [4]. Since path-loss of mmWave channels is much larger than that of microwave channels, large-scale antenna arrays will be an indispensable component in mmWave communication systems for providing large array gains to combat severe path loss [5]. However, due to the stringent requirement on a mmWave device, the equipment of large-scale antenna arrays particularly requires the use of cheap and power-efficient RF amplifiers [6], whose linear range is often quite limited. Consequently, PAPR is also a critical issue in mmWave communications. Conventional multi-antenna transmission technologies, for example linear beamforming/precoding, usually generate high PAPR and cause lower power efficiency.

To address this issue, constant envelope (CE) precoding was proposed to control the PAPR in large-scale multi-antenna systems [3] and [7]. In a CE precoding scheme, the instantaneous power of each antenna is restricted to be a constant, irrespective of channel state information as well as transmitted symbols. In this case, only phase variation of each antenna is used to form the desired signal at the receiver. Hence, CE precoding can sacrifice a part of transmit diversity gain to make it possible to use nonlinear but cheap and highly power-efficient switched-mode power amplifiers [8], [9]. Furthermore, in mmWave communication systems, CE precoding can be implemented in the analog domain with analog phase shifters, whose elements are all constrained to be of constant modulus [10].

CE precoding provides an attractive signal processing way to manage power efficiency and reduce the implementation costs for large-scale multi-antenna systems. However, finding the optimal CE precoder is a challenging problem, due to the highly nonlinear property of CE precoding [3]. In single-user multiple-input multiple-output (MIMO) or multi-user multiple-input single-output (MISO) systems, the CE precoding problems are often formulated into least square problems with constant-modulus constraints, which are, however, nonconvex. To further increase the power efficiency of large-scale multi-antenna

systems, antenna subset selection (ASS) arises as a useful technique to combine the desired signal without using all antennas [11]–[13]. However, the joint optimization of CE precoding and ASS leads to an even more difficult problem where no satisfactory solution has been presented.

In the literature, there are only a few works studying CE precoding. In [7], a two-level algorithm, consisting of depth-first-search and gradient descent iteration, was proposed to design CE beamforming for a large-scale single-user MISO system. In [14], a conventional gradient descent approach was used to solve the CE precoding problem for multi-user MIMO systems. Later, in [15] an improved CE precoder based on cross-entropy optimization (CEO) was developed for massive MIMO systems. Note that these CE precoding algorithms are based on either the gradient descent method or the CEO method. Recently, the noise-free receive signal region of CE precoding was completely characterized in [16].

The gradient descent method, by its nature, generally cannot guarantee to find the globally optimal solution to a nonconvex optimization problem that may have many local optima. The achieved solution is sensitive to the selection of the initial point. As for the CEO method, it is actually a randomized method and requires numerous samplings and iterations but only shows asymptotic convergence properties in terms of samplings and iterations [17]. Unfortunately, CE precoding is equivalent to solving a nonlinear least squares (NLS) problem that is nonconvex and has multiple local minima. Consequently, neither the gradient descent method nor the CEO method can efficiently find the globally optimal solution.

When it comes to joint optimization of the CE precoder and ASS, the problem is even more difficult. Indeed, the ASS problem, when combined with CE precoding, is inherently a nonlinear combinatorial optimization problem (0-1 integer programming). A common approach to address 0-1 integer programming is to explore its dual problem [18], [19], which, however, cannot guarantee to find the optimal solution and may introduce high complexity. The joint optimization of CE precoding and ASS was only studied in [8], [9]. Specifically, [8] studied ASS for maximizing the received SNR with imperfect channels, while [9] studied the minimum-power ASS, but only a suboptimal method was proposed.

In this paper, we study the CE precoding and ASS optimization problem from a novel geometric perspective. The main contributions of this paper are summarized as follows:

- Using the geometric language, we analyze and compare two ASS design philosophies, namely constellation-oriented and symbol-oriented ASS. We show that symbol-oriented ASS has great advantages in minimizing the total transmit power and providing robustness to channel uncertainty.
- Then, we develop a novel ASS algorithm, jointly used CE precoding, to minimize the transmit power while satisfying the QoS requirement. The proposed ASS algorithm is guaranteed to find the globally optimal ASS solution for both perfect and imperfect channels. In particular, for imperfect channels, our ASS algorithm enjoys the advantage of no need to estimate the noise power and the bound of channel uncertainty.
- The proposed ASS algorithm can be implemented with only addition and comparison operations. Its complexity is at most quadratic in the number of antennas in the worst case. Superiority of our proposed geometric algorithms is verified by comprehensive numerical results.

We note that the focus of this paper is to provide the optimal symbol-oriented CE precoding and ASS solution, while the constellation-oriented CE precoder was obtained in [8], [9]. Furthermore, to the best of our knowledge, the joint optimization of CE precoding and ASS, aiming to achieve the minimum transmit power, is still an open problem. We solve this problem by proposing an efficient method to find the globally optimal CE-ASS solution for both perfect and imperfect channels.

The remainder of this paper is organized as follows. The system model and problem formulation are described in Section II. Then, in Section III, a novel CE precoding algorithm is presented and discussed. Section IV presents the adaptive ASS algorithm with its optimality and implementation. Simulation results and conclusions of this paper are given in Sections V and VI, respectively.

Symbol Notations: Bold uppercase, lowercase and squiggle letters represent matrices, vectors and sets, respectively. Notation $\mathcal{I}$, in particular, represents an index set. The cardinality of set $\mathcal{I}$ is denoted by $\text{card}(\mathcal{I})$. All-one and all-zero vectors of length $k$ are denoted by $\mathbf{1}_k$ and $\mathbf{0}_k$, respectively. Superscript $T$ represents the transpose operator of vectors or matrices. The modulus of complex number $z$ is denoted by $|z|$. For $z \in \mathbb{C}$, notation $z$ also denotes the vector in the complex plane, to underline its geometric mean, though $z$ is just a scalar.

II. PROBLEM FORMULATION

A. System Model and CE Precoding Problems

We consider a standard single-user MISO system with $N$ transmit antennas. The channel vector is denoted by $\mathbf{h} \in \mathbb{C}^N$, where the $i$-th element $h_i$ stands for the channel coefficient between the $i$-th transmit antenna and the receive antenna. The received signal is given by

$$y = \mathbf{h}^T \mathbf{x} + n$$

where $n$ denotes the complex circular symmetry AWGN distributed as $n \sim \mathcal{C}\mathcal{N}(0, \sigma_n^2)$.\n
We assume that channel state information (CSI) is known perfectly by the transmitter. The symbol constellation is denoted by $\mathcal{U}$ and has unit average power, i.e.,

$$\frac{1}{\text{card}(\mathcal{U})} \sum_{u \in \mathcal{U}} |u|^2 = 1. \quad (2)$$

Note that most commonly used modulation schemes, such as QPSK and 16-QAM, satisfy this assumption.

In a CE scheme, the task of the precoder is to design the transmitted signal $x$ under the constant modulus constraint, given the channel vector $h$ and transmitted information symbol $u \in \mathcal{U}$. Mathematically, the constant modulus constraint forces the transmitted signal $x_i$ of the $i$-th antenna to take the form of

$$x_i = \sqrt{\frac{P_t}{N}} e^{j\theta_i}, \quad \forall i \in \{1, 2, \ldots, N\} \quad (3)$$

where $P_t$ and $\theta_i$ denote the total transmit power and the phase of $x_i$, respectively.

With fixed transmit power, one can choose the phase of each $x_i$ so that the received signals are properly combined to meet the QoS requirement. For clarity, such a CE precoding scheme is illustrated in Fig. 1. Mathematically, CE precoding is equivalent to solving the following multivariate nonlinear equation

$$\sqrt{\frac{P_t}{N}} \sum_{i=1}^{N} h_i e^{j\theta_i} = \alpha u, \quad (4)$$

where $\alpha > 0$ is a constant reflecting the QoS requirement at the receiver and $u \in \mathcal{U}$ represents the transmitted symbol drawn from the constellation [7], [8].

It seems that the CE precoding and ASS algorithms will happen every time either the channel or the transmitted symbol changes. In general, like all precoding algorithms based on instantaneous channel state information, if the channel changes, the CE precoding and ASS algorithms should run again. However, if only the transmitted symbol changes, a more efficient mechanism is recommended. For simplicity, we consider a TDD-based system where the channel remains unchanged in every coherence block or time-slot [20], [21]. This block is divided into 3 phases, as shown in Fig. 2. The first phase (S1) is used for training and estimating channel and so on. The second and third phases (S2 and S3) are used for downlink and uplink data transmissions, respectively.

