Bifurcation analysis and control of chaos for a hybrid ratio-dependent three species food chain

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, a hybrid ratio-dependent three species food chain model with time delay is studied by using the theory of functional differential equation and Hopf bifurcation, the condition on which positive equilibrium exists and the quality of Hopf bifurcation are given. Chaotic solutions are observed and are controlled by delay parameter. Finally, we indicate that when the delay passes through certain critical values, chaotic oscillation is converted into a stable state or a stable periodic orbit.

1. Introduction

Recently, the topics of chaos and chaotic control are growing rapidly in many different fields such as biological systems, ecological and chemical systems, and so forth [3,8,9,12]. The desirability of chaos depends on the particular application. Therefore, it is important that the chaotic response of a system can be controlled. Many researchers have proposed chaos control and synchronization schemes in recent years [1,2,4,5,7]. For example, Song and Wei in [11] investigated the chaos phenomena of Chen's system using the method of delayed feedback control. Their results show that, when the controlling parameter \( K \) to be some value, taking the delay \( s \) as the bifurcation parameter, when \( s \) pass through a certain critical value, the stability of the equilibrium will be changed from unstable to stable, chaos vanish and a periodic solution emerge. Wang and Pang in [13] propose a three species food chain model as following

\[
\begin{align*}
X(T) &= r_1 X(T) - a_1 X^2(T) - a_2 X(T) Y(T), \\
Y(T) &= r_2 Y(T) - d_1 Y^2(T) - \frac{b Y(T)}{X(T)} Z(T), \\
Z(T) &= \frac{b Y(T)}{X(T)} Z(T) - d_2 Z(T),
\end{align*}
\]

where \( X, Y \) and \( Z \) stand for the population (or density) of the prey, predator and top predator, respectively. \( r_1, r_2, a_1, a_2, d_1, b, \delta, k \) and \( d_2 \) are positive constants, \( r_1, r_2 \) are the growth rate of prey and predator, respectively. \( d_1, d_2 \) are the death rate of predator and top predator, respectively. \( \frac{b}{a_2} \) is the carrying capacity of prey in the absence of predation. \( \frac{b}{a_1} \) is the carrying capacity of the middle predator. Since \( a_2 X \) and \( \frac{b Y}{X} \) are the functional response of Holling type I and II respectively, they called system (1.1) a hybrid ratio-dependent food chain. We choose

\[
t = r_1 T, \quad X(t) = \frac{r_1}{a_1 X(T)}, \quad Y(t) = \frac{k}{r_1 \delta} Y(T), \quad Z(t) = \frac{1}{r_1 \delta} Z(T),
\]

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the system (1.1) takes the nondimensional form
\[
\begin{align*}
\dot{x}(t) &= 1 - x(t) + ax(t)y(t), \\
\dot{y}(t) &= ry(t) - \alpha x(t)y^2(t) - \frac{by(t)}{1 + my(t)}z(t), \\
\dot{z}(t) &= \frac{by(t)}{1 + my(t)}z(t) - dz(t),
\end{align*}
\] (1.2)
where \( a = \frac{a_0}{C_0}, r = \frac{r_1}{C_0}, c = \frac{a_1d_1}{C_0}, m = \frac{m_1}{C_0}, d = \frac{d_2}{C_0} \). Taking \( b \) as the bifurcation parameter, they get the condition under which Hopf bifurcation occurs. At the end of their paper they give an example showing that when \( m \) taking some values, the positive equilibrium will lose its stability, and a periodic solution occurs. Further increasing the value of \( m \), they get a typical chaotic phenomena. In this paper, we add a time-delayed force \( K(y(t) - y(t - \tau)) \) to the second equation of system (1.2), that is the following delayed feedback control system
\[
\begin{align*}
\dot{x}(t) &= 1 - x(t) + ax(t)y(t), \\
\dot{y}(t) &= ry(t) - \alpha x(t)y^2(t) - \frac{by(t)}{1 + my(t)}z(t) + K(y(t) - y(t - \tau)), \\
\dot{z}(t) &= \frac{by(t)}{1 + my(t)}z(t) - dz(t),
\end{align*}
\] (1.3)
where \( K \) denote the capture coefficient when \( K < 0 \) (or release coefficient when \( K > 0 \)). By choosing \( \tau \) as bifurcation parameter, we get the condition under which Hopf bifurcation occurs. At last we will give a example showing that when \( K \) fixed, with \( \tau \) increasing, the stability of the positive equilibrium will changed, and chaos vanish, a periodic solution occurs. However, to the best of our knowledge, it is the first try to introduce a time-delayed force to control the chaos of a tree species food chain. This paper is organized as follows: In Section 2, we first focus on the stability and Hopf bifurcation of the positive equilibrium. In Section 3, we derive the direction and stability of Hopf bifurcation by using normal form and central manifold theory. Finally in Section 4, an example is given for showing the effects of chaotic control.

