A class of \( p-q \)-Laplacian type equation with concave–convex nonlinearities in bounded domain

Honghui Yin\(^a,b\), Zuodong Yang\(^a,c,*\)

\(^a\) School of Mathematical Sciences, Nanjing Normal University, Jiangsu Nanjing 210046, China
\(^b\) School of Mathematical Sciences, Huaiyin Normal University, Jiangsu Huaian 223001, China
\(^c\) College of Zhongbei, Nanjing Normal University, Jiangsu Nanjing 210046, China

A R T I C L E   I N F O

Article history:
Received 26 March 2010
Available online 6 May 2011
Submitted by H. Liu

Keywords:
p–q-Laplacian
Critical exponent
Concave–convex nonlinearities
Weak solution

A B S T R A C T

In this paper, our main purpose is to establish the existence of multiple solutions of a class of \( p-q \)-Laplacian equation involving concave–convex nonlinearities:

\[
\begin{cases}
-\Delta_p u - \Delta_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*-2}u + \lambda f(x, u), & x \in \Omega, \\
\quad u = 0, & x \in \partial\Omega
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( \lambda, \theta > 0 \), \( 1 < r < q < p < N \) and \( p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian of \( u \). We prove that for any \( \lambda \in (0, \lambda^*) \), \( \lambda^* > 0 \) is a constant, there is a \( \theta^* > 0 \) such that for every \( \theta \in (0, \theta^*) \), the above problem possesses infinitely many weak solutions. We also obtain some results for the case \( 1 < q < p < r < p^* \). The existence results of solutions are obtained by variational methods.

1. Introduction

In this paper, we are interested in finding multiple nontrivial weak solutions to the following nonlinear elliptic problem of \( p-q \)-Laplacian type involving the critical Sobolev exponent

\[
\begin{cases}
-\Delta_p u - \Delta_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*-2}u + \lambda f(x, u), & x \in \Omega, \\
\quad u = 0, & x \in \partial\Omega
\end{cases}
\]  \hfill (1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( \lambda, \theta > 0 \), \( 1 < r < q < p < N \) and \( p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian of \( u \).

Problem (1.1) comes, for example, from a general reaction–diffusion system

\[
u_t = \text{div}[H(u)\nabla u] + c(x, u)
\]  \hfill (1.2)

where \( H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2} \). This system has a wide range of applications in physics and related science such as biophysics, plasma physics and chemical reaction design. In such applications, the function \( u \) describes a concentration, the first term on the right-hand side of (1.2) corresponds to the diffusion with a diffusion coefficient \( H(u) \); whereas the second
one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term \( c(x, u) \) has a polynomial form with respect to the concentration \( u \).

Recently, the stationary solution of (1.2) was studied by many authors, that is many works considered the solutions of the following problem

\[
-\text{div}[H(u)\nabla u] = c(x, u). \tag{1.3}
\]

In the present paper we are concerning problem (1.1), a special case of problem (1.3) in a bounded domain.

If \( p = q = 2 \), (1.1) can be reduced to

\[
\begin{cases}
-\Delta u = \theta V(x)|u|^{p-2}u + |u|^{2^*-2}u + \lambda f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega
\end{cases}
\]

which is a normal Schrödinger equation and has been widely studied, see [1–4].

The solutions of problem \((P_{\theta V, \lambda f})\) correspond to the critical points of the energy functional

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\theta}{r} \int_{\Omega} V(x)|u|^r dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx - \lambda \int_{\Omega} F(x, u) dx
\]

defined on \( W^{1,2}_0(\Omega) \), where \( F(x, s) = \int_0^s f(x, t) dt \).

If \( r = 2 \), the pioneer result of Brézis and Nirenberg [5] studied problem \((P_{\theta 0})\) and shows that if \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) in the zero Dirichlet boundary condition, \( N > 3 \) and \( 0 < \theta < \lambda_1 \), problem \((P_{\theta 0})\) possesses a positive solution in \( W^{1,2}_0(\Omega) \), while \( N = 3 \), problem \((P_{\theta 0})\) possesses a positive solution in \( W^{1,2}_0(\Omega) \), if \( \theta^* < \theta < \lambda_1 \) for some \( 0 < \theta^* < \lambda_1 \). Later in [6], H. Brézis proposed the problem of infinitely many solutions to problem \((P_{\theta 0})\) with \( r = 2, N \geq 4 \) and \( \alpha(x) > 0 \) somewhere in \( \Omega \). Recently, H. Liu in [7] studied problem \((P_{\theta V,\lambda})\) with \( V(x) = \frac{1}{|x|^2} \), \( F(x, u) = \lambda f(x)u + \nu g(x)u^q \) and \( r = 2 \), where \( 0 \leq \theta < \theta^* = \frac{N-2}{2} \), \( 0 < q < 1 \), \( f(x) \), \( g(x) \) are positive measurable functions, with other suitable conditions, it shows that there exists \( v^* > 0 \) such that for any \( v \in (0, v^*) \), problem \((P_{\theta V,\lambda})\) has at least two positive solutions \( u_1, u_2 \) with \( \sup_{u_1} < 0 < \inf u_2 \).

The typically difficulty in dealing with problem \((P_{\theta V,\lambda})\) is that the corresponding functional \( I(u) \) doesn’t satisfy a \((PS)\)-condition due to the lack of compactness of the embedding: \( H^1_\theta \hookrightarrow L^2(\Omega) \). Hence we couldn’t use the standard variational methods.

However, if \( 1 < r < 2 \), the situation is quite different. In [8] proved that problem \((P_{\theta 0})\) has infinitely many solutions satisfying \( I(u) < 0 \), provided that \( \theta > 0 \) is close to zero. The main essence is that when \( 1 < r < 2 \), the functional \( I(u) \) is sublinear, when \( \theta \) is small enough, \( I(u) \) satisfies the \((PS)_c\)-condition for \( c < 0 \). So we can look for critical points of negative critical values of \( I(u) \). Also, in [9] T.F. Wu considered problem \((P_{\theta 0,\lambda})\) with the decomposition of the Nehari manifold via the combination of concave and convex nonlinearities. It shows that there exists \( \lambda_0 > 0 \) such that for \( \lambda \in (0, \lambda_0) \), the equation \((P_{\theta 0,\lambda})\) has at least two positive solutions.

For a general case of \((P_{\theta V,\lambda})\), we consider \( p \)-Laplacian problem

\[
\begin{cases}
-\Delta_p u = \theta V(x)|u|^{p-2}u + |u|^{2^*-2}u + \lambda f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega
\end{cases}
\]

which is a special case of (1.1) when \( p = q \). The corresponding energy functional is

\[
J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\theta}{r} \int_{\Omega} V(x)|u|^r dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} dx - \lambda \int_{\Omega} F(x, u) dx
\]

defined on \( W^{1,p}_0(\Omega) \), and \( F(x, u) \) is as above.