For each transmitted signal $\alpha u \in \{\alpha u | u \in \mathcal{U}\}$, the ASS and CE precoding algorithms are used to generate corresponding pattern vector of ASS $q_{\alpha u}$ and precoding vector of CE $\theta_{\alpha u}$. The $i$-th component of $q_{\alpha u}$ takes value 1 or 0, corresponding to the $i$-th antenna being selected (used) or not. The collection of such pairs denoted by $\mathcal{C} = \{(q_{\alpha u}, \theta_{\alpha u}) | u \in \mathcal{U}\}$ are stored in the cache. In the phase of downlink transmission, for each transmitted signal $\alpha u$, the corresponding pair $(q_{\alpha u}, \theta_{\alpha u})$ is taken out from the cache and transmitted from the corresponding antennas specified by pattern vector of ASS $q_{\alpha u}$ and precoded by $\theta_{\alpha u}$. This process continues until the phase S2 is over. In the next (coherence) block, ASS and CE precoding algorithms are used to update $\mathcal{C}$ according to $\alpha$ and $\mathcal{U}$. Note that during downlink data transmission, there is no need to call the ASS or CE precoding algorithms. Therefore, the alternative mode of transmission has much higher efficiency.

### B. Nonlinear Least Square Strategy

Unfortunately, though CE precoding is equivalent to solving multivariate nonlinear (4), it is difficult to solve the multivariate nonlinear equation directly. The difficulties are twofold. Firstly, it is unknown whether this equation admits a solution or not, i.e., the feasibility of such an equation. Secondly, due to the nonlinearity between $\alpha u$ and $\theta$, finding the exact solution is hard. These difficulties make the problem even more challenging when ASS is jointly considered, which often leads to an exhaustive search for ASS along with solving equation (4) numerous times.

For ease of understanding, we take the widely used nonlinear least square problem formulation [3], [7] to address CE precoding and further explain the idea of geometric methods. The nonlinear least square problem for CE precoding is given by

$$\min_{\theta} e(\theta) = \left| \alpha u - \sqrt{\frac{P_t}{N}} \sum_{i=1}^{N} h_i e^{j\theta_i} \right|^2. \quad (5)$$

It is easily seen that if the equation (4) is feasible, the problem (5) is equivalent to (4). Then, one can exploit conventional optimization methods to address (5) instead.

For example, in [7], the gradient descent method was proposed to solve problem (5) for large $N$. Specifically, one can sequentially update a phase angle at each iteration while keeping the other phase angles fixed and gradually decrease the objective function $e(\theta)$. At each iteration, an one-dimensional optimization problem is solved. The corresponding iterative procedure
can be expressed by
\[ \theta_k^{(n+1)} = \theta_k^{(n)} - \mu \Im \{ t_k r_k^* e^{j\theta_k} \}, \quad k = 1, \ldots, N \] (6)
where \( \mu \) is a step size, and \( t_k \) and \( r_k \) are given respectively by
\[ t_k = h_k \sqrt{\frac{h_k}{N}} \]
\[ r_k = \alpha u - \sqrt{\frac{h_k}{N}} \sum_{i=1, i \neq k}^N h_i e^{j\theta_i}. \]

However, the conventional gradient-based methods, e.g., the gradient descent method above, have several drawbacks. First, the nonlinear least square problem (5) is indeed a nonconvex problem and the gradient-based methods generally are only able to reach a locally optimal solution but there is no guarantee for a globally optimal solution. Second, the gradient-based methods usually need many iterations and increasing the computational complexity especially when \( N \) is large. Third, by dealing with problem (5), one actually cannot know whether the original problem (4) is feasible or not. Fourth, the gradient-based methods cannot be jointly used with ASS, without leading to extremely high computational complexity.

To overcome these difficulties, in this paper, we will investigate CE precoding from a new perspective. Specifically, we will review the original CE precoding problem (4) from a geometric point of view, and propose novel and efficient methods for CE precoding along with ASS.

III. GEOMETRIC CE PRECODING

We start by reinterpreting CE precoding from the perspective of the geometry. Based on the geometric intuition, we further derive the necessary and sufficient conditions of the feasibility of the CE problem. Then, a new solution to the CE precoding problem is developed by using triangle construction.

A. Geometric Explanation

For the sake of convenience, we first give the following definition.

**Definition 1:** The constant envelope combination (CEC) of \( d \in \mathbb{C} \) by \( \{ b_i \in \mathbb{C} \mid i = 1, 2, \ldots, N \} \) is defined by the following equation
\[ d = \sum_{i=1}^N b_i e^{j\theta_i}, \quad \theta_i \in [-\pi, \pi), i = 1, \ldots, N. \] (7)

According to Definition 1, CE precoding of \( d = \alpha u \) is equivalent to a CEC of \( \alpha u \) by \( \{ \sqrt{P_T/N} b_i \mid i = 1, \ldots, N \} \). Equation (7) can be rewritten as
\[ d e^{j\pi} + \sum_{i=1}^N m_i e^{j\phi_i} = 0, \quad \phi_i = \theta_i + \psi_i \] (8)
where \( m_i = |b_i| \) and \( \psi_i = \arg(b_i) \) are the modulus and argument of \( b_i \), respectively. Let \( v_i = m_i e^{j\phi_i} \) (i = 1, \ldots, N) and \( d e^{j\pi} = |d| e^{j(\psi_i + \pi)} \) denote the vectors in the complex plane. Then, \( \sum_{i=1}^N v_i = \sum_{i=1}^N m_i e^{j\phi_i} \) is the sum of \( N \) vectors in the complex plane in terms of the parallelogram principle. An example of \( N = 4 \) is shown in Fig. 3 for clarity. It is worth emphasizing that \( de^{j\pi} \) and \( v_i (i = 1, \ldots, N) \) represent vectors in the complex plane.

If there exists a \( \phi = [\phi_1, \ldots, \phi_N] \in [0, 2\pi)^N \) such that
\[ d = \sum_{i=1}^N m_i e^{j\phi_i}, \quad \phi_i = \theta_i + \psi_i, \] (9)
then we have \( de^{j\pi} + \sum_{i=1}^N v_i^* = 0 \), corresponding to a zero vector in the complex plane. It is clear that \( de^{j\pi} = -d \) and \( v_i^* (i = 1, \ldots, N) \) can form a polygon in this circumstance, as is shown in Fig. 4. In other words, the existence of CE precoding is equivalent to that of polygon construction. Without loss of generality, it is sufficient to consider \( N \) vectors, i.e., \( v_i (i = 1, \ldots, N) \).

If all included angles between vector \( v_i (i = 1, \ldots, N) \) and the \( \Re \)-axis are available, then \( \phi_i (i = 1, \ldots, N) \) can also be obtained by rotating all the vectors appropriately. Clearly, the less the angles we need to calculate, or equivalently, the less edges the polygon has, the less complex the polygon construction is. Apparently, the simplest polygon with least edges is (in fact reduces to) a line segment. The vectors in the complex plane can form a line segment if and only if the sum of the lengths of some vectors is equal to the sum of the lengths of the others. Note that these vectors, observed from (4), correspond to different channels, which are continuous random variables, so are their lengths. Consequently, the probability for the vectors being able to form a line segment tends to be zero.

Therefore, in practice, the simplest polygon that are possible to be constructed would be a triangle. In this case, one can exploit simple geometric tools, such as the Law of Cosines, to solve the polygon construction problem. Intuitively, a triangle is

One should understand the polygon as a generalized polygon, i.e., it may be nonconvex or intersected with itself. However, we can always construct a corresponding convex polygon instead.
available by rearranging, rotating and combining vectors \( v_i (i = 1, \ldots, N) \) appropriately in the right circumstances (see Fig. 5).

The following lemma provides the mathematical foundation for our intuition and observation.

**Lemma 1**: Suppose that \( \{ m_i | i = 1, \ldots, N, N \geq 3 \} \) satisfy the following two conditions:

1) \( m_1 \geq m_2 \geq \cdots \geq m_N > 0 \).

2) \( m_1 \leq \sum_{i=2}^{N} m_i \).