2. Stability analysis and Hopf bifurcation

In this section, by analyzing the characteristic equation of the linearized system of system (1.3) at the positive equilibrium, we investigate the stability of the positive equilibrium and the existence of the local Hopf bifurcations occurring at the positive equilibrium. To guarantee that system (1.3) has always a positive equilibrium, throughout this section, we assume that the coefficients of system (1.3) satisfy the following condition:
\[(H_1) \quad b > b_0 = \frac{d(1 + my_1)}{y_1},\]
where \( y_1 = \frac{1}{\tau C_0} \). Clearly, under the hypothesis \((H_1)\), system (1.3) has a unique positive equilibrium \( E^\ast(x^\ast, y^\ast, z^\ast) \), where
\[
\begin{align*}
x^\ast &= \frac{b - md}{b - md - ad}, \quad y^\ast = \frac{d}{b - md}, \quad z^\ast = \frac{rb - rd(m + a) - dc}{(b - md)(b - ma - md)},
\end{align*}
\]
Under the hypothesis \((H_1)\), the linearized system of (1.3) is
\[
\dot{u} = Au + Bu(t - \tau),
\] (2.1)
where \( u(t) = (x, y, z)^T, A = (a_{ij})_{3 \times 3}, B = (b_{ij})_{3 \times 3}; a_{11} = -1 + ay, a_{12} = ax, a_{21} = -cy^2, a_{22} = K + r - 2cyy^2 - \frac{by}{1 + my}, a_{32} = -K, \) all the others of \( a_{ij} \) and \( b_{ij} \) are 0. The characteristic equation of system (2.1) is
\[
\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + (K\lambda^2 + a_4\lambda)e^{-\tau\lambda} = 0,
\] (2.2)
where \( a_1 = -a_{11} - a_{22}, a_2 = a_{11}a_{22} - a_{12}a_{21} - a_{22}a_{12}, a_3 = a_{11}a_{22}a_{32}, a_4 = -a_{11}K \).
The equilibrium \( E^\ast(x^\ast, y^\ast, z^\ast) \) is stable if all roots of (2.2) have negative real parts. Thus, we need to investigate the distribution of roots of Eq. (2.2). Obviously, \( i\omega(\omega > 0) \) is a root of Eq. (2.2) if and only if \( \omega \) satisfies
\[
-i\omega^3 - a_1\omega^2 + ia_2\omega + a_3 + (-K\omega^2 + ia_4\omega)(\cos\tau - i\sin\tau) = 0.
\] (2.3)
Separating the real and imaginary parts, we have
\[
\begin{align*}
-\omega^3 + a_2\omega &= -a_4\cos\tau - K\omega^2\sin\tau, \\
a_1\omega^2 + a_3 &= -a_4\omega\sin\tau + K\omega^2\cos\tau,
\end{align*}
\] (2.4)
which implies
\[
\omega^6 + (a_1^2 - K^2 - 2a_2)\omega^4 + (a_1^2 - 2a_1a_3 - a_3^2)\omega^2 + a_3^2 = 0.
\] (2.5)
Let \( z = \omega^2 \) and denote
\[
p = a_1^2 - K^2 - 2a_2, \quad q = a_1^2 - 2a_1a_3 - a_3^2, \quad r = a_3^2.
\] (2.6)
Then, Eq. (2.5) becomes

\[ z^3 + pz^2 + qz + r = 0. \]  

(2.7)

Denote

\[ h(z) = z^3 + pz^2 + qz + r. \]

(2.8)

By (2.8), we have

\[ \frac{dh(z)}{dz} = 3z^2 + 2pz + q. \]

Since \( r = a_2^2 \geq 0 \), we know that when \( \Delta = p^2 - 3q \leq 0 \), Eq. (2.8) has no positive roots for \( z \in [0, \infty) \). On the other hand, when \( \Delta = p^2 - 3q > 0 \), the following equation

\[ 3z^2 + 2pz + q = 0 \]

(2.9)

has two real roots

\[ z_1 = \frac{1}{3} (-p + \sqrt{\Delta}) \quad \text{and} \quad z_2 = \frac{1}{3} (-p - \sqrt{\Delta}). \]

(2.10)

Obviously, \( h'(z_1) = 2\sqrt{\Delta} > 0 \) and \( h'(z_2) = -2\sqrt{\Delta} < 0 \). It follows that \( z_1 \) and \( z_2 \) are the local minimum and the local maximum of \( h(z) \), respectively. Hence, we have the following Lemma.

**Lemma 2.1.** Suppose that \( \Delta > 0 \). Then Eq. (2.7) has positive roots if and only if \( z_1 > 0, h(z_1) \leq 0 \). From Lemma 2.1 and the above discussions, we obtain the following Lemma.