Problem \((E_{\theta V,\lambda f})\) was also studied by many authors, many results valid for problem \((P_{\theta V,\lambda f})\) has been extended to problem \((E_{\theta V,\lambda f})\). For example, when \( \Omega \) is bounded in \( \mathbb{R}^N \), the existence of nontrivial solution of problem \((E_{\theta 0})\) was studied (see e.g. [10]). In [11] J.G. Azvero and I.P. Aloson proved that when \( 1 < r < p \) and \( \theta > 0 \) is small enough, problem \((E_{\theta 0})\) has infinitely many solutions. Recently, in [12] H. Liu studied problem \((E_{\theta V,\lambda f})\) with \( 0 \leq \theta < \theta^* = \frac{(N-2)^2}{4} \), \( 1 < p < N \) and shows that there exists \( \lambda^* > 0 \) such that for any \( \lambda \in (0, \lambda^*) \), problem \((E_{\theta V,\lambda f})\) has at least two positive solutions.

The main difficulty in extending the results for problem \((P_{\theta V,\lambda f})\) to the corresponding results for problem \((E_{\theta V,\lambda f})\) is that \( W^{1,p}_0(\Omega) \) is not a Hilbert space in general, even if the \((PS)\)-sequence \( \{u_n\} \) of \( J(u) \) is bounded, we cannot ensure

\[
|\nabla u_n|^p \to |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^{p^*}(\Omega)
\]

for some subsequence \( \{u_n\} \) of \( \{u\} \). To over this difficulties we use the Concentration–Compactness Principle as in [10]. As for \( p-q\)-Laplacian equation, more analysis is needed.
We recall some results about problem (1.3) now. V. Benci, A.M. Micheletti and D. Visetti in [13] studied the solutions of (1.3) on a bounded domain $\Omega \subset \mathbb{R}^N$ with
\[ c(x, u) = -p(x)|u|^{p-2}u - q(x)|u|^{q-2}u + \lambda g(x)|u|^{r-2}u \]
for $1 < p < r < q$ and $r < p^*$. In [14], M. Wu and Z. Yang proved the existence of a nontrivial solution to problem (1.3) with
\[ c(x, u) = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u - f(x, u) \]
in the whole space $\mathbb{R}^N$, where $a(x), b(x)$ are positive functions, also when $a(x) \equiv m$, $b(x) \equiv n$ are positive constants, it was proved in [15] that problem (1.3) has a nontrivial solution. Recently in [16], G. Li and G. Zhang studied problem (1.3) involving critical exponent with
\[ c(x, u) = |u|^{p^*-2}u + \theta|u|^{r-2}u \]
by using Lusternik–Schnirelman’s theory (see also in [11]). It shows that when $\theta > 0$, $1 < r < q < p < N$, there is a $\theta_0 > 0$ such that problem (1.3) possesses infinitely many weak solutions in $W^{1,p}_0(\Omega)$ for any $\theta \in (0, \theta_0)$.

Motivated by [12,14,16], and borrowed the methods in [11], we consider the more general problem (1.1). For the functions $V(x), f(x, t)$, we add the following assumptions:

(D1) suppose $V(x) \in L^\infty(\Omega)$ and $V(x) > \sigma > 0$ in $\Omega$, and there exists $\eta > 0$ such that
\[ \int_\Omega V(x)|u|^p \, dx \leq \eta \int_\Omega |\nabla u|^p \, dx, \quad \forall u \in W^{1,p}_0(\Omega); \]

(D2) $|f(x, t)| \leq a_1|t|^{r-1} + a_2|t|^\xi - 1$, for $\forall x \in \Omega$, $t \in R$, where $a_1, a_2 > 0$ and $1 < \xi < p^*$;

(D3) there exist $a_3 > 0$ and $s \in (1, p)$ such that
\[ f(x, t)t - p^*F(x, t) \geq -a_3|t|^s, \quad \forall x \in \Omega, \ t \in R \]
where $F(x, t) = \int_0^t f(x, \tau) \, d\tau$;

(D4) $f(x, t) > 0$ for $\forall x \in \Omega$, $t \in R^+$, and $f(x, t) = -f(x, -t)$, for $\forall x \in \Omega, \ t \in R$.

Extend and generalize some results in [11,12,16], we obtain our main result.

**Theorem 1.1.** Assume $1 < r < q < p < N$, and (D1)–(D4) hold. Then there is a $\lambda^* > 0$ and for any $\lambda \in (0, \lambda^*)$, there exists a $\theta^* > 0$ such that for any $\theta \in (0, \theta^*)$, problem (1.1) possesses infinitely many weak solutions in $W^{1,p}_0(\Omega)$.

**Remark 1.2.** In [17] need $f(x, u)$ satisfies the Ambrosetti–Rabinowitz condition, we can easily see that condition (D3) is weaker than the Ambrosetti–Rabinowitz condition.

**Remark 1.3.** Assumption (D1) is already used in [18] in order to prove multiple results for a class of semilinear elliptic equations, also see [12] for quasilinear case.

**Remark 1.4.** Here we give some examples of the nonlinearity satisfying (D2)–(D4).

1. $f(x, t) = |t|^{s-2}t$ with $1 < s < p$;
2. $f(x, t) = (\varepsilon \cos t)|t|^{s-2}t$ with $1 < s < p$ and $\varepsilon > 1$.

The present paper is organized as follows, in Section 2, we give some preliminary results, in Section 3, we will prove the main result, and we will give some results of problem (1.1) for the case $1 < q < p < r < p^*$ in Section 4.

### 2. Preliminaries results

In what follows, we denote by $\| \cdot \|_p, \; | \cdot |_p$ the norm on $W^{1,p}_0(\Omega)$ and $L^p(\Omega)$ respectively, that is,
\[ \|u\|_p = \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \quad |u|_p = \left( \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \]
and define $S$ as the usually Sobolev constant as follows

\[ S = \left( \int_\Omega |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}} \]
Throughout this paper, we denote weak converge by $\rightharpoonup$, and denote strong converge by $\rightarrow$, also we denote positive constants (possibly different) by $C_i$.

The energy functional associated with problem (1.1) is defined by

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{\theta}{r} \int_{\Omega} V(x)|u|^r \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \lambda \int_{\Omega} F(x,u) \, dx.$$ 

Obviously, $E(u)$ is even and it is well known that $E(u) \in C^1(W^{1,p}_0(\Omega),R)$ and nontrivial critical points of $E(u)$ are weak solutions of problem (1.1).

First, we define the Palais–Smale (PS)-sequence, (PS)-value, and (PS)-conditions in $W^{1,p}_0(\Omega)$ for $E$ as follows.

**Definition 2.1.** (I) For $c \in R$, a sequence $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a (PS)$_c$-sequence for $E$ if $E(u_n) = c + o(1)$ and $E'(u_n) = o(1)$ strongly in $W'$ as $n \to \infty$, where $W'(\Omega)$ is the dual of $W^{1,p}_0(\Omega)$.

(II) $c \in R$ is a (PS)-value in $W^{1,p}_0(\Omega)$ for $E$ if there exists a (PS)$_c$-sequence in $W^{1,p}_0(\Omega)$ for $E$.

(III) $E$ satisfies the (PS)$_c$-condition in $W^{1,p}_0(\Omega)$ for $E$ if every (PS)$_c$-sequence in $W^{1,p}_0(\Omega)$ for $E$ contains a convergent subsequence.