Then, there exists at least one index \( i \in \{2, \ldots, N-1\} \) such that

\[
  m_1 \geq \frac{1}{n} \sum_{i=2}^{n} m_i - \frac{N}{n+1} m_i .
\]

**Proof**: See Appendix A. \( \blacksquare \)

**Remark 3.1**: Lemma 1 implies that it is possible to construct a triangle by vectors \( v_i (i = 1, \ldots, N) \) if appropriate conditions are satisfied. To illustrate this viewpoint, we consider three line segments \( a, b \) and \( c \) with lengths \( m_1, \sum_{i=2}^{n} m_i \) and \( \sum_{i=n+1}^{N} m_i \), respectively. The two conditions of Lemma 1 imply that \( a \geq |b-c| \) holds. If both \( a > |b-c| \) and \( a < b+c \) hold, from elementary geometry, a triangle can be constructed by the three line segments. If \( a = |b-c| \) or \( a = b+c \), then the triangle degenerates to one segment, i.e., two segments and the third one are coincident.

**Remark 3.2**: The first condition in Lemma 1 requires that none of the \( N \) vectors is a zero vector. The probability of this event is 1 when each element of \( \{ v_i | i = 1, \ldots, N \} \) is selected from a continuous distribution such as complex Gaussian distribution. Even if there are zero vectors, one can just remove them from the vector set. It also suggests that the considered complex vectors shall be sorted by their lengths.

**Remark 3.3**: The second condition in Lemma 1 is in fact a necessary condition for the feasibility of the CE problem (4). By replacing \( \{ m_1, \ldots, m_N \} \) with \( \{ |a| |u|, \sqrt{P_T/N} |h_1|, \ldots, \sqrt{P_T/N} |h_N| \} \), one can see that (4) is infeasible if this condition fails. CE precoding and ASS are meaningless in this case. In fact, we will show that it is also a sufficient condition, under which we are always able to find the optimal CE precoder via triangle construction.

Similar to the single user case, there also exists a corresponding geometric interpretation for multi-user case, which may provide some useful insights. Since we aim to provide CE precoding and ASS solutions for the single user case and the approach provided in this paper cannot be directly applied to the multi-user case, where triangle construction often fails. We defer the extension to the multi-user case to future work from high dimensional geometry theory.

**B. Necessary and Sufficient Condition for CEC**

Based on Lemma 1, we are able to provide the necessary and sufficient conditions for CE precoding, as stated in Theorem 1.

**Theorem 1**: Suppose that \( N \geq 3 \) and \( \{ v_i | i = 1, \ldots, N \} \) satisfy \( |v_1| \geq \cdots \geq |v_N| > 0 \). Then, \( \forall k \in \{1, \ldots, N\}, v_k \) is a CEC of the other vectors i.e., \( \{ v_i | i = 1, \ldots, N, i \neq k \} \), if and only if

\[
  |v_1| \leq \sum_{i=2}^{N} |v_i|. \tag{11}
\]

**Proof**: See Appendix B. \( \blacksquare \)

**Remark 3.4**: It is clear that Theorem 1 holds true for \( N = 2 \) as well, though we assume that \( N \) is no less than 3.

**Remark 3.5**: Theorem 1 implies, in fact, the mutuality of CEC among the combining vectors. In other words, if there exists a CEC for one vector, then the existence applies to the other combining vectors. It is important to underscore this property since it plays a key role in designing the ASS algorithm in Section IV. For convenience narration in Section IV, if the geometric figure formed by \( \{ v_i \} \) in Fig. 4 is a polygon (or triangle), we call the corresponding CEC as PCEC, i.e., polygon constant envelope combination (or TCEC, i.e., triangle constant envelope combination).

**Remark 3.6**: It should be noted that the proof of Theorem 1 is constructive and implies a precoding approach. To illustrate this point, we just rewrite \( d = \sum_{i=1}^{N} m_i e^{j\theta_i} \) as \( d = e^{j\pi} + \sum_{i=1}^{N} m_i e^{j\theta_i} = 0 \). So we can conclude that the existence of CE precoding is equivalent to that of polygon or triangle construction in the complex plane. The detail for CE precoding is deferred to the next subsection.

Based on Theorem 1, we can immediately provide a simple necessary and sufficient condition to judge the feasibility of problem (4), as stated in the following corollary.

**Corollary 1**: The following equation

\[
  d = \sum_{i=1}^{N} b_i e^{j\phi_i}, \quad x_i \in [0, 2\pi) \tag{12}
\]

is feasible if and only if

\[
  2a_{\text{max}} \leq \sum_{i=1}^{N+1} a_i , \tag{13}
\]

where \( a_{\text{max}} = \arg \max_{1 \leq i \leq N+1} a_i \) with \( a_{N+1} = |d|, a_i = |b_i|, (i = 1, \ldots, N) \).

**Remark 3.7**: By means of classified discussions, one can prove that Corollary 1 is essentially equivalent to Theorem 1 of [8]. Therefore, Corollary 1 in fact provides another equivalent judging condition of feasibility of CE precoding. However, the results and derivations in [8] do not provide any physical meaning or insight, while our geometric method clearly shows the physical meaning of the optimal structure of CE precoding.

Corollary 1 indicates that the CE precoding problem (4) is feasible if and only if the maximum modulus among \( \{a_{\text{ei}} \)
\[ g, i \text{ is not more than the sum of the } + |i|, ..., N \]  
and 
\[ = (17) \] 
\[ = |i| \geq \cdots \geq |N| \]  
and define 
\[ v_i = 1 \]

C. CE Precoding Algorithm

Based on Lemma 1 and Theorem 1, we propose a CE precoding or phase recovery algorithm with linear complexity in the problem size, i.e., \( N \).

For simplicity, let \( d = au \) and \( b_i = \sqrt{P_i/N} |h_i| \). Without loss of generality, we assume that \( |b_1| \geq \cdots \geq |b_k| \geq |d| \geq \sum_{i=k+1}^{N} |b_i| \). If \( |b_1| > |d| + \sum_{i=k+1}^{N} |b_i| \), then Theorem 1 asserts that there is no \( \theta = [\theta_1, \ldots, \theta_N] \) such that
\[ d = \sum_{i=1}^{N} b_i e^{i \theta_i} . \]

Next, we assume \(|d| + \sum_{i=k+1}^{N} |b_i| \geq |b_1| \) and define \( v_i \) as
\[ v_i = \begin{cases} 1 & i = 1, \ldots, k \\ d & i = k + 1 \\ b_{i-1} & i = k + 2, \ldots, N + 1 \end{cases} \quad (14) \]
Lemma 1 and the apparent inequality \(|v_1| \leq \sum_{i=2}^{N+1} |v_i| \) imply that there is an index \( n \) such that \(|a - b| \leq c \leq |a - b| \) with \( c = |v_1| \), \( a = \sum_{i=2}^{n} |v_i| \) and \( b = \sum_{i=n+1}^{N+1} |v_i| \). Therefore, (1) the three line segments can construct a triangle; or (2) two segments and the third one coincide. Similar to the proof of Theorem 1, let \( 1 \) and \( \alpha \) denote the included angles of edges \((a, c)\) and \((b, c)\) which can be calculated by applying the Law of Cosines, respectively; \( (2) g_1 = e^{i \alpha}, g_2 = -ae^{(i-\alpha)\beta} \) and \( g_5 = be^{i(\alpha-\beta)} \). It is clear that \( g_1 + g_2 + g_3 = 0 \) (See Fig. 15). Therefore, we obtain the following equation
\[ v_1 e^{-i \gamma_1} + \sum_{i=2}^{n} v_i e^{i(-\gamma_i - \pi + \beta)} + \sum_{i=n+1}^{N+1} v_i e^{i(-\gamma_i + \pi - \alpha)} = 0, \quad (15) \]
where \( \gamma_i \) denotes the argument of \( v_i \), i.e., \( \gamma_i = \arg v_i \).

With \( \phi_i \) defined as
\[ \phi_i = \begin{cases} -\gamma_1 & i = 1 \\ -\gamma_i - \pi + \beta & i = 2, \ldots, n \\ -\gamma_i + \pi - \alpha & i = n + 1, \ldots, N + 1 \end{cases} \quad (16) \]
Equation (15) can be rewritten as
\[ v_{k+1} e^{i \phi_{k+1}} + \sum_{i=1, i \neq k+1}^{N+1} v_i e^{i \phi_i} = 0, \quad (17) \]
which is equivalent to
\[ d = v_{k+1} + \sum_{i=1, i \neq k+1}^{N+1} v_i e^{i(\phi_i + \pi - \phi_{k+1})} . \quad (18) \]

Algorithm 1: Geometric CE Precoding (GCEP) for problem (4).

1: **Input:** \( d = au, b_i = \sqrt{P_i/N} |h_i|, i = 1, \ldots, N \), with \( |b_1| \geq \cdots \geq |b_k| \geq |d| \geq \sum_{i=k+1}^{N} |b_i| \).