**Lemma 2.2.** For the polynomial Eq. (2.7), we have the following results.

(i) If \( \Delta = p^2 - 3q \leq 0 \), then Eq. (2.7) has no positive roots.

(ii) If \( \Delta = p^2 - 3q > 0 \), then Eq. (2.7) has positive roots if and only if \( z_1 = \frac{1}{3} (-p + \sqrt{\Delta}) > 0 \) and \( h(z_1) \leq 0 \). Since \( r = a_2^2 \geq 0 \), Eq. (2.7) has at most two positive roots. Without loss of generality, we assume that it has two positive roots, denoted by \( z_1 \) and \( z_2 \), respectively. Then, Eq. (2.5) has two positive roots

\[ \omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}. \]

By (2.4), we have

\[ \cos \omega \tau = \frac{K(a_1 - a_1 \omega^2) - a_4(a_2 - \omega^2)}{K^2 \omega^2 + a_4^2}. \]

Thus, if we denote

\[ \tau_k^{(j)} = \frac{1}{\omega_k} \left\{ \cos^{-1} \left( \frac{K(a_1 - a_1 \omega_k^2) - a_4(a_2 - \omega_k^2)}{K^2 \omega_k^2 + a_4^2} \right) + 2\pi j \right\}, \]

(2.11)

where \( k = 1, 2; \ j = 0, 1, 2, \ldots, \) then \( \pm i\omega_k \) is a pair of purely imaginary roots of Eq. (2.2) with \( \tau_k^{(j)} \). Define

\[ \tau_0 = \tau_{k_0}^{(0)} = \min_{k \in \{1, 2\}} \{ \tau_k^{(0)} \}, \quad \omega_0 = \omega_{k_0}. \]

(2.12)

Note that when \( \tau = 0 \), Eq. (2.2) becomes

\[ \lambda^3 + (K + a_1) \lambda^2 + (a_2 + a_4) \lambda + a_3 = 0. \]

(2.13)

Till now, we can employ a result from Ruan and Wei [10] to analyze Eq. (2.2), which is stated as follows.

**Lemma 2.3.** Consider the exponential polynomial

\[ P(\lambda, e^{-i\tau_1}, \ldots, e^{-i\tau_m}) = z^m + p_1^{(0)} z^{m-1} + \cdots + p_{n-1}^{(0)} \lambda + p_n^{(0)} + p_1^{(1)} \lambda^{n-1} + \cdots + p_{n-1}^{(1)} \lambda + p_n^{(1)} e^{-i\tau_1} + \cdots + p_{m-1}^{(m)} \lambda^{m-1} + \cdots + p_n^{(m)} e^{-i\tau_m}, \]

where \( \tau_1 \geq 0, (i = 1, 2, \ldots, m) \) and \( p_i^{(j)} (i = 0, 1, \ldots, m; j = 1, 2, \ldots, n) \) are constants. As \( (\tau_1, \tau_2, \ldots, \tau_m) \) vary, the sum of the order of the zeros of \( P(\lambda, e^{-i\tau_1}, \ldots, e^{-i\tau_m}) \) on the open right half plane can change only if a zero appears on or crosses the imaginary axis. Using Lemmas 2.2 and 2.3, we can obtain the following results on the distribution of roots of the transcendental Eq. (2.2).

**Lemma 2.4.** For the third degree exponential polynomial Eq. (2.2), we have
Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq. (2.2) near $\tau = \tau_0^0$ satisfying
$$\alpha(\tau_0^0) = 0, \quad \omega(\tau_0^0) = \omega_k.$$

Then, the following transversality condition holds.

**Lemma 2.5.** Suppose that $z_k = \omega_k^2$ and $h'(z_k) \neq 0$, where $h(z)$ is defined by (2.8). Then, $\frac{d(\text{Re}(\tau_0^0))}{dt} \neq 0$, and the sign of $\frac{d(\text{Re}(\tau_0^0))}{dt}$ is consistent with that of $h'(z_k)$.

**Proof.** Substituting $\lambda(\tau)$ into Eq. (2.2) and differentiating the resulting equation in $\tau$, we obtain

$$\{3\lambda^2 + 2a_1\lambda + a_2 + [2K\lambda + a_4 - \tau(K\lambda^2 + a_4\lambda)]e^{i\tau}\} \frac{d\lambda}{dt} = \lambda(K\lambda^2 + a_4\lambda)e^{-i\tau}.$$ 

Thus,

$$\frac{d\lambda}{dt} = \frac{(3\lambda^2 + 2a_1\lambda + a_2)e^{i\tau}}{\lambda(K\lambda^2 + a_4\lambda)} + \frac{2K\lambda + a_4}{\lambda(K\lambda^2 + a_4\lambda)} - \frac{\tau}{\lambda}.$$ 