Now we give some results for the proof of Theorem 1.1.

**Lemma 2.2** (Concentration–Compactness Principle). (See [19].) Let $p < N$ and $\{u_n\}$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ converging weakly to some $u$ such that $|Du_n|^p$ converges weakly to $\mu$ and $|u_n|^p$ converges weakly to $\nu$ where $\mu$, $\nu$ are bounded nonnegative measures on $\mathbb{R}^N$. Also assume that $|u_n|^p$ is a tight sequence, i.e., there is a sequence $\{y_k\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$, there is an $R = R(\varepsilon) > 0$ such that for any $k$ we have

$$\int_{\mathbb{R}^N \setminus B(y_k,R)} |u_n|^p \, dx < \varepsilon.$$ 

Then we have

(i) There exists some at most countable set $J$, distinct points $\{x_j : j \in J\} \subset \mathbb{R}^N$, $\{\mu_j\} \subset (0, \infty)$, $\{\nu_j\} \subset (0, \infty)$ such that

$$\nu = |u|^p + \sum_{j \in J} \nu_j \delta_{x_j} \quad \mu = |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j},$$

where $\delta_{x_j}$ is the Dirac measure at $x_j$.

(ii) In addition, $S |\nabla_j|^p \leq \mu_j$, where $S$ is the best Sobolev constant as above with $\Omega$ be replaced by $\mathbb{R}^N$.

(iii) If $u \equiv 0$ and

$$\mu(\mathbb{R}^N)^{\frac{1}{p}} \leq S^\frac{1}{p} \nu(\mathbb{R}^N)^{\frac{1}{p}},$$

then $J$ is a singleton, i.e., $\nu$ is concentrated at a single point.

**Remark 2.3.** Since the space $W^{1,p}_0(\Omega)$ is not a Hilbert space for $1 < p < N$ and it does not satisfy Brézis–Lieb’s Lemma (see [20]), except for $p = 2$. But we have the following result holds (see [14]).

$$\|u_n\|^p \geq \|u_n - u\|^p + \|u\|^p$$

where $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a bounded sequence such that $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega)$ and $u_n \to u$ a.e. in an open set $\Omega \subset \mathbb{R}^N$.

If $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a (PS)$_c$-sequence of $E(u)$, then we have the following lemma.

**Lemma 2.4.** Suppose (D1)–(D2) hold, and $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a bounded (PS)$_c$-sequence of $E(u)$ for some $c \in R$, then there exist a $u \in W^{1,p}_0(\Omega)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{in} \quad W^{1,p}_0(\Omega),$$

and

$$\nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in} \quad \Omega,$$

$$|\nabla u_n|^{p-2} \nabla u_n \rightharpoonup |\nabla u|^{p-2} \nabla u \quad \text{in} \quad L^{\frac{p}{p-1}}(\Omega),$$

$$|\nabla u_n|^{q-2} \nabla u_n \rightharpoonup |\nabla u|^{q-2} \nabla u \quad \text{in} \quad L^{\frac{q}{q-1}}(\Omega).$$
Proof. Combined with condition (D1), (D2) and Lemma 2.2, modifying the proof of Lemma 2.3 in [16], we can easily get the result. □

We still need the following results.

Lemma 2.5. Suppose condition (D1)–(D3) hold, then there exist constants $C_1, C_2 > 0$ such that any (PS)$_c$-sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ of $E(u)$ contains a convergent subsequence when

$$c < \frac{1}{N} S_0^\frac{n}{p} - C_1 \theta^\alpha - C_2 \lambda^\beta,$$

where $\alpha = \frac{\alpha}{q'}, \beta = \frac{p^*}{p-2}$ and $C_1, C_2$ will be determined later.

Proof. Suppose $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a (PS)$_c$-sequence of $E(u)$, i.e.,

$$E(u_n) = c + o(1), \quad E'(u_n) = o(1). \quad (2.1)$$

We now show that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Assume that $\|u_n\|_p \to \infty$ by contradiction. Let $\hat{u}_n = \frac{u_n}{\|u_n\|_p}$, clearly, $\|\hat{u}_n\|_p = 1$ is bounded in $W_0^{1,p}(\Omega)$. We may assume that

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } W_0^{1,p}(\Omega),$$

$$\hat{u}_n \to \hat{u} \text{ in } L^q(\Omega), \quad 1 \leq s < p^*.$$

From (2.1) and $\|u_n\|_p \to \infty$, we have

$$\frac{1}{p} \int_\Omega |\nabla \hat{u}_n|^p dx + \frac{1}{q} \|u_n\|_q^{-p} \int_\Omega |\nabla \hat{u}_n|^q dx - \frac{\theta}{r} \|u_n\|_r^{-p} \int_\Omega V(x) |\hat{u}_n|^r dx$$

$$- \frac{1}{p^*} \|u_n\|_p^{p^* - p} \int_\Omega |\nabla \hat{u}_n|^p dx - \frac{\lambda}{p} \|u_n\|_p^{-p} \int_\Omega F(x,u_n) dx = o(1),$$

$$\int_\Omega |\nabla \hat{u}_n|^p dx + \|u_n\|_q^{-p} \int_\Omega |\nabla \hat{u}_n|^q dx - \theta \|u_n\|_r^{-p} \int_\Omega V(x) |\hat{u}_n|^r dx$$

$$- \|u_n\|_p^{p^* - p} \int_\Omega |\nabla \hat{u}_n|^p dx - \lambda \|u_n\|_p^{-p} \int_\Omega f(x,u_n) u_n dx = o(1),$$

and

$$\int_\Omega V(x) |\hat{u}_n|^q dx \to \int_\Omega V(x) |\hat{u}|^q dx.$$

Together with (D3), we obtain

$$\left(\frac{p^*}{p} - 1\right) \int_\Omega |\nabla \hat{u}_n|^p dx \leq \left(1 - \frac{p^*}{q}\right) \|u_n\|_q^{-p} \int_\Omega |\nabla \hat{u}_n|^q dx + \left(\frac{p^*}{r} - 1\right) \theta \|u_n\|_r^{-p} \int_\Omega V(x) |\hat{u}_n|^r dx$$

$$+ \alpha_3 \|u_n\|_p^{-p} \int_\Omega |\hat{u}|^q dx \to 0 \quad \text{as } n \to \infty,$$

which contradicts the fact $\|\hat{u}_n\|_p = 1$. So $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

By Lemma 2.4, we may assume there exists a $u \in W_0^{1,p}$, and extracting a subsequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}$, and from (D1)–(D2), we may also assume

$$\int_\Omega f(x,u_n) u_n dx = \int_\Omega f(x,u) u dx + o(1),$$

$$\int_\Omega F(x,u_n) dx = \int_\Omega F(x,u) dx + o(1),$$

$$\int_\Omega V(x)|u_n|^q dx = \int_\Omega V(x)|u|^q dx + o(1).$$

(2.2)
A standard argument shows that \( u \) is a critical point of \( E \), i.e., \( u \) is a weak solution to problem (1.1), and

\[
E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{\theta}{r} \int_{\Omega} V(x)|u|^r \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \lambda \int_{\Omega} F(x, u) \, dx = c,
\]

\[
E'(u) = \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |\nabla u|^q \, dx - \theta \int_{\Omega} V(x)|u|^r \, dx - \int_{\Omega} |u|^{p^*} \, dx - \lambda \int_{\Omega} f(x, u) u \, dx = 0.
\]