2: Let \( v_i = b_i (i = 1, \ldots, k), v_{k+1} = d, \gamma_i = \arg v_i, s_1 = 0 \\
\quad v_i = b_{i-1} (i = k + 2, \ldots, N + 1), s_2 = \sum_{i=k+2}^{N+1} |v_i| . \)

3: If \(|v_1| > s_2 \) return (there is no solution).

4: \( \phi = \arccos \frac{\sum_{i=2}^{N+1} |v_i|^2 - s_2^2}{2 |v_1| s_2} \)
\( \phi_1 = -\gamma_1, \phi_i = -\gamma_i - \pi + \beta (i = 2, \ldots, n) \)
\( \phi_i = -\gamma_i + \pi - \alpha (i = n + 1, \ldots, N + 1) \)

5: **Output:** \( \theta_i = \phi_i - \phi_{k+1} + \pi (i = 1, \ldots, k) \)
\( \theta_i = \phi_{i+1} - \phi_{k+1} + \pi (i = k + 1, \ldots, N) \)

Furthermore, we have
\[ d = \sum_{i=1, i \neq k+1}^{N+1} v_i e^{i(\phi_i + \pi - \phi_{k+1})} + \sum_{i=k+2}^{N+1} v_i e^{i(\phi_i + \pi - \phi_{k+1})} \quad (19) \]
\[ = \sum_{i=1}^{k} v_i e^{i(\phi_i + \pi - \phi_{k+1})} + \sum_{i=k+1}^{N} v_i e^{i(\phi_i + \pi - \phi_{k+1})} \]
\[ = \sum_{i=1}^{k} v_i e^{i(\phi_i + \pi - \phi_{k+1})} + \sum_{i=k+1}^{N} v_i e^{i(\phi_i + \pi - \phi_{k+1})} \]
where the last equality follows from (14).

Therefore, the solution for problem (4) is \( \theta = [\theta_1, \ldots, \theta_N] \)
with \( \theta_i \) given by
\[ \theta_i = \begin{cases} \phi_1 - \phi_{k+1} + \pi & i = 1, \ldots, k \\ \phi_{i+1} - \phi_{k+1} + \pi & i = k + 1, \ldots, N \end{cases} \quad (20) \]

The proposed precoding approach is formally described in Algorithm 1.\(^2\) We judge the feasibility of problem (4) and find the three edges of the CE triangle in step 2 and 3 of Algorithm 1, respectively. In step 4 and 5, we calculate the included angles, adjust the phase, and eventually recover the phase.

From the intuition of CE precoding, higher order modulation may lead to a higher probability that there is no CE precoding vector, under the condition that the number of transmit antennas and the minimal distance between different symbols \( \min\{|u - u'| | u \neq u', u \in U, u' \in U\} \) among constellation\( U \)

\(^2\)Note that there is no need to perform sorting operation in Algorithm 1, since ASS is before CE precoding and the sorting operation has already been performed by Algorithm 2. Please see Section IV for more details.
are both fixed. Note that antenna correlation in large-scale multi-antenna systems cannot be ignored in some cases. However, it is generally difficult to analytically characterize the influence of antenna correlations on CE precoding, even for MISO systems. On the other hand, as long as the necessary and sufficient condition of CE precoding is satisfied with high probability, the CE precoding scheme is then applicable, even if correlation exists.

IV. GEOMETRIC ANTENNA SUBSET SELECTION

In this section, we consider using ASS in conjunction with CE precoding to improve power efficiency of the large-scale multi-antenna system and robustness to channel uncertainty. We show that the minimum power ASS problem can also be addressed in a geometric way. Then, an efficient algorithm is developed to achieve the globally optimal solution.

A. Minimum Power Antenna Subset Selection

As a cost of joint optimization, ASS requires additional computation and storage resources. For example, it needs to store the collection of ASS pattern vectors \( \{ \mathbf{q}_u \} \in \mathcal{U} \), while periodically updating \( \{ \mathbf{q}_u \} \in \mathcal{U} \) also consumes some computation resources. However, ASS techniques are able to improve power efficiency by exploiting the spatial redundancy of multi-antenna systems. For example, it is possible to use fewer antennas, thus less transmit power, to meet the predefined QoS requirement. The geometric interpretation of CE precoding, i.e., polygon/triangle construction, paves a natural path to devising power efficient ASS.

From the previous section, in CE precoding, the transmitted information symbol \( d \) is a CEC of \( \{ v_i \} \), as shown in Fig. 6. One can observe that \( d \) can be possibly combined by a subset \( \mathcal{S} \subset \{ v_i \} \), but not necessarily by all elements in \( \{ v_i \} \). For example, in Fig. 6, the subset \( \mathcal{S} = \{ v_1, v_3 \} \) is enough to build the CE precoder. The direct meaning of using less elements in \( \{ v_i \} \) is that it is possible to use less antennas and thus less power to transmit information symbols. This observation suggests an interesting way for ASS from the geometric perspective, i.e., using as less edges as possible to build a triangle for CE precoding.

Before formally formulating the ASS optimization problem, it is necessary to discuss two different ASS design philosophies: designs for an entire constellation and for any individual symbol, namely constellation-oriented and symbol-oriented designs. We will show that the symbol-oriented ASS has more advantages in reducing transmit power and improving robustness to channel uncertainty than the constellation-oriented ASS. For ease of understanding, we still use the geometric language of CE precoding.

Reducing Total Transmit Power: Assume that constellation \( \mathcal{U} \) is the normalized 16-QAM and perfect channel vector \( \mathbf{h} = [h_1, \ldots, h_N] \) is available. We let \( u_{\min} = \arg \min_{u \in \mathcal{U}} |u| \) and \( u_{\max} = \arg \max_{u \in \mathcal{U}} |u| \) and further assume that \( \alpha \) satisfies

\[
\alpha |u_{\max}| > \max \left\{ \sqrt{P_T/N|h_1|}, \ldots, \sqrt{P_T/N|h_N|} \right\}.
\]

From Theorem 1, \( \alpha u_{\max} \) is the longest edge of constructed triangle.

Let \( q = [q_1, \ldots, q_N]^T \), where \( q_i = 1 \) or 0, corresponding to the \( i \)-th antenna being selected or not. The problem of minimizing total transmit power is equivalent to the following optimization problem

\[
\min_q \quad 1^T q
\]

s.t. \( \alpha |u_{\max}| \leq \sum_{i=1}^N \sqrt{P_T/N|h_i|q_i} \)

\[q_i \in \{0, 1\}, \quad i = 1, \ldots, N \tag{21}\]

where \( 1 \) is an \( N \)-dimensional vector with all entries equal to one.

Let the solution of problem (21) be \( q_0 \). However, for \( \alpha u_{\min} \), the solution is far from satisfactory since

\[
\sum_{i=1}^N \sqrt{P_T/N|h_i|q_{0,i}} \geq \alpha |u_{\max}| \gg \alpha |u_{\min}|.
\]

This implies that we may find another vector \( q_1 \) such that \( \sum_{i=1}^N q_{1,i} < \sum_{i=1}^N q_{0,i} \) and the CE precoding conditions are satisfied. Therefore, the use of \( q_0 \) for \( \alpha u_{\min} \) implies a waste of transmit power. Moreover, the situation gets even worse for higher order modulation or larger \( \alpha \). Note that a larger \( \alpha \) implies larger SNR and high quality of communication. If ASS is designed for any single symbol, this problem can be preventable.