(2.14)

It follows from (2.4) that

$$\left[\lambda(K\lambda^2 + a_4\lambda)\right]_{\tau=\tau_0^0} = -a_4\omega_k^2 - iK\omega_k^2,$$

$$\left[3\lambda^2 + 2a_1\lambda + a_2\right]e^{i\tau}\left[\tau-\tau_0^0\right] = (a_2 - 3a_0^2 + 2ia_1\omega_k)(\cos \omega_k \tau_k^0 + i \sin \omega_k \tau_k^0),$$

$$[2K\lambda + a_4]_{\tau=\tau_0^0} = a_4 + 2iK\omega_k.$$ 

From 2.14, 2.15 and 2.6, we can obtain

$$\lim_{\tau \to \tau_0^0} \frac{\text{Re}(\lambda(\tau))}{d\tau} = \text{Re}\left[\frac{3\lambda^2 + 2a_1\lambda + a_2}{\lambda(K\lambda^2 + a_4\lambda)} + \frac{2K\lambda + a_4}{\lambda(K\lambda^2 + a_4\lambda)}\right]$$

$$= \frac{1}{A} \left(-a_4\omega_k^2 (a_2 - 3a_0^2) \cos \omega_k \tau_k^0 - K\omega_k^2 (a_2 - 3a_0^2) \sin \omega_k \tau_k^0 + a_4\omega_k^2 2a_1 \sin \omega_k \tau_k^0 - 2a_1 K\omega_k^2 \cos \omega_k \tau_k^0 - a_2^2 \omega_k^2 - 2K^2 \omega_k^2\right)$$

$$= \frac{1}{A} \left(3a_0^2 \omega_k^2 + 2(a_1^2 - K^2 - 2a_2)\omega_k^4 + (a_2^2 - a_0^2 - 2a_1a_3)\omega_k^6\right) = \frac{1}{A} \left(3a_0^2 \omega_k^2 + 2a_1a_3 + q\omega_k^4\right) = \frac{2}{A} \dot{z}_k h'(z_k),$$

where $A = a_0^2 \omega_k^4 + K^2 \omega_k^2 > 0$. Thus, we have

$$\text{sign}\left[\left.\frac{\text{Re}(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_0^0}\right] = \text{sign}\left[\left.\frac{\text{Re}(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_0^0}\right]^{-1} = \text{sign}\left[\frac{z_k}{A} h'(z_k)\right].$$

Notice that $A, z_k > 0$, we conclude that the sign of $\frac{d(\text{Re}(\tau_0^0))}{dt}$ is determined by that of $h'(z_k)$. This proves the Lemma. Note that when $\tau = 0$, Eq. (2.2) becomes Eq. (2.13), Routh–Hurwitz criterion implies that if

$$(H_2) \quad K + a_1 > 0, a_2 + a_4 > 0 \quad and \quad (K + a_1)(a_2 + a_4) - a_3 > 0,$$

then all roots of Eq. (2.2) with $\tau = 0$ have negative real parts. If

$$(H_3) \quad K + a_1 > 0, a_2 + a_4 > 0 \quad and \quad (K + a_1)(a_2 + a_4) - a_3 < 0,$$

then Eq. (2.13) have one negative real root and one pair of conjugate complex roots with positive real parts. Thus, from Lemmas 2.3 and 2.5, we have the following Theorem.

**Theorem 2.6.** Suppose that $(H_1)$ and $(H_2)$ hold.

(i) If $\Delta = p^2 - 3q < 0$, all roots of Eq. (2.2) have negative real parts for all $\tau \geq 0$, and the positive equilibrium $E^*$ is asymptotically stable for all $\tau \geq 0$.

(ii) If $\Delta = p^2 - 3q > 0, z_1^0 > 0$ and $h(z_1^0) < 0$, then $h(z)$ has at least one positive root $z_k$, and all roots of Eq. (2.2) have negative real parts for $\tau \in [0, \tau_k^0]$, and the positive equilibrium $E^*$ is asymptotically stable for $\tau \in [0, \tau_k^0]$. 

(iii) If all the conditions in (ii) are hold and \( h'(z_0) \neq 0 \), then system (1.3) undergoes a Hopf bifurcation at the positive equilibrium \( E^* \) when \( \tau = \tau^{(0)}_k (j = 0, 1, 2, \ldots) \).

**Theorem 2.7.** Suppose that \((H_1)\) and \((H_3)\) hold.

(i) If \( \triangle = p^2 - 3q \leq 0 \), then Eq. (2.2) have some roots with positive real parts for all \( \tau \geq 0 \), and the positive equilibrium \( E^* \) is unstable for all \( \tau \geq 0 \).