Now, we only need to show that \( u_n \to u \) in \( W^{1,p}_0 \), as usually we set \( v_n = u_n - u \), combine with Lemma 2.4, Brézis–Lieb’s Lemma (see [20]), (2.1) and (2.2), we have that

\[
\frac{1}{p} \int_{\Omega} |\nabla v_n|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v_n|^q \, dx - \frac{\theta}{r} \int_{\Omega} V(x)|v_n|^r \, dx - \frac{1}{p^*} \int_{\Omega} |v_n|^{p^*} \, dx - \lambda \int_{\Omega} F(x, v_n) \, dx
\]

\[
+ \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx = c + o(1). \tag{2.3}
\]

Similarly, we also have that

\[
\int_{\Omega} |\nabla v_n|^p \, dx + \int_{\Omega} |\nabla v_n|^q \, dx - \theta \int_{\Omega} V(x)|v_n|^r \, dx - \int_{\Omega} |v_n|^{p^*} \, dx - \lambda \int_{\Omega} f(x, v_n) v_n \, dx
\]

\[
+ \int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |\nabla u|^q \, dx - \int_{\Omega} |u|^{p^*} \, dx = o(1). \tag{2.4}
\]

From (2.4) and \( E'(u) = 0 \) we have

\[
\int_{\Omega} |\nabla v_n|^p \, dx + \int_{\Omega} |\nabla v_n|^q \, dx - \int_{\Omega} |v_n|^{p^*} \, dx = o(1).
\]

Assume

\[
\|v_n\|_p^p = a + o(1), \quad \|v_n\|_q^q = b + o(1), \quad \|v_n\|_{p^*}^{p^*} = l + o(1).
\]

By the Sobolev inequality, we have

\[
a \geq Sl^\beta \geq Sa^\beta. \tag{2.5}
\]

If \( a = 0 \), then we complete the proof, if \( a > 0 \), then (2.5) implies that

\[
a \geq S \frac{n}{p^*},
\]

combined with (2.3), (D1)–(D3), and use the Young inequality, as \( n \to \infty \), we have

\[
c = \frac{a}{p} + \frac{b}{q} - \frac{1}{p^*} + \frac{1}{p} \|v_n\|_p^p + \frac{1}{q} \|v_n\|_q^q - \frac{1}{p^*} |v_n|^{p^*} - \frac{\theta}{r} \int_{\Omega} V(x)|v_n|^r \, dx - \lambda \int_{\Omega} F(x, v_n) \, dx
\]

\[
\geq \frac{1}{N} S \frac{n}{p^*} + \frac{1}{N} |v_n|^{p^*} - \theta \left( \frac{1}{r} - \frac{1}{p} \right) \int_{\Omega} V(x)|v_n|^r \, dx - \lambda \int_{\Omega} F(x, v_n) \, dx - \frac{1}{p} \int_{\Omega} f(x, v_n) v_n \, dx + \frac{1}{p^*} \|v_n\|_q^q
\]

\[
\geq \frac{1}{N} S \frac{n}{p^*} + \frac{1}{N} |v_n|^{p^*} - \theta \eta \left( \frac{1}{r} - \frac{1}{p} \right) \|v_n\|_p^p - \lambda \frac{a_3}{p} |v_n|^{p^*} + \frac{1}{q} \|v_n\|_q^q
\]

\[
\geq \frac{1}{N} S \frac{n}{p^*} + \frac{1}{N} |v_n|^{p^*} - \theta \delta \|v_n\|_q^q - \theta \delta \lambda C_0 |v_n|^{p^*} + \left( \frac{1}{q} - \frac{1}{p} \right) \|v_n\|_q^q
\]

\[
= \frac{1}{N} S \frac{n}{p^*} + \frac{1}{N} |v_n|^{p^*} - C_1 \theta \delta \frac{a^3}{p} - \lambda C_0 |v_n|^{p^*}
\]

where we choose \( \delta = \frac{1}{r} - \frac{1}{p} \), then we have \( C_1 = C_1(r, q, p, \eta, |\Omega|) > 0 \) and \( C_0 = C_0(s, a_3, p^*, |\Omega|) > 0 \) are constants independent of \( \theta, \lambda \).
Consider the function 
\[ g(x) = \frac{1}{N} x^p - \lambda C_0 x^q, \]
the function obtains its minimum (for \( x > 0 \)) at point \( x_0 = (\frac{C_0 N}{p^*})^\frac{1}{p^* - q} \), then we have 
\[ g(x) \geq g(x_0) = -C_2 \lambda \frac{p^*}{p^* - q}, \]
where 
\[ C_2 = C_0 \frac{p^* - q}{p^*} \left( \frac{C_0 N}{p^*} \right)^\frac{1}{p^* - q} > 0. \]
Then we obtain
\[ c \geq \frac{1}{N} S \eta - C_1 \theta \frac{q}{p^*} - C_2 \lambda \frac{p^*}{p^* - q}, \]
which contradicts the assumption. So we have \( a = 0 \), and complete the proof. \( \square \)

Now we set \( \lambda_0 = \left( \frac{1}{C_2 N} \frac{p^*}{p^* - q} \right) \), we have \( \frac{1}{N} S \eta - C_1 \theta \frac{q}{p^*} - C_2 \lambda_0 \frac{p^*}{p^* - q} > 0 \) for any \( \lambda \in (0, \lambda_0) \).

The following is the classical Deformation Lemma:

**Lemma 2.6.** (See [21].) Let \( f \in C^1(X, R) \) and satisfy the (PS)\(^{-}\)-condition. If \( c \in R \) and \( N \) is any neighborhood of \( K_c \subset \{ u \in X \mid f(u) = c, f'(u) = 0 \} \), there exists \( \eta(t, x) \equiv \eta_1(x) \subset C([0, 1] \times X, X) \) and constants \( \bar{\epsilon} > \epsilon > 0 \) such that

1. \( \eta_0(x) = x \) for all \( x \in X \),
2. \( \eta_1(x) = x \) for all \( x \in f^{-1}[c - \bar{\epsilon}, c + \bar{\epsilon}] \),
3. \( \eta_1(x) \) is a homeomorphism of \( X \) onto \( X \) for all \( t \in [0, 1] \),
4. \( f(\eta_1(x)) \leq f(x) \) for all \( x \in X \), \( t \in [0, 1] \),
5. \( \eta_1(A_{c + \epsilon} - N) \subset A_{c + \epsilon} \), where \( A_{c} = \{ x \in X \mid f(x) \leq c \} \) for any \( c \in R \),
6. \( K_c = \emptyset, \eta_1(A_{c + \epsilon}) \subset A_{c - \epsilon} \),
7. if \( f \) is even, \( \eta_1 \) is odd in \( x \).