Improving Robustness to Channel Uncertainty: ASS designed for an entire constellation will introduce performance loss in terms of robustness to channel uncertainty. To see this, we consider the imperfect CSIT model

\[
y = \sum_{i=1}^N (\hat{h}_i + \Delta h_i) x_i + n, \tag{22}\]

where \( \hat{h}_i \) and \( \Delta h_i \) denote estimated channel coefficient and channel uncertainty of \( i \)-th antenna, respectively. For simplicity, we model the channel uncertainty \( \Delta \mathbf{h} = [\Delta h_1, \ldots, \Delta h_N] \) as a Gaussian random vector, i.e., \( \Delta \mathbf{h} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_N) \). The transmitter treats the estimated channel as the real channel and performs CE precoding by satisfying the equation \( \alpha \mathbf{u} = \sum_{i=1}^N \sqrt{P_T/N} \hat{h}_i q_i e^{j \theta_i} \). The received signal given by (22)
can be rewritten as
\[ y = \alpha u + \sqrt{P_T/N} \sum_{i=1}^{N} \Delta h_i q_i e^{j\theta_i} + n. \]

Since the receiver estimates information symbol \( u \) based on \( y/\alpha \), it is desired to minimize the interference and noise, i.e.,
\[ I = \mathbb{E}(|y/\alpha - u|^2). \]
The SNR defined as \( 1/I \) is given by
\[ \rho = \frac{\alpha^2}{\sigma_n^2 + P_T/N \sigma_h^2 \sum_{i=1}^{N} q_i}. \]  

(23)

Apparently, the use of less transmitted antennas implies higher SNR under the condition that CE precoding is feasible. With the assumption \( \alpha|u_{\text{max}}| > \max\{\sqrt{P_T/N}|h_i|| i = 1, \ldots, N\} \), the robust problem formulation of the constellation-oriented ASS design is then given by
\[
\begin{align*}
\min_{q} \quad & 1^T q \\
\text{s.t.} \quad & \alpha|u_{\text{max}}| \leq \sqrt{P_T/N} |\hat{h}_i| q_i \\
& q_i \in \{0, 1\}, \quad i = 1, \ldots, N.
\end{align*}
\]

(24)

Let the solution of problem (24) be \( q_0 \). However, for \( \alpha|u_{\text{min}}| \), the solution is usually too conservative since
\[ \sum_{i=1}^{N} \sqrt{P_T/N} |\hat{h}_i| q_{0,i} \geq \alpha|u_{\text{max}}| \gg \alpha|u_{\text{min}}|. \]

This implies that we may find another vector \( q_1 \) such that \( \sum_{i=1}^{N} q_{1,i} < \sum_{i=1}^{N} q_{0,i} \), and the CE precoding conditions are satisfied. It is therefore possible to reduce total transmit power and increase SNR, i.e., \( \rho \) in (23), by using \( q_1 \).

Therefore, the symbol-oriented ASS algorithms enjoy the advantages in both reducing total transmit power and improving robustness to channel uncertainty, especially for larger \( \alpha \) and higher order of modulation. For this reason, we will adopt the symbol-oriented ASS design philosophy in this paper.

In the following, we formally formulate the ASS optimization problem, aiming to obtain the minimum power and maximum robustness solution. Note that the only difference between problems (21) and (24) is that perfect channel vector \( h \) in (21) is replaced by estimated channel vector \( \hat{h} \) in (24). The two optimization problems take essentially the same form. For this reason, we only consider the case where imperfect channel vector \( \hat{h} \) is available. In fact, our ASS algorithm is general-purpose and we can just replace \( \hat{h} \) by \( h \) if \( h \) is available.

First, we consider the special case where only one antenna is used for CE precoding. In this case, according to (4), a CE precoder exists if and only if there exists an index \( i \in \{1, \ldots, N\} \) such that \( d = \sqrt{P_T/N}|h_i| \). However, the probability of this event is zero in practice. Therefore, it is reasonable to assume that at least two antennas are used. Now, given the intended signal \( d = \alpha u \) and the transmit power \( P_T = P_T/N \) at each antenna, the problem of minimizing the total transmit power (or maximizing robustness to channel uncertainty) via ASS can be formulated as the following optimization problem:
\[
\begin{align*}
\min_{q} \quad & 1^T q \\
\text{s.t.} \quad & d = \sum_{i=1}^{N} \sqrt{P_T/N} |\hat{h}_i| q_i e^{j\theta_i}, \quad \text{for some } \theta_i \in [0, 2\pi) \\
& q \in \{0, 1\}^N.
\end{align*}
\]

(25)

Apparently, in contrast to the ASS formulation in [8], one significant advantage of the ASS formulation (25) is that there is no need to estimate the levels of noise and channel uncertainty \( \sigma_n^2 \) and \( \sigma_h^2 \), which avoids introducing new error or the other uncertainties.

One can see that the ASS optimization problem (25), is a difficult combinatorial problem. The main difficulty lies in that it has an indefinite number of constraints, since the phase variables are not determined yet. Consequently, one cannot use conventional integer programming methods, such as the Hungarian algorithm, to solve (25), but often requires exhaustive search, which is, however, impractical for larger \( N \). To reduce the number of enumerations in exhaustive search, one may use the branch and bound that still has exponential complexity and requires good bounds at each iteration, which may be difficult to obtain [22]–[24].

To avoid exhaustive search, we propose a novel algorithm to obtain the optimal ASS solution in a geometric way. For this purpose, we rely on two properties provided in Section III. One is the necessary and sufficient conditions for CEC, i.e., (11), which enables us to use only addition and comparison operations to verify whether there exists a CEC of the desired symbol. The other one, from Theorem 1, is the mutuality of CEC among all the combining elements, which enables us to treat the combining elements equally and shift the focus from transmitted information symbol, i.e., target, to the other combining elements, thus reducing the number of enumerations in the algorithm.

B. Optimal ASS Algorithm

The main idea to solve the ASS problem is as follows. Intuitively, we increase step by step the edges used for constructing a triangle. In other words, we first find out an initial solution and refine this solution by increasing an edge once a time. We first define set \( S = \{d, b_1, \ldots, b_N\} \), where \( b_i = \sqrt{P_T/N} h_i \). With the assumption that one optimal solution corresponding to a subset \( S_0 \) of \( S \) is available, we pick out a vector \( v \in S \setminus S_0 \) whose length is minimal among \( S \setminus S_0 \) but larger than that of any vector in \( S_0 \) and then add \( v \) into \( S_0 \). Then we try to find a better solution within \( S_0 \cup \{v\} \) and refine the stored solution. Similarly, by adding a new vector in the previous set and refining the previous solution step by step, we can finally find the global optimal solution.

For this purpose, the vectors in set \( S \) are sorted from the largest to the smallest according to their lengths. Without loss of generality, we assume that \( |b_1| \geq \cdots \geq |b_k| \geq |d| \geq |b_{k+1}| \geq \cdots \geq |b_N| \). We define set \( S_i (i = 1, \ldots, k + 1) \) as below:
\[
S_i = \{d, b_i, b_{i+1}, \ldots, b_N\}.
\]

(26)
For clarity, the structure and inclusion relation between each other of sets \( S_1, \ldots, S_{k+1} \) are illustrated in Fig. 7. Note that for any non-empty subset \( S_0 \) of \( S \), there always exists a unique set \( S_0 \subseteq \{ S_i | i = 1, \ldots, k + 1 \} \) such that \( S_0 \subseteq S \) and the cardinality of \( S_0 \) is minimal. Since \( S_0 \subseteq S \), the optimal solution within \( S_0 \) is not superior to the one within \( S_0 \). Therefore, it is enough for us to consider only finite \( k + 1 \) sets \( S_1, \ldots, S_{k+1} \).

To solve problem (25), we first find the optimal solutions within sets \( S_1, \ldots, S_{k+1} \), i.e., searching the minimum power ASS solutions from \( S_1, \ldots, S_{k+1} \), respectively. Denote the optimal solution within \( S_l (l = 1, \ldots, k + 1) \) by \( \mathcal{T}_l \) (\( l = 1, \ldots, k + 1 \)). The globally optimal solution corresponds to the minimal cardinality set among \( \{ \mathcal{T}_l | l = 1, \ldots, k + 1 \} \). Before proceeding, we provide a useful lemma on the relationship between PCEC and TCEC, as stated in the following lemma.

**Lemma 2:** If there exists a PCEC of \( d \) by \( \{ b_i | i \in I \} \), where \( I \) is an index set, then there must exist a TCEC of \( d \) by using the same elements, i.e., \( \{ b_i | i \in I \} \).

**Proof:** See Appendix C.

**Remark 4.1:** Lemma 2 indicates that a PCEC of \( d \) also implies a TCEC of \( d \). Therefore, it suffices to consider TCEC, i.e., triangle CE construction, without affecting the optimality of the solution. Lemma 2 also indicates that \( d \) must belong to one of the three edges (i.e., either an edge or a part of an edge) of the CE triangle. Let \( s = \max \{ |d_i|, |b_i|, \ldots, |b_N| \} \). According to Corollary 1, CEC of \( d \) does not exist if \( 2s > |d| + \sum_{i=1}^{N} |b_i| \) and ASS is meaningless in this case.