(ii) If \( \triangle = p^2 - 3q > 0 \), \( z_1 > 0 \) and \( h(z_1) \leq 0 \), then \( h(z) \) has at least one positive root \( z_0 \), and Eq. (2.2) have some roots with positive real parts for \( \tau \in (0, \tau^{(0)}_k) \), and the positive equilibrium \( E^* \) is unstable for \( \tau \in (0, \tau^{(0)}_k) \).

(iii) If all the conditions in (ii) are hold and \( h'(z_0) \neq 0 \), then system (1.3) undergoes a Hopf bifurcation at the positive equilibrium \( E^* \) when \( \tau = \tau^{(0)}_k (j = 0, 1, 2, \ldots) \).

### 3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained the conditions under which system (1.3) undergoes Hopf bifurcation at \( \tau = \tau^{(0)} (j = 0, 1, 2, \ldots) \). In this section, we assume that system (1.3) undergoes a Hopf bifurcation at the positive equilibrium \( E^* \) when \( \tau = \tau^{(0)} (j = 0, 1, 2, \ldots) \), i.e., a family of periodic solutions bifurcate from the positive equilibrium \( E^* \). In this section, using the normal form theory and center manifold reduction due to Hassard et al. [6], we are able to determine the Hopf bifurcation direction, i.e., make clear whether the bifurcating branch of periodic solution exists locally for \( \tau > \tau^{(0)} \) or \( \tau < \tau^{(0)} \), and determine the properties of these bifurcating periodic solutions.

Let \( x_1 = x - x', x_2 = y - y', x_3 = z - z', \dot{x}_3(t) = x_3(\tau t), \tau = \tau^{(0)} + \mu \), where \( \tau^{(0)} \) is defined by (2.11). For convenience, drop the bar and let \( p(x) = x_3 \max \). then the system (1.3) can be written as an FDE in \( C = \mathbb{C}([-1,0], \mathbb{R}^3) \) as

\[
\dot{x}(t) = L_\mu(x(t)) + f(x(t)),
\]

where \( x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3 \), and \( L_\mu : C \to \mathbb{R}^3, f : \mathbb{R} \times C \to \mathbb{R}^3 \) are given respectively by

\[
L_\mu(\phi) = (\tau^{(0)} + \mu)A\phi(0) + (\tau^{(0)} + \mu)B\phi(-1)
\]

and

\[
f(\mu, \phi) = (\tau^{(0)} + \mu) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},
\]

where

\[
f_1 = a\phi_1(0)\phi_2(0), \\
f_2 = -l_1\phi_1(0)\phi_2(0) - l_2\phi_2(0) + l_1\phi_1(0)\phi_2(0) - l_5\phi_2(0)\phi_3(0) - l_6\phi_2(0)\phi_3(0) - l_7\phi_2^2(0) + \cdots, \\
f_3 = l_4\phi_2(0)\phi_3(0) + l_5\phi_2^2(0)\phi_3(0) + l_6\phi_2(0)\phi_3(0) + l_7\phi_2^2(0) + l_8\phi_2(0) + \cdots
\]

and

\[
\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in \mathbb{R}^3, ~ l_1 = 2cy' - e, ~ l_2 = cx + \frac{1}{2}p''(y')z', ~ l_3 = c, ~ l_4 = pF(y'), ~ l_5 = \frac{1}{2}p''(y'), ~ l_6 = \frac{1}{2}p''(y')z', ~ l_7 = \frac{3}{4}p''(y')z'.
\]

From the discussions in Section 2, we know that if \( \mu = 0 \), then system (3.1) undergoes a Hopf bifurcation at the positive equilibrium \( E^* \) and the associated characteristic equation of system (3.1) has a pair of simple imaginary roots \( \pm \tau^{(0)} \omega_0 \). By the Riesz representation theorem, there exists a function \( \eta(\theta, \mu) \) of bounded variation for \( \theta \in [-1,0] \), such that

\[
L_\mu \phi = \int_{-1}^{0} d\eta(\theta, 0) \phi(\theta), \quad \text{for } \phi \in C.
\]

In fact, we can choose

\[
\eta(\theta, \mu) = (\tau^{(0)} + \mu)A\delta(\theta) - (\tau^{(0)} + \mu)B\delta(\theta + 1),
\]

where \( \delta \) is Dirac-delta function. For \( \phi \in C([-1,0], \mathbb{R}^3) \), Define

\[
A(\mu) \phi = \left\{ \frac{d\phi(\theta)}{d\theta}, \quad \theta \in [-1,0), \right. \\
\int_{-1}^{0} d\eta(s, \mu) \phi(s), \quad \theta = 0
\]