**Remark 2.7.** Lemma 2.6 is also true if \( f \) satisfies the (PS)\(_k\)-condition for \( c < c_0 \) for some \( c_0 \in R \).

At the end of this section, we recall some concepts in minimax theory.

Let \( X \) be a Banach space, and
\[ \Sigma = \{ A \subset X \mid |0| \} \]
as the closed, \( -A = A \}
and
\[ \Sigma_k = \{ A \in \Sigma \mid \gamma(A) \geq k \}, \]
where \( \gamma(A) \) is the \( Z_2 \) genus of \( A \), that is
\[ \gamma(A) = \begin{cases} \infty & \text{if there exist odd, continuous } h: A \to R^n \setminus \{0\}, \\ +\infty & \text{if it doesn't exist odd, continuous } h: A \to R^n \setminus \{0\}, \forall n \in Z_+, \\ 0 & \text{if } A = \emptyset. \end{cases} \]
The main properties of genus are contained in the following lemma.

**Lemma 2.8.** (See [22].) Let \( A, B \in \Sigma \). Then

1. if there exists \( f \in C(A, B) \), odd, then \( \gamma(A) \leq \gamma(B) \).
2. if \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \).
3. if there exists an odd homeomorphism between \( A \) and \( B \), then \( \gamma(A) = \gamma(B) \).
4. if \( S^{N - 1} \) is the sphere in \( R^N \), then \( \gamma(S^{N - 1}) = N \).
5. \( \gamma(A \cup B) \leq \gamma(A) + \gamma(B) \).
6. if \( \gamma(A) < \infty \), then \( \gamma(A - B) \geq \gamma(A) - \gamma(B) \).
7. if \( A \) is compact, then \( \gamma(A) < \infty \), and there exists \( \delta > 0 \) such that \( \gamma(A) = \gamma(N_\delta(A)) \), where \( N_\delta(A) = \{ x \in X \mid d(x, A) \leq \delta \} \).
8. if \( X_0 \) is a subspace of \( X \) with codimension \( k \), and \( \gamma(A) > k \), then \( A \cap X_0 \neq \emptyset \).
3. Proof of Theorem 1.1

We will prove the existence of infinitely many solutions for problem (1.1) in this section. We try to use Lusternik–Schnirelman’s theory for \(Z_2\)-invariant functional (see [22]). But since the functional \(E(u)\) defined in Section 2 is not bounded from below, so we follow [11] (or see [16]) to consider a truncated functional \(E_\infty(u)\) which will be constructed later, since the nonlinearities are more complicated than it in [11] or [16], we need more careful analysis in the construction of \(E_\infty(u)\).

At first, let’s consider the functional \(E(u)\), using Sobolev’s inequality with the hypothesis \(1 < r < q < p < N\), we obtain

\[
E(u) \geq \frac{1}{p} \|u\|^p_p - \frac{1}{p^* S^p} \|u\|^{p^*}_p - \frac{\eta \theta}{r} \|u\|^{r}_r - \lambda \int_\Omega F(x, u) \, dx
\]

\[
\geq \frac{1}{p} \|u\|^p_p - \frac{1}{p^* S^p} \|u\|^{p^*}_p - \frac{\eta \theta}{r} |\Omega| \frac{p^*}{r} \|u\|^{p^*}_p - \lambda \frac{\alpha_1}{\zeta} |\Omega| \frac{p^*}{\zeta} S^{-\frac{2}{\zeta}} \|u\|^{p^*}_p - \lambda \frac{\alpha_2}{\xi} \frac{\alpha_3}{\xi} S^{-\frac{2}{\xi}} \|u\|^{p^*}_p
\]

\[
= C_3 \|u\|^p_p - C_4 \|u\|^{p^*}_p - C_5 \|u\|^{p^*}_p - C_6 \lambda \|u\|^{p^*}_p - C_7 \lambda \|u\|^{p^*}_p
\]

where \(C_3 = \frac{1}{p}\), \(C_4 = \frac{1}{p^* S^p}\), \(C_5 = \frac{\eta \theta}{r} |\Omega| \frac{p^*}{r}\), \(C_6 = \frac{\alpha_1}{\zeta} |\Omega| \frac{p^*}{\zeta} S^{-\frac{2}{\zeta}}\), \(C_7 = \frac{\alpha_2}{\xi} \frac{\alpha_3}{\xi} S^{-\frac{2}{\xi}}\) are all positive constants.

We now consider function

\[
h(x) = C_3 x^p - C_4 x^{p^*} - C_5 \theta x^r - C_6 \lambda x^s - C_7 \lambda x^t, \quad x > 0.
\]

By the hypothesis \(1 < r < p < N\), we easily know that there exists a positive constant \(\lambda^* \leq \lambda_0\) such that for any \(\lambda \in (0, \lambda^*)\), \(h(x) = C_3 x^p - C_4 x^{p^*} - C_5 \theta x^r - C_6 \lambda x^s - C_7 \lambda x^t\) can take positive value for some \(x > 0\). Then for any \(\lambda \in (0, \lambda^*)\), there exists a \(\theta^* = \theta^*(\lambda) > 0\) such that for any \(\lambda \in (\theta^*, \theta^*)\), the following results hold:

(a) \(h(x)\) reaches its positive maximum;

(b) \(\frac{1}{p} S^p - C_1 \theta^0 x^r - C_2 \lambda x^s > 0\), where \(C_1, C_2, \alpha, \beta\) are given in Lemma 2.5.

From the structure of \(h(x)\), we see that there are finite positive solutions of \(h(x) = 0\), assume the positive solutions as follows

\[
0 < R_1 < R_2 < \cdots < R_m < \infty.
\]

Then we can easily know that

\[
h(x) \begin{cases} 
< 0, & x \in (0, R_1) \cup (R_2, R_3) \cup \cdots \cup (R_m, \infty), \\
> 0, & x \in (R_1, R_2) \cup (R_3, R_4) \cup \cdots \cup (R_{m-1}, R_m). 
\end{cases} \quad (3.1)
\]

From now on we denote \((0, R_1) \cup (R_2, R_3) \cup \cdots \cup (R_m, \infty)\) by \(A\), and denote \(B = A \setminus (R_m, \infty)\). We let \(\tau : R^+ \to [0, 1]\) be \(C^\infty\) function such that

\[
\tau(x) = 1 \quad \text{if} \quad x \in B,
\]

\[
\tau(x) = 0 \quad \text{if} \quad x \in (R_m, \infty).
\]

Let \(\phi(u) = \tau(\|u\|_p)\), we consider the truncated functional

\[
E_\infty(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^{q} \, dx - \frac{\theta}{r} \int_\Omega V(x) |u|^r \, dx - \frac{1}{p^* S^p} \int_\Omega |u|^{p^*} \phi(u) \, dx - \lambda \int_\Omega F(x, u) \phi(u) \, dx
\]

similar as above, we consider the function

\[
\tilde{h}(x) = C_3 x^p - C_4 x^{p^*} \tau(x) - C_5 \theta x^r - C_6 \lambda x^s \tau(x) - C_7 \lambda x^t \tau(x),
\]

and have that

\[
E_\infty(u) \geq \tilde{h}(\|u\|_p). \quad (3.2)
\]

By farther analysis, we can see \(\tilde{h}(x) \geq h(x)\), for all \(x > 0\); and \(\tilde{h}(x) = h(x)\), for \(x \in B\); and \(\tilde{h}(x) \geq 0\), for \(x > R_m\). So we have that \(E(u) = E_\infty(u)\) when \(\|u\|_p \in B\), and since \(\tau \in C^\infty\), we get \(E_\infty(u) \in C^1(W_{0}^{1,p}(\Omega), R)\). Also we obtain the following results.