To solve optimization problem (25), we first consider the subset \( S_{k+1} \). Suppose that there exists a CEC of \( d \) by a subset of \( \{ b_{k+1}, b_{k+2}, \ldots, b_N \} \subseteq S_{k+1} \), i.e.,

\[
d = \sum_{i \in \mathcal{I}} b_i e^{j\theta_i},
\]

where \( \mathcal{I} \) is an index set such that \( \mathcal{I} \subset \{ k+1, \ldots, N \} \). Applying the triangle inequality, so we have

\[
|d| \leq \sum_{i \in \mathcal{I}} |b_i|.
\]

Note that \( |b_{k+1}| \geq |b_{k+2}| \geq \cdots \geq |b_N| \), so we have

\[
|d| \leq \sum_{i \in \mathcal{I}} |b_i| \leq \sum_{i=k+1}^{\text{\#card}(\mathcal{I})} |b_i|.
\]

According to Theorem 1, there must exist a CEC of \( d \) by \( \{ b_{k+1}, b_{k+2}, \ldots, b_{\text{\#card}(\mathcal{I})} \} \), where the total transmit power does not change since the antenna number is still \( \text{\#card}(\mathcal{I}) \). This implies that within \( S_{k+1} \), the minimum power ASS optimization is equivalent to the following problem:

\[
\begin{align*}
\min & \quad n \\
\text{s.t.} & \quad |d| \leq \sum_{i=1}^{n} |b_{k+i}| \\
& \quad 2 \leq n \leq N-k.
\end{align*}
\]

No more than \( N-k-1 \) addition operations are enough to solve problem (28). If \( |d| \leq |b_{k+1}| + |b_{k+2}| \), we can immediately terminate and proceed to pre-code with \( \{ b_{k+1}, b_{k+2} \} \). Let the optimal solution of problem (28) be \( n_{k+1} \), the corresponding optimal ASS solution within \( S_{k+1} \) is given by \( I_{k+1} = \{ b_{k+1}, \ldots, b_{k+n_{k+1}} \} \). We set the cardinality of \( I_{k+1} \) to \( \infty \) if (28) is infeasible, i.e., \( n_{k+1} = \text{\#card}(I_{k+1}) = \infty \).

Next, we consider the subset \( S_k \). Note that if the CEC of \( d \) does not include \( b_k \), the power minimization or cardinality minimization problem will reduce exactly to (28). Therefore, it suffices to consider the non-trivial case where the CEC of \( d \) contains \( b_k \). According to the mutuality of CEC, if there exists a CEC of \( d \), there must exist a CEC of \( b_k \), which is guaranteed if \( |b_k| \leq |d| + \sum_{i=1}^{n} |b_{k+i}| \) for some \( n \). Consequently, the power minimization problem for \( S_k \) is given by

\[
\begin{align*}
\min & \quad n \\
\text{s.t.} & \quad |b_k| \leq |d| + \sum_{i=1}^{n} |b_{k+i}| \\
& \quad 1 \leq n \leq \min(N-k, n_{k+1}-1).
\end{align*}
\]

No more than \( \min(N-k, n_{k+1}-1) \) addition operations are enough to solve problem (29). If \( |b_k| \leq |d| + |b_{k+1}| \), we can terminate and proceed to pre-code with \( \{ b_{k}, b_{k+1} \} \). Similarly, let the optimal solution of problem (29) be \( n_k \), the corresponding optimal ASS solution within \( S_k \) is given by \( I_k = \{ b_k, b_{k+1}, \ldots, b_{k+n_k} \} \). We set the cardinality of \( I_k \) to \( \infty \) if (29) is infeasible, i.e., \( n_k = \text{\#card}(I_k) = \infty \).

Now, we examine the set \( S_l \) in turn for \( l = k-1, k-2, \ldots, 1 \). Similarly, we solve the following a sequence of problems (denoted as \( P_l \) for \( l = k-1, \ldots, 1 \))

\[
\begin{align*}
\min & \quad n \\
\text{s.t.} & \quad |b_l| \leq |d| + \sum_{i=1}^{n} |b_{l+i}| \\
& \quad 1 \leq n \leq \min(N-l, n_{l+1}).
\end{align*}
\]

No more than \( \min(N-l, n_{l+1}) \) addition operations are enough to solve problem \( P_l \). Let the optimal solution of problem \( P_l \) be \( n_l \), the corresponding optimal ASS solution within \( S_l \) is given by \( I_l = \{ b_l, \ldots, b_{l+n_l} \} \). If the cardinality of \( I_l \) is \( 2 \), i.e., \( n_l = 1 \), then we can immediately terminate and proceed to pre-code with \( \{ b_l, b_{l+1} \} \). Similarly, we set the cardinality of \( I_l \) to \( \infty \) if (30) is infeasible.

Finally, we only need to solve the following very easy problem to obtain the global optimal solution of problem (25)

\[
I^* = \arg \min_{l=1, \ldots, k+1} \text{\#card}(I_l).
\]
The pseudo code of the proposed algorithm is provided in Algorithm 2. In step 2 of Algorithm 2, we examine the subset \( S_{k+1} \), and we examine the subsets \( S_k, \ldots, S_1 \) in turn in steps 3-4. We find and output the optimal antenna subset in step 5 and step 6, respectively.

Note that the solution corresponding to problem (25) is jointly optimal. In fact, the joint optimization of CE precoding and ASS can be formulated as follows:

\[
\begin{align*}
\min_{\theta, q} & \quad \sum_{i=1}^{N} \sqrt{P_0} \hat{h}_i q_i e^{j\theta_i} \\
\text{s.t.} & \quad d = \sum_{i=1}^{N} \sqrt{P_0} \hat{h}_i q_i e^{j\theta_i} \\
& \quad q_i \in \{0, 1\}^N.
\end{align*}
\]  

(31)

Let the optimal solution of problem (25) be \( \mathbf{q}^* \). There must exist a vector \( \theta^* = [\theta_1^*, \ldots, \theta_N^*] \) such that \( d = \sum_{i=1}^{N} \sqrt{P_0} \hat{h}_i q_i^* e^{j\theta_i^*} \). In other words, \( (\theta^*, \mathbf{q}^*) \) is also a feasible solution of problem (31). Considering that the feasible region of problem (31) is the same as that of problem (25), we can conclude that \( (\theta^*, \mathbf{q}^*) \) is also the optimal solution to problem (31) since the objective functions of problems (25) and (31) are the same.

**Remark 4.2:** It is clear that the number of addition operations is no more than \( \sum_{i=1}^{N} n_i \leq \sum_{i=1}^{N} (N - l) = N(N - 1)/2 \). In other words, the ASS algorithm performs addition operations at most \( N(N - 1)/2 \) times, which has complexity \( \mathcal{O}(N^2) \). Taking into account the sorting operation, whose complexity is \( \mathcal{O}(N \log_2 N) \), the total complexity is still \( \mathcal{O}(N^2) \). Note that in Algorithm 2, we only use addition and comparison operations.

Note that the estimation of complexity is very conservative, since the required number is \( N(N - 1)/2 \) only in the worst case. The worst case occurs only when \( \alpha = \min \{|b_i| : i = 1, \ldots, N\} \). With the assumption that \( \hat{h}_i (i = 1, \ldots, N) \) are independent and distributed as \( CN(0, 1) \), the probability \( P(\alpha | u < \min \{|b_1|, \ldots, |b_N|\}) \) can be calculated as

\[
\begin{align*}
P(\alpha | u < \min \{|b_1|, \ldots, |b_N|\}) & = \prod_{k=1}^{N} P(\alpha | u < \sqrt{P_0/N} | \hat{h}_k|) \\
& = \exp(-\alpha^2 |u|^2 N^2 / P_0).
\end{align*}
\]

Since CE precoding is intended for large-scale antenna array systems, the probability \( P(\alpha | u < \min \{|b_1|, \ldots, |b_N|\}) \) is very small. Even in the worst case, the number of additions is not necessarily \( N(N - 1)/2 \), since the number of needed additions to solve subproblem \( P_1 \) is very likely smaller than \( N - l \).

Actually, the number of addition operations is small, especially when (1) QoS is large; or (2) QoS is relatively small and some channel elements have relatively strong amplitudes. In fact, in the two cases, we have (1) \( d \geq |b_i| \geq \cdots \geq |b_N| \); or (2) \( |b_i| \geq \cdots \geq |b_{k_0}| \geq |b_d| \geq \cdots \geq |b_N| \) and \( k_0 \) is relative small. To obtain the optimal ASS solution, we only need to solve the following optimization problem

\[
\begin{align*}
\min_{n} & \quad n \\
\text{s.t.} & \quad |d| \leq \sum_{i=1}^{n} |b_i| \\
& \quad \leq n \leq N, \\
\end{align*}
\]

(32)

and/or subproblem \( P_1 (l = 1, \ldots, k_0) \)

\[
\begin{align*}
\min_{n} & \quad n \\
\text{s.t.} & \quad |b_i| \leq |d| + \sum_{i=1}^{n} |b_{i+1}| \\
& \quad 1 \leq n \leq N - l.
\end{align*}
\]

(33)

Apparently, the number of addition operations is small in the two cases.