and

\[
R(\mu) \phi = \left\{ 0, \quad \theta \in [-1,0), \\
f(\mu, \phi), \quad \theta = 0
\right.
\]
Thus, we can choose $g$ where
\[
A^*\psi(s) = \begin{cases} \frac{d\psi(s)}{ds}, & s \in (0, 1], \\
\int_{-1}^{0} d\eta^2(t, 0)\psi(-t), & s = 0 
\end{cases}
\]
and a bilinear inner product
\[
\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\theta - \xi) d\eta(\theta)\phi(\xi)d\xi.
\]
where $\eta(\theta) = \eta(0, 0)$. Denote $A = A(0)$, then, $A$ and $A^*$ are adjoint operators. By Theorem 2.6, we know that $\pm i\tau^{(j)}q_0$ are eigenvalues of $A$. Thus, they are also eigenvalues of $A^*$. Suppose that $q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0 \tau^{(j)}\theta}$ is the eigenvector of $A$ corresponding to $i\tau^{(j)}q_0$. Then, $Aq(\theta) = i\tau^{(j)}q_0 q(\theta)$. It follows from the definition of $A$ and (3.5) that
\[
\tau^{(j)} \begin{pmatrix} i\omega_0 - a_{11} \\
-a_{21} \\
i\omega_0 - a_{22} + Ke^{-i\omega_0 \tau^{(j)}} \\
0 \\
i\omega_0 
\end{pmatrix} q(0) = \begin{pmatrix} 0 \\
0 \\
0 
\end{pmatrix},
\]
which yields
\[
q(0) = (1, \alpha, \beta)^T = \left(1, \frac{i\omega_0 - a_{11}}{a_{12}}, \frac{i\omega_0 a_{22} - a_{11} a_{23}}{i\omega_0 a_{21}}\right)^T.
\]
Similarly, it can be verified that $q'(s) = D(1, \alpha', \beta') e^{i\omega_0 \tau^{(j)}s}$ is the eigenvector of $A^*$ corresponding to $-i\omega_0 \tau^{(j)}$, where
\[
\alpha' = \frac{-i\omega_0 - a_{11}}{a_{12}}, \quad \beta' = \frac{i\omega_0 a_{22} + a_{11} a_{23}}{i\omega_0 a_{21}}.
\]
By (3.7), we get
\[
\langle q'(s), q(\theta) \rangle = D(1, \alpha', \beta')(1, \alpha, \beta)^T - \int_{-1}^{0} \int_{0}^{\theta} D(1, \alpha', \beta') e^{-i(\alpha - \beta)\omega_0 \tau^{(j)}s} d\eta(\theta)(1, \alpha, \beta)^T e^{i\omega_0 \tau^{(j)}s} d\xi = D \left\{ 1 + \alpha\alpha' + \beta\beta' - \int_{-1}^{0} (1, \alpha', \beta') d\eta(\theta)(1, \alpha, \beta)^T \right\} = D \left\{ 1 + \alpha\alpha' + \beta\beta' - K\tau^{(j)}\alpha\alpha' e^{-i\omega_0 \tau^{(j)}} \right\}.
\]
Thus, we can choose
\[
D = \frac{1}{1 + \alpha\alpha' + \beta\beta' - K\tau^{(j)}\alpha\alpha' e^{-i\omega_0 \tau^{(j)}}},
\]
such that $\langle q'(s), q(\theta) \rangle = 1$. In the following, we follow the ideas in Hassard et al. [6] and by using the same notations as there to compute the coordinates describing the center manifold $C_0$ at $\mu = 0$. Let $x_t$ be the solution of Eq. (3.1) when $\mu = 0$. Define
\[
Z(t) = \langle q', x_t \rangle, \quad W(t, \theta) = x_t(\theta) - 2\text{Re}\{Z(t)q(\theta)\}.
\]
On the center manifold $C_0$, we have
\[
W(t, \theta) = W(Z(t), Z(t), \theta) = W_{20}(\theta) \frac{Z^2}{2} + W_{11}(\theta)ZZ + W_{02}(\theta) \frac{Z^2}{2} + W_{30}(\theta) \frac{Z^3}{6} + \cdots,
\]
where $z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q$ and $\bar{q}$. Note that $W$ is real if $x_t$ is real. We consider only real solutions. For the solution $x_t \in C_0$ of (3.1), since $\mu = 0$, we have
\[
\dot{z} = i\omega_0 \tau^{(j)}z + q'(0, \theta) f(0, W(Z(t), Z(t), \theta) + 2\text{Re}\{Z(t)q(\theta)\}) = i\omega_0 \tau^{(j)}z + q'(0) f(0, W(Z(t), Z(t), \theta) + 2\text{Re}\{Z(t)q(\theta)\}) = i\omega_0 \tau^{(j)}z + q'(0) f(z, \bar{z}) = i\omega_0 \tau^{(j)}z + g(z, \bar{z}),
\]
where
\[
g(z, \bar{z}) = q'(0) f(z, \bar{z}) = g_{20}(\theta) \frac{Z^2}{2} + g_{11}(\theta)ZZ + g_{02}(\theta) \frac{Z^2}{2} + \cdots
\]
By (3.8), we have \( x(t) = (x_1(t), x_2(t), x_3(t))^T = W(t, \theta) + zq(\theta) + \tilde{z}q(\theta) \), and then
\[
\begin{align*}