**Lemma 3.1.** (1) If \(E_\infty(u) < 0\), then \(\|u\|_p \in B\), and \(E(v) = E_\infty(v)\) for all \(v\) in a small enough neighborhood of \(u\).

(2) For any \(\lambda \in (0, \lambda^*)\), there exists a \(\theta^* > 0\), such that when \(\theta \in (0, \theta^*)\), \(E_\infty(u)\) satisfies the (PS)_c-condition for \(c < 0\).
Proof. We prove (1) by contradiction, assume $E_{\infty}(u) < 0$ and $\|u\|_p \in R^+ \setminus B$. Then if $\|u\|_p \in R^+ \setminus A$, by (3.1), (3.2), we see that

$$E_{\infty}(u) \geq \bar{h}(\|u\|_p) \geq h(\|u\|_p) \geq 0.$$ 

If $\|u\|_p \in (R_m, \infty)$, by (3.2) and the above analysis, we also have that

$$E_{\infty}(u) \geq \bar{h}(\|u\|_p) \geq 0.$$ 

Thus $\|u\|_p \in B$, (1) holds.

Now, we prove (2), let $\theta^*$ be related to $\lambda \in (0, \lambda^*)$ as above. If $c < 0$ and $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a $(PS)_c$-sequence of $E_{\infty}$, then we may assume that $E_{\infty}(u_n) < 0$ and $E'_{\infty}(u_n) = o(1)$, by (1), $\|u_n\|_p \in B$, hence $E(u_n) = E_{\infty}(u_n)$ and $E'(u_n) = E'_{\infty}(u_n)$.

Since (b) holds when $\theta \in (0, \theta^*)$. By Lemma 2.5, $E(u)$ satisfies the $(PS)_c$-condition for $c < 0$. Thus $E_{\infty}(u)$ satisfies the $(PS)_c$-condition for $c < 0$, (2) holds.

Now we prove our main result via genus.

Proof of Theorem 1.1. Let $c_k = \inf_{u \in E_{\infty}(u)} \sup_{u \in A} E_{\infty}(u)$, $K_c = \{u \in W^{1,p}_0(\Omega) \mid E_{\infty}(u) = c, \ E'_{\infty}(u) = 0\}$, and suppose that $\theta \in (0, \theta^*)$, $\theta^*$ is as above.

We claim that if $k, l \in N$ are such that $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \geq l + 1$.

In fact, we assume

$$E_{\infty}^{-\epsilon} = \{u \in W^{1,p}_0(\Omega) \mid E_{\infty}(u) \leq -\epsilon\},$$

we will show for any $k \in N$, there exist an $\epsilon = \epsilon(k) > 0$, such that

$$\gamma(E_{\infty}^{-\epsilon}(u)) \geq k.$$ 

Fix $k \in N$, denote $X_k$ be a $k$-dimensional subspace of $W^{1,p}_0(\Omega)$, choose $u \in X_k$, with $\|u\|_p = 1$, for $0 < \rho < R_1$, we have

$$E(\rho u) = E_{\infty}(\rho u) = \frac{1}{p} \rho^p + \frac{\rho^q}{q} \int_{\Omega} |\nabla u|^q dx - \frac{\theta \rho^p}{r} \int_{\Omega} V(x)|u|^r dx - \frac{\rho^p}{p^*} \int_{\Omega} |u|^r dx - \lambda \int_{\Omega} F(x, \rho u) dx. \ (3.3)$$

For $X_k$ is a finite dimension space, all the norms in $X_k$ are equivalent. So we can define

$$\alpha_k = \sup_{u \in X_k} \left\{ \int_{\Omega} |\nabla u|^q dx \mid \|u\|_p = 1 \right\} < \infty, \ (3.4)$$

$$\beta_k = \inf_{u \in X_k} \left\{ \int_{\Omega} |u|^r dx \mid \|u\|_p = 1 \right\} > 0, \ (3.5)$$

$$\gamma_k = \inf_{u \in X_k} \left\{ \int_{\Omega} |u|^r dx \mid \|u\|_p = 1 \right\} > 0, \ (3.6)$$

from (3.3)–(3.6), we have

$$E_{\infty}(\rho u) \leq \frac{1}{p} \rho^p + \alpha_k \frac{\rho^q}{q} - \sigma \gamma_k \rho^r \frac{\theta \rho^p}{r} - \beta_k \frac{\rho^p}{p^*}.$$ 

For any $\epsilon > 0$ and a $0 < \rho < R_1$, such that $E_{\infty}(\rho u) \leq -\epsilon$ for $u \in X_k$, $\|u\|_p = 1$, let $S_\rho = \{u \in W^{1,p}_0(\Omega) \mid \|u\|_p = \rho\}$, then $S_\rho \cap X_k \subset E_{\infty}^{-\epsilon}$. By Lemma 2.8, we obtain that

$$\gamma(E_{\infty}^{-\epsilon}(u)) \geq \gamma(S_\rho \cap X_k) = k. \ (3.7)$$

Since $E_{\infty}$ is continuous and even, with (3.7), we have $E_{\infty}^{-\epsilon} \in \Sigma_c$ and $c = c_k \leq -\epsilon < 0$. As $E_{\infty}$ is bounded from below, we see that $c = c_k > -\infty$ (this is the main reason that we consider $E_{\infty}$ instead of $E$). Then by Lemma 2.5, $E_{\infty}$ satisfies the $(PS)_c$-condition and it is easy to see that $K_c$ is a compact set.

Now we prove our claim by contradiction, suppose on the contrary $\gamma(K_c) \leq l$. By Lemma 2.8, there is a closed and symmetric set $U$ with $K_c \subset U$ and $\gamma(U) \leq l$. Since $c < 0$, we also can assume that the closed set $U \subset E_{\infty}^{-\epsilon}$. By Lemma 2.6, there exists an odd homeomorphism

$$\eta : W^{1,p}_0(\Omega) \rightarrow W^{1,p}_0(\Omega)$$

such that $\eta(E_{\infty}^{-\epsilon} - U) \subset E_{\infty}^{-\epsilon}$ for some $0 < \delta < -c$. 