As a byproduct of deriving Algorithm 2, we are able to characterize the structure of the optimal ASS solution, as shown in Proposition 1.

**Proposition 1:** Suppose that the CE precoding problem (4) is feasible and \( |b_1| \geq |b_2| \geq \cdots \geq |b_N| > 0 \). Then, the optimal solution to the minimum power ASS problem (25) is given by the following form:

\[
\mathbf{q}^* = [0_1^T, 1_1^T, \ldots, 0_{N-1-j}^T, 0_j^T]^T
\]

(34)

for some \( i, j \in \{0, \ldots, N - 2\} \) with \( i + j \leq N - 2 \).

**Proof:** See Appendix D.

**Remark 4.3:** It is interesting to see that the optimal solution structure revealed in Proposition 1 coincides with that in [8], although our work and [8] considered two different problems. Proposition 1 also implies an optimal ASS algorithm with complexity no more than \( \mathcal{O}(N^2) \). Since the structure of the optimal solution takes the form of (34) and the number of this form is \( \sum_{i=1}^{N-1} (N - i) = N(N - 1)/2 \approx \mathcal{O}(N^2) \). Hence, no more than \( N(N - 1)/2 \) enumerations are enough to find the optimal solution. Note that in practice the complexity of Algorithm 2 is less than that of the ASS algorithm based on Proposition 1,
because the latter has to conduct $N(N - 1)/2$ enumerations every time.

**Remark 4.4:** The symbol error rate (SER) constraint $f(\rho) > b$ can also be included in (25), where $f$ is a monotone function of the upper bound of symbol error probability [25] and $b$ is the threshold of SER constraint. Thanks to (23), this constraint can be equivalently written as $1^T q < C$ for some constant $C$. Consequently, the ASS problem with the SER constraint can still be solved efficiently by the proposed ASS algorithm.

V. Simulation Results

This section illustrates the performance of the proposed algorithms. The simulation setting is given as follows. The transmitted information symbol is selected uniformly and randomly from the constellation of 16-QAM and the noise is a circular symmetric complex Gaussian random variable distributed as $CN(0, 1)$. We consider two types of channel models that have been used in the literature [8]. The first model is uncorrelated Rayleigh fading channel with each element of the channel vector distributed as $CN(0, 1)$ in i.i.d. manner. The second model takes LOS components into consideration with $M$ elements of the channel vector distributed as $CN(\mu, 1)$ and the other elements distributed as $CN(0, 1)$. We set $M = 8$ and $\mu = 8$ in the simulation. The relationship between QoS, i.e., $\alpha$ in (4), and antenna number $N$ is linear. For the sake of convenience, we call the two channel models above as channel model 1 and channel model 2. The processor of used computer is Intel(R) Core(TM) i3-4150 CPU. Except for matlab built-in functions, no other software package is used in the simulation. All simulation results are averaged over 16000 independent channel realizations for the two channel models, except for Figs. 12 and 13.

For comparison, we call our proposed CE precoding and ASS algorithms GCEP and GASS, corresponding to Algorithms 1 and 2 respectively. We compare our CE precoding method with the existing nonlinear least square CE precoding method via gradient descent approach [7], which is termed NLSP. To show the optimality of the geometric CE precoding algorithm, we choose the average CE precoding error as a performance measure, which is defined as

$$ e_p = \frac{1}{K} \sum_{k=1}^{K} e_k(\theta) $$

where $e_k(\theta)$ defined in (5) is the $k$th CE precoding error and $K$ is the number of simulations. As for the comparison of ASS performance, it is well known that dual-based methods are widely used to address integer programming [26] and were proposed to search antenna subset selection for CE precoding in [8] and [9]. We call the dual-based antenna subset selection algorithm DASS, for brevity and refer interesting readers to [8], [26] for details. The performance measure for the comparison of ASS algorithms is average transmit power.

The average CE precoding error of the two CE precoding algorithms, i.e., NLSP and GCEP, is shown in Fig. 8. One can clearly see that the average CE precoding error of NLSP is much larger than that of GCEP. In fact, GCEP is an exact phase recovery algorithm, and thus the CE precoding error is zero exactly if an infinite calculation precision is available. The CE precoding error of GCEP shown in Fig. 8 arises mainly from the calculation of nonlinear functions such as $\cos(x)$ and $e^x$, and its precision is determined by the machine precision in the simulation. In contrast, the NLSP CE precoding algorithm based on gradient descent search is easily trapped in a local optimum, which leads to large errors. Fig. 8 also indicates that the average CE precoding error is not sensitive to the type of channel model and the number of antennas.

Fig. 9 shows the average transmit power of the two ASS algorithms for different channel models, where the transmit power of per-antenna is $-8$ dB and the number of the total transmit antennas is $N = 128$. The total transmit power is 20.287 if all transmit antennas are active. The average transmit power of GASS is less than that of DASS in both two channel models. This is because GASS can provide the globally optimal ASS solution, while DASS generally can only provide a suboptimal solution.
To further illustrate the performance of the proposed ASS algorithm, the distribution of active antennas for high QoS ($\alpha = 15$ dB) is shown in Figs. 10 and 11 for channel model 1 and channel model 2, respectively. The two figures all show that the ASS algorithms can reduce greatly the number of active antennas, i.e., transmit power accordingly. In addition, from the two figures, it can be seen that there are three peaks for GASS. This is because there are three different amplitude values for 16-QAM constellation and GASS is adaptive to each transmitted information symbol. In contrast, DASS is designed for the entire constellation, so there is only one peak. The two figures also show that GASS uses less active antennas than DASS to meet the same QoS requirement, thus using less transmit power and improving power efficiency.

Interestingly, one can also observe that both GASS and DASS algorithms use fewer active antennas in channel model 2 than in channel model 1. This is because the channel gains in channel model 2 are stronger than those in channel model 1, so the ASS algorithms are able to use fewer channels (i.e., antennas) to meet the QoS requirement. Yet, in both channel models, GASS use fewer antennas than DASS.

To further illustrate the superiority of the symbol-oriented ASS design philosophy and the performance of the proposed ASS algorithm, symbol error rate (SER) corresponding to channel model 1 and channel model 2 under perfect and imperfect channel conditions is plotted in Figs. 12 and 13, respectively. The noise power is fixed to be 1, i.e., $\sigma_n^2 = 1$. We consider three cases where perfect channel vector ($\sigma_h^2 = 0$), imperfect channel vector with small uncertainty level ($\sigma_h^2 = 0.02$), and imperfect channel vector with large uncertainty level ($\sigma_h^2 = 0.2$) are available at the transmitter.

It is observed from Figs. 12 and 13 that the symbol-oriented ASS design philosophy has greater advantage in improving robustness to channel uncertainty under different channel
VI. CONCLUSION

In this paper, we proposed a new design of CE precoding and antenna subset selection for large-scale MISO systems from a geometric perspective. We transformed the CE precoding problem into a triangle construction problem and provided the necessary and sufficient condition for the feasibility of CE precoding. A novel geometric approach was proposed to achieve the optimal CE precoder with linear complexity. Furthermore, also in a geometric way, we developed an adaptive ASS algorithm possessing great advantages in minimizing total transmit power and improving robustness to channel uncertainty. Finally, simulation results were provided to demonstrate the effectiveness of proposed algorithms.

APPENDIX A

PROOF OF LEMMA 1

We would like to use induction to prove Lemma 1.

Step 1: For $N = 3$, we clearly have

$$m_1 \geq m_2 - m_3 = |m_2 - m_3| \geq 0.$$

So, $n = 2$ is the index we desire.

Step 2: Next, supposing that inequality (10) holds for $N(N > 3)$, we need to prove the existence of index $n$ when $m_1 \geq m_2 \geq \cdots \geq m_{N+1}$ and $m_1 \leq \sum_{i=2}^{N+1} m_i$. We shall discuss the following two cases to complete step 2.

Case 1: If $m_1 \geq \sum_{i=2}^{N} m_i$, then

$$m_1 \geq \sum_{i=2}^{N} m_i \geq \sum_{i=2}^{N} m_i - m_{N+1} \geq 0.$$

Therefore, $n = N$ is the index we desire.