    x_{11}(0) &= z + z + W_{11}^{(0)}(0) z^2 + W_{11}^{(1)}(0) z^3 + o((z, z)^3), \\
    x_{21}(0) &= z x + z^2 + W_{21}^{(0)}(0) z^3 + W_{21}^{(1)}(0) z^2 + o((z, z)^2), \\
    x_{31}(0) &= z^2 + z + W_{31}^{(0)}(0) z^2 + W_{31}^{(1)}(0) z + o((z, z)).
\end{align*}
\]
It follows together with (3.3) that
\[
g(z, z) = \tilde{q}(0)f_0(z, z) = \mathcal{D} \tau(1, \tilde{x}^*, \tilde{\beta}^*)
\begin{pmatrix}
    f_1(0) \\
    f_2(0) \\
    f_3(0)
\end{pmatrix}
\]
\[
= \mathcal{D} \tau(0) [ax_{11}(0)x_{21}(0) + \tilde{x}^*(l_1x_{11}(0)x_{21}(0) - l_2x_{21}(0)x_{11}(0) - l_3x_{11}(0)x_{31}(0) - l_4x_{21}(0)x_{31}(0) - l_5x_{31}(0)x_{21}(0) - l_6x_{31}(0)x_{11}(0) + l_7x_{21}(0)x_{11}(0) + l_8x_{21}(0)x_{31}(0) + l_9x_{31}(0)x_{11}(0) + l_{10}x_{21}(0)x_{31}(0) + \cdots].
\]
Comparing the coefficients with (3.10), we have
\[
g_{20} = 2\mathcal{D} \tau(0) [(ax - l_1x^*) + (l_2x^* \tilde{\beta}^* + l_3x^2 \tilde{\beta}^* - l_4x^2 \tilde{x}^* - l_5x^2 \tilde{x}^*)],
\]
\[
g_{21} = 2\mathcal{D} \tau(0) [(ax - l_1x^*) + (l_2x^* \tilde{\beta}^* + l_3x^2 \tilde{\beta}^* - l_4x^2 \tilde{x}^* - l_5x^2 \tilde{x}^*)],
\]
\[
g_{22} = 2\mathcal{D} \tau(0) [(ax - l_1x^*) + l_2x^* \tilde{x}^* + l_3x^2 \tilde{x}^* - l_4x^2 \tilde{x}^* - l_5x^2 \tilde{x}^*],
\]
\[
g_{21} = 2\mathcal{D} \tau(0) [(l_1x^* - l_2x^*)W_{11}^{(0)}(0) + 4x(l_1 \tilde{\beta}^* - l_2 \tilde{x}^*)W_{11}^{(0)}(0)]
\]
\[
+ 2\mathcal{D} \tau(0) [(l_1 \tilde{\beta}^* - l_2 \tilde{x}^*)W_{11}^{(0)}(0) + l_4(\tilde{\beta}^* - \tilde{x}^*)(\tilde{\beta}^* + 2l_2 \tilde{x}^* \tilde{\beta}^* + l_2x^2 \tilde{x}^*)] + \mathcal{D} \tau(0) [(a - l_1 \tilde{x}^*)W_{11}^{(0)}(0)]
\]
\[
+ 2\mathcal{D} \tau(0) [(l_1 \tilde{\beta}^* - l_2 \tilde{x}^*)W_{11}^{(0)}(0) + l_4(\tilde{\beta}^* - \tilde{x}^*)(\tilde{\beta}^* + 2l_2 \tilde{x}^* \tilde{\beta}^* + l_2x^2 \tilde{\beta}^*)] + (a - l_1 \tilde{x}^*)W_{11}^{(0)}(0) + W_{20}^{(2)}(0) - 2l_2x^2 \tilde{x}^*.
\]
In order to determine \( g_{21} \), in the sequel, we need to compute \( W_{20}(\theta) \) and \( W_{11}(\theta) \). From (3.6) and (3.8), we have
\[
W = \tilde{W} = W_3 \tilde{W} + \tilde{W}_3,
\]
thus,
\[
(A - 2i\tau(0)\omega_0)W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta).
\]
Since (3.11), For \( \theta \in [-1, 0) \), we have
\[
\begin{align*}
    H(z, z, \theta) &= -\tilde{q}(0)f_0q(\theta) - q^*(0)f_0\tilde{q}(\theta) = -gq(\theta) - \tilde{g}\tilde{q}(\theta), \\
    H_{20}(\theta) &= -g_{20}q(\theta) - g_{02}\tilde{q}(\theta), \quad H_{11}(\theta) = -g_{11}\tilde{q}(\theta) - \tilde{g}_{11}q(\theta).
\end{align*}
\]
From (3.14), (3.15) and the definition of \( A \), we can get
\[
W_{20}(\theta) = 2i\tau(0)\omega_0W_{20}(\theta) + g_{20}q(\theta) + g_{02}\tilde{q}(\theta).
\]
Notice that \( q(\theta) = q(0)e^{i\omega_0\theta} \), we have
\[
W_{20}(\theta) = -\frac{i\omega_0}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\omega_0}{3\tau(0)\omega_0} \tilde{q}(0)e^{i\omega_0\theta} + E_1e^{i\omega_0\theta},
\]
where \( E_1 = (E_1^1, E_1^2, E_1^3) \in \mathbb{R}^3 \) is a constant vector. In the same way, we can also obtain
\[
W_{11}(\theta) = -\frac{i\omega_0}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\omega_0}{3\tau(0)\omega_0} \tilde{q}(0)e^{i\omega_0\theta} + E_2e^{i\omega_0\theta},
\]
where \( E_2 = (E_2^1, E_2^2, E_2^3) \in \mathbb{R}^3 \) is also a constant vector. In what follows, we will seek appropriate \( E_1 \) and \( E_2 \). From the definition of \( A \) and (3.14), we obtain
\[
\int_{-1}^0 dq(\theta)W_{20}(\theta) = 2i\tau(0)\omega_0W_{20}(0) - H_{20}(0)
\]
and