Author's personal copy
From the definition of \( c = ck + l \), we know that there is an \( A \in \Sigma_{k+l} \) such that

\[
\sup_{u \in A} E_\infty(u) < c + \delta
\]

i.e., \( A \subset E_{\infty}^{c+\delta} \), and

\[
\eta(A - U) \subset \eta(E_{\infty}^{c+\delta} - U) \subset E_{\infty}^{c-\delta},
\]

that means

\[
\sup_{u \in \eta(A - U)} E_\infty(u) \leq c - \delta.
\]

Again by Lemma 2.8, we have

\[
\gamma(\eta(A - U)) \geq \gamma(\bar{A} - U) \geq \gamma(A) - \gamma(U) \geq k.
\]

Thus we have \( \eta(A - U) \in \Sigma_{k} \) and \( \sup_{u \in \eta(A - U)} E_\infty(u) \geq c_k = c \), which contradicts to (3.8). So we have proved our claim.

Now let’s complete the proof of Theorem 1.1. If for all \( k \in N \), we have \( \Sigma_{k+1} \subset \Sigma_{k} \), \( c_k \leq c_{k+1} < 0 \). If all \( c_k \) are distinct, then \( \gamma(K_c) \geq 1 \), and we see that \( \{c_k\} \) is a sequence of distinct negative critical values of \( E_\infty \); if for some \( k_0 \), there is an \( l \geq 1 \) such that \( c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l} \), then by the claim, we have

\[
\gamma(K_c) \geq l + 1,
\]

which shows that \( K_c \) contains infinitely many distinct elements.

By Lemma 3.1, we know \( E(u) = E_\infty(u) \) when \( E_\infty(u) < 0 \), so we show that there are infinitely many critical points of \( E(u) \). Theorem 1.1 is proved. \( \square \)

4. Some Results of Problem (1.1) for the Case \( 1 < q < p < r < p^* \)

In this section, we will extend some results in [11] for problem \( (E_{\theta,0}) \) to (1.1). We study problem (1.1) with \( 1 < q < p < r < p^* \), and will show that there exists a nontrivial solution \( u \) of (1.1) by the following general version of the Mountain Pass Lemma (see [23]).

**Lemma 4.1.** Let \( E \) be a functional on a Banach space \( X, E \in C^1(X, R) \). Let us assume that there exist \( \rho, R > 0 \) such that

(i) \( E(u) > \rho, \forall u \in X \) with \( \|u\|_p = R \),

(ii) \( E(0) = 0, \) and \( E(v_0) < \rho \) for some \( v_0 \in X \), with \( \|v_0\|_p > R \).

Let us define \( \Gamma = \{y \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = v_0\} \), and

\[
\alpha = \inf_{y \in \Gamma} \max_{t \in [0, 1]} E(y(t)).
\]

Then there exists a sequence \( \{u_n\} \subset X \), such that \( E(u_n) \to \alpha \), and \( E'(u_n) \to 0 \) in \( X^* \) (dual of \( X \)) as \( n \to \infty \).

We modify condition (D2) and (D3) with

\[
\text{(D2)}' \quad |f(x, t)| \leq a_1|t|^\gamma - 1 + a_2|t|^\xi - 1, \text{ for } \forall x \in \Omega, t \in R, \text{ where } a_1, a_2 > 0 \text{ and } p < \gamma, \xi < p^*.
\]

\[
\text{(D3)}' \quad f(x, t) \text{ satisfies the Ambrosetti–Rabinowitz condition: There exists a } \theta \geq p, \text{ such that}
\]

\[
f(x, t)t - \partial F(x, t) \geq 0, \quad \forall x \in \Omega, t \in R
\]

where \( F(x, t) = \int_0^t f(x, \tau) d\tau \).

Combining (D3)' with \( 1 < q < p < r < p^* \), we can easy to see \( E(u) \) verifies (i) and (ii).

Now similar to Lemma 2.5 in Section 2, we have the following result.

**Lemma 4.2.** Suppose condition (D1), (D2)' and (D3)' hold, then any (PS)_c-sequence \( \{u_n\} \subset W_0^{1,p}(\Omega) \) of \( E(u) \) contains a convergent subsequence when

\[
c < \frac{1}{N} \frac{1}{\bar{\tau}}.
\]

Author's personal copy
\textbf{Proof.} Combining assumptions (D1), (D2)', (D3)' with $1 < q < p < r < p^*$, the proof is similar to Lemma 2.5. \qed

Then we have

\textbf{Theorem 4.3.} If $1 < q < p < r < p^*$, (D1), (D2)', (D3)' and (D4) hold, then there is a $\theta > 0$, such that for any $\theta > \theta_*$ and any $\lambda > 0$, problem (1.1) has a nontrivial solution.

\textbf{Proof.} From (4.1) and (4.2), we only need to show

$$\alpha < \frac{1}{N} S^\frac{N}{p},$$  \hfill (4.3)

then Lemma 4.1 and Lemma 4.2 give the existence of the critical point of $E$.

To obtain (4.3), let us choose $u_0 \in W^{1,p}_0(\Omega)$, with

$$|u_0|^p = 1, \quad \lim_{t \to \infty} E(tu_0) = -\infty,$$

then there exists a $t_{0\lambda} > 0$ such that $sup_{t > 0} E(tu_0) = E(t_{0\lambda}u_0)$ holds, and then $t_{0\lambda}$ satisfies

$$0 = t_{0\lambda}^{-1} \int_\Omega |\nabla u_0|^p dx + t_{0\lambda}^{q-1} \int_\Omega |\nabla u_0|^q dx - \theta t_{0\lambda}^{-1} \int_\Omega V(x)|u_0|^\theta dx - \lambda \int_\Omega f(x, t_{0\lambda}) u_0 dx + \lambda \int_\Omega f(x, t_{0\lambda}) u_0 dx$$

then we get

$$t_{0\lambda}^{p-r} \int_\Omega |\nabla u_0|^p dx + t_{0\lambda}^{q-r} \int_\Omega |\nabla u_0|^q dx - t_{0\lambda}^{p-r} - t_{0\lambda}^{q-r} \lambda \int_\Omega f(x, t_{0\lambda}) u_0 dx = \lambda \int_\Omega V(x)|u_0|^\theta dx$$

from (D2)' and $1 < q < p < r < p^*$, we get $t_{0\lambda} \to 0$ as $\theta \to \infty$. Then there exists $\theta_* > 0$ such that for any $\theta > \theta_*$ and any $\lambda > 0$, we have

$$sup_{t \geq 0} E(tu_0) < \frac{1}{N} S^\frac{N}{p}.$$

Now we take $v_0 = t_0 u_0$ with $t_0$ large enough to verify $E(v_0) < 0$, we get

$$\alpha \leq \max_{t \in [0, 1]} E(\gamma_0(t))$$

where $\gamma_0(t) = tv_0$. Therefore,

$$\alpha \leq \sup_{t \geq 0} E(tv_0) < \frac{1}{N} S^\frac{N}{p}$$

then we have proved (4.3), that's complete the proof. \qed

\textbf{Remark 4.4.} It is easy to see that Theorem 4.3 also holds for the case $1 < q < r < p < N$.

Now let's assume $1 < q < \frac{N(p-1)}{N-1} < p \leq \max\{|p, p^* - \frac{q}{p^*+1}\} < r < p^*$, and define, for $\epsilon > 0$,

$$u_\epsilon(x) = \frac{\psi(x)}{\epsilon + |x|^{\frac{p}{p^*+1}}}, \quad v_\epsilon(x) = \frac{u_\epsilon(x)}{|u_\epsilon(x)|^{p*}}$$

where $\psi(x) \in C^\infty_0(B(0,2R))$ is such that $0 \leq \psi(x) \leq 1$, and $\psi(x) \equiv 1$ on $B(0, R)$.