Case 2: If $m_1 \leq \sum_{i=2}^{N} m_i$, we have

$$m_1 \geq m_2 \geq \cdots \geq m_{N} \geq 0 \text{ and } m_1 \leq \sum_{i=2}^{N} m_i.$$

Applying inductive assumption, there must exist an index $k$ such that

$$m_1 \geq \left| \sum_{i=2}^{k} m_i - \sum_{i=k+1}^{N} m_i \right|.$$

Subcase I of Case 2: If

$$c_0 = \sum_{i=2}^{k} m_i - \sum_{i=k+1}^{N} m_i \geq 0,$$

we consider the following two cases.

1. If $c_0 \geq m_{N+1}$, then

$$m_1 \geq c_0 \geq m_{N+1} = \sum_{i=2}^{k} m_i - \sum_{i=k+1}^{N+1} m_i \geq 0.$$
Hence, \( k \) is the index needed.

2: If \( m_{N+1} \geq c_0 \), then

\[
m_1 - m_{N+1} \geq 0 \geq c_0 - m_{N+1} \geq -m_{N+1}.
\]

Hence, we have

\[
0 \leq m_{N+1} - c_0 = -\sum_{i=2}^{k} m_i + \sum_{i=k+1}^{N+1} m_i \leq m_{N+1} \leq m_1,
\]

As a result, \( k \) is the index we want.

Subcase 2 of Case 2: If

\[
d_0 = \sum_{i=k+1}^{N} m_i - \sum_{i=2}^{k} m_i \geq 0,
\]

we consider the following two cases.

I: If \( m_{N+1} \leq m_1 - d_0 \), i.e.,

\[
m_{N+1} + d_0 \leq m_1,
\]

by substituting (37) into (38), we obtain

\[
0 \leq \sum_{i=k+1}^{N+1} m_i - \sum_{i=2}^{k} m_i \leq m_1.
\]

Therefore, \( k \) is the index required.

2: If \( m_{N+1} \geq m_1 - d_0 \), we denote \( a \) and \( b \) as

\[
a = \sum_{i=1}^{N} m_i, \quad b = \sum_{i=2}^{k} m_i.
\]

Then, \( 0 \leq d_0 = a - b \leq m_1 \). Now, we proceed to show that

\[
\left| \sum_{i=k+2}^{N+1} m_i - \sum_{i=2}^{k} m_i \right| \leq m_1
\]

or equivalently

\[
-m_1 \leq \sum_{i=k+2}^{N+1} m_i - \sum_{i=2}^{k} m_i \leq m_1.
\]

To begin with, we verify the right hand side of (39). Since

\[
\sum_{i=k+2}^{N+1} m_i - \sum_{i=2}^{k} m_i = m_{N+1} - 2m_{k+1} + \sum_{i=k+1}^{N} m_i - \sum_{i=2}^{k} m_i
\]

\[
= m_{N+1} + d_0 - 2m_{k+1} \leq m_{k+1} + d_0 - 2m_{k+1}
\]

\[
= d_0 - m_{k+1} \leq d_0 \leq m_1,
\]

the right hand side of (39) holds.

For the left hand side of (39), we need to show

\[
m_{N+1} + d_0 - 2m_{k+1} \geq -m_1
\]

or equivalently

\[
m_{N+1} + a - b + m_1 - 2m_{k+1} \geq 0.
\]

Since \( m_{N+1} + d_0 \geq m_1 \), i.e., \( m_{N+1} + a \geq b + m_1 \), we have

\[
m_{N+1} + a - b + m_1 - 2m_{k+1} \geq b + m_1 - b + m_1
\]

\[
-2m_{k+1} = 2m_1 - 2m_{k+1} \geq 0,
\]

which proves the left hand side of (39). Hence, \( k + 1 \) is the index we desire.

Considering all the cases above and using induction, we have proven the lemma.

**APPENDIX B**

**PROOF OF THEOREM 1**

**Sufficiency:** According to Lemma 1, there must exist an index \( n (2 \leq n \leq N - 1) \) such that

\[
|v_3| \geq \left| \sum_{i=2}^{n} |v_i| - \sum_{i=n+1}^{N} |v_i| \right|.
\]

Combining with condition (11), we have

\[
\sum_{i=2}^{n} |v_i| + \sum_{i=n+1}^{N} |v_i| \geq \sum_{i=2}^{n} |v_i| - \sum_{i=n+1}^{N} |v_i|.
\]

Let \( c = |v_1|, \ a = \sum_{i=2}^{n} |v_i| \) and \( b = \sum_{i=n+1}^{N} |v_i| \). Then, \( a, b \) and \( c \) can form a triangle, as depicted in Fig. 15. Let \( \beta \) and \( \alpha \) denote the included angles of edges \((a, c)\) and \((b, c)\), respectively.

Applying the Law of Cosines directly, we have

\[
\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos \beta = \frac{a^2 + c^2 - b^2}{2ac}.
\]

Furthermore, we assume that the edge \( c \) is parallel to \( Re \)-axis and define the vectors \( g_1, g_2 \) and \( g_3 \) in the complex plane as

\[
g_1 = ce^{j0},
\]

\[
g_2 = ae^{j(-\pi + \beta)},
\]

\[
g_3 = be^{j(-\pi - \alpha)}.
\]

Obviously, we have \( g_1 + g_2 + g_3 = 0 \), which results in

\[
|v_1| + \sum_{i=2}^{n} |v_i|e^{j(-\pi + \beta)} + \sum_{i=n+1}^{N} |v_i|e^{j(\pi - \alpha)} = 0.
\]

Let \( v_i = |v_i|e^{j\gamma_i} \), i.e., \( |v_i| = v_i e^{j\theta_i} \), substituting the term \( |v_i| \) into (42), we have

\[
v_1 e^{-j\gamma_1} + \sum_{i=2}^{n} v_i e^{j(-\gamma_i - \pi + \beta)} + \sum_{i=n+1}^{N} v_i e^{j(-\gamma_i + \pi - \alpha)} = 0.
\]
Moving the target term from the left side of (43) to the right side and multiplying both sides by $e^{j\theta_i}$ where $t$ depends on the argument of the target term, we obtain the desired CEC. Hence, $v_k$ is a CEC of $\{v_i|i\neq k\}$.

Note that if $\sum_{i=2}^n |v_i| + \sum_{i=n+1}^N |v_i| = |v_1|$ or $|v_1| = \sum_{i=2}^n |v_i| - \sum_{i=n+1}^N |v_i|$ in (40), (42) still holds, despite failing to constitute a triangle under these circumstances.

(Necessity): Since $v_1$ is a CEC of $\{v_i|i\neq 1\}$, there must exist a vector $\theta = [\theta_1, \theta_2, \ldots, \theta_N] \in [0,2\pi)^N$ such that

$$v_1 = \sum_{i=2}^N v_ie^{j\theta_i}.$$ 

Applying the triangle inequality directly, we have

$$|v_1| = \sum_{i=2}^N |v_ie^{j\theta_i}| \leq \sum_{i=2}^N |v_i|.$$ 

**Proof of Lemma 2**

Due to the existence of the PCEC of $d$ by $\{b_i|i \in I\}$, there must exist a $\theta_i \in [0,2\pi]$, $i \in I$ such that

$$d = \sum_{i \in I} b_ie^{j\theta_i}.$$ 

We assume $|b_i| = \arg\max_{i \in I} |b_i|$ and consider the following two cases:

1) $|d| \leq |b_1|$. Through simple algebraic manipulation, we have

$$b_1 = \sum_{i \in I,i \neq t} b_ie^{j(\theta_i + \pi - \theta_t)} + \delta e^{-j\theta_t}.$$ 

According to the triangle inequality, we obtain

$$|b_1| \leq |d| + \sum_{i \in I,i \neq t} |b_i|.$$ 

Applying Theorem 1, we can conclude that there exists a TCEC of $d$ by the same elements, i.e., $\{b_i|i \in I\}$.

2) $|d| > |b_1|$. Applying the triangle inequality, we have

$$|d| = \sum_{i \in I} b_ie^{j\theta_i} \leq \sum_{i \in I} |b_i|.$$ 

Applying Theorem 1, we can assert that there exists a TCEC of $d$ by the same elements, i.e., $\{b_i|i \in I\}$.

**Appendix D**

**Proof of Proposition 1**

Since the optimal solution is one of $\{I_l|i = 1, \ldots, k+1\}$ that has the minimal cardinality and $I_l$ ($l = 1, \ldots, k+1$) all take the form of $[0^T, 1_{N-i-j-1}^T, 0^T]^T$. Therefore, the optimal solution must take the form of $[0^T, 1_{N-i-j}^T, 0^T]^T$ as well.

**References**


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