We obtain the following estimates (see [24]).

$$\int_\Omega |u_\epsilon|^t dx = \begin{cases} K_1 \epsilon^{\frac{N(p-1) - t(N-p)}{p}} + O(1), & t > \frac{N(p-1)}{N-p}, \\ K_1 \ln \epsilon + O(1), & t = \frac{N(p-1)}{N-p}, \\ O(1), & t < \frac{N(p-1)}{N-p}, \end{cases}$$  \hfill (4.4)

$$\int_\Omega |\nabla u_\epsilon|^t dx = \begin{cases} K_2 \epsilon^{\frac{t(N(p-1) - t(N-p))}{p}} + O(1), & t > \frac{N(p-1)}{N-p}, \\ K_2 \ln \epsilon + O(1), & t = \frac{N(p-1)}{N-p}, \\ O(1), & t < \frac{N(p-1)}{N-p}, \end{cases}$$  \hfill (4.5)
In particular, we have
\[ \int_{\Omega} |\nabla u_\varepsilon|^p \, dx = K_2 \varepsilon^{\frac{p-N}{p}} + O(1) \] (4.6)
and
\[ \left( \int_{\Omega} |u_\varepsilon|^p \, dx \right)^{\frac{p}{p'}} = K_3 \varepsilon^{\frac{p-N}{p}} + O(1), \] (4.7)
\[ \int_{\Omega} |u_\varepsilon|^p \, dx = \begin{cases} K_1 \varepsilon^{\frac{p-N}{p}} + O(1), & p^* < N, \\ K_1 |\ln \varepsilon| + O(1), & p^2 = N, \\ O(1), & p^2 > N \end{cases} \] (4.8)
where \( K_1, K_2, K_3 \) are positive constants independent of \( \varepsilon \), and \( S = \frac{K_2}{K_3} \) is the best Sobolev constant given in Section 2.

Then we can prove the following stronger result.

**Theorem 4.5.** If \( 1 < q < \frac{N(p-1)}{N-1} < p \leq \max \{ p, p^* - \frac{N}{p-1} \} < \frac{p^*}{2} \), (D1), (D2)', (D3)' and (D4) hold, then for any \( \theta > 0, \lambda > 0 \), problem (1.1) has a nontrivial solution.

**Proof.** Set
\[ g(t) = E(tv_\varepsilon) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p \, dx + \frac{t^q}{q} \int_{\Omega} |\nabla v_\varepsilon|^q \, dx - \frac{\theta t^r}{r} \int_{\Omega} V(x)|v_\varepsilon|^r \, dx - \frac{t^{p^*}}{p^*} + \lambda \int_{\Omega} F(x, t v_\varepsilon) \, dx. \]

Then there exists a \( t_\varepsilon > 0 \) such that \( \sup_{t \geq 0} E(tv_\varepsilon) = E(t_\varepsilon v_\varepsilon) \) hold, and then \( t_\varepsilon \) satisfies
\[ 0 = g'(t_\varepsilon) = t_\varepsilon^{p-1} \int_{\Omega} |\nabla v_\varepsilon|^p \, dx + t_\varepsilon^{q-p} \int_{\Omega} |\nabla v_\varepsilon|^q \, dx - \theta t_\varepsilon^{r-1} \int_{\Omega} V(x)|v_\varepsilon|^r \, dx - t_\varepsilon^{p^*-1} + \lambda \int_{\Omega} f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \, dx. \] (4.9)
then we have
\[ \int_{\Omega} |\nabla v_\varepsilon|^p \, dx + t_\varepsilon^{q-p} \int_{\Omega} |\nabla v_\varepsilon|^q \, dx > t_\varepsilon^{p^*-p}. \]

From (4.4)-(4.8) we can know
\[ \int_{\Omega} |\nabla v_\varepsilon|^p \, dx = S + O(\varepsilon^{\frac{N-p}{p}}), \quad \int_{\Omega} |\nabla v_\varepsilon|^q \, dx = O(\varepsilon^{\frac{p(N-p)}{2}}). \]
Let \( \varepsilon \) be small enough, then there exists \( T_1 > 0 \) such that
\( t_\varepsilon \leq T_1. \)

Also, from (4.9) we obtain
\[ \int_{\Omega} |\nabla v_\varepsilon|^p \, dx < \theta t_\varepsilon^{r-p} \| V(x) \|_{\infty} \int_{\Omega} |v_\varepsilon|^r \, dx + t_\varepsilon^{p^*-p} + \lambda t_\varepsilon^{p^*-p} \int_{\Omega} f(x, t_\varepsilon v_\varepsilon) v_\varepsilon \, dx. \] (4.10)

From (4.4)-(4.8), (D2)' and (4.10), choose \( \varepsilon \) small enough, then there exists \( T_2 > 0 \) such that
\( t_\varepsilon \geq T_2. \)

Now we consider
\[ h(t) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p \, dx - \frac{t^{p^*}}{p^*}. \]
the function attains its maximum at \( t_0 = \left( \int_{\Omega} |\nabla v_\varepsilon|^p \, dx \right)^{\frac{1}{p'}} \), and again combine with (4.4)-(4.8), we have
\[ g(t) \leq h(t) + \frac{t^q}{q} \int_\Omega |\nabla v(t\epsilon)|^q dx - \frac{\theta T^r}{r} \int_\Omega V(x)|v(t\epsilon)|^r dx \]
\[ \leq h\left( \left( \int_\Omega |\nabla v(t\epsilon)|^p dx \right)^{\frac{1}{p}} \right) + \frac{(T_1)^q}{q} \int_\Omega |\nabla v(t\epsilon)|^q dx - \frac{\theta (T_2)^r}{r} \int_\Omega |v(t\epsilon)|^r dx \]
\[ \leq \frac{1}{N} S_{S_1} + c_8 \epsilon^{\frac{N-p}{p}} + c_9 \epsilon^{\frac{q(N-p)}{p^*}} - c_{10}\epsilon^{\frac{p-1}{p}(N-r\frac{N-p}{p^*})} \]

where \(c_8, c_9, c_{10}\) are positive constants independent of \(\epsilon\). Since \(1 < q < \frac{N(p-1)}{N-p} < p \leq \max\{p, p^* - \frac{q}{p^*}\} < r < p^*,\) we obtain that
\[ \frac{N-p}{p} > \frac{q(N-p)}{p^2} > \frac{p-1}{p} \left( N - r \frac{N-p}{p} \right), \]
then choose \(\epsilon\) small enough, we get \(g(t) = \sup_{t \geq 0} E(v(t\epsilon)) < \frac{1}{N} S_{S_1}\), by Lemma 4.1 and Lemma 4.2, we complete the proof. \(\square\)

Acknowledgment

The authors are grateful to an anonymous referee for several comments and suggestions which contributed to improve this paper.

References