\[
\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0),
\]  
(3.20)

where \( \eta(\theta) = \eta(0, \theta) \). From (3.11) and (3.12), we have

\[
H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau(0) \begin{pmatrix}
\alpha \xi - l_2 \xi^2 - l_4 \xi \beta \\
l_2 \xi \beta + l_4 \xi^2
\end{pmatrix}
\]  
(3.21)

and

\[
H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau(0) \begin{pmatrix}
\frac{2aRe(\xi)}{2l_4 Re(\xi) + 2l_8 \xi \bar{\xi}} \\
-2l_4 Re(\xi) - 2l_2 \xi \bar{\xi} - 2l_4 Re(\xi \beta)
\end{pmatrix}.
\]  
(3.22)

Substituting (3.17) and (3.21) into (3.19) and noticing that

\[
\left( i\omega \tau(0) I - \int_{-1}^{0} e^{i\omega \tau(0) \theta} d\eta(\theta) \right) q(0) = 0
\]

and

\[
\left( -i\omega \tau(0) I - \int_{-1}^{0} e^{-i\omega \tau(0) \theta} d\eta(\theta) \right) \bar{q}(0) = 0.
\]
we obtain

\[
\left(2i\omega_0 \tau_{i}^j\right) \frac{1}{1} - \int_{-1}^{0} e^{2i\omega_0 \tau^{(j)} \delta \eta(\theta)} d\eta(\theta) \quad E_1 = 2\tau^{(j)} \begin{pmatrix} a\xi \\ -l_1\alpha - l_2\alpha^2 - l_4\alpha\beta \\ l_4\alpha\beta + l_9\alpha^2 \end{pmatrix},
\]

which leads to

\[
\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & -a_{13} \\
-a_{21} & 2i\omega_0 - a_{22} + Ke^{-2ir^{(i)}\omega_0} & 0 \\
0 & -a_{32} & 2i\omega_0 \end{pmatrix} \quad E_1 = 2\begin{pmatrix} a\xi \\ -l_1\alpha - l_2\alpha^2 - l_4\alpha\beta \\ l_4\alpha\beta + l_9\alpha^2 \end{pmatrix}.
\]

It follows that

\[
E^{(1)}_1 = \frac{2}{\tau_1} \begin{pmatrix} a\xi \\ -l_1\alpha - l_2\alpha^2 - l_4\alpha\beta \\ l_4\alpha\beta + l_9\alpha^2 \end{pmatrix} - \begin{pmatrix} -a_{12} \\ -a_{12} \\ -a_{12} \end{pmatrix},
\]

\[
E^{(2)}_1 = \frac{2}{\tau_1} \begin{pmatrix} 2i\omega_0 - a_{11} & a\xi & 0 \\
-a_{21} & -l_1\alpha - l_2\alpha^2 - l_4\alpha\beta & -a_{23} \\
0 & l_4\alpha\beta + l_9\alpha^2 & 2i\omega_0 \end{pmatrix},
\]

\[
E^{(3)}_1 = \frac{2}{\tau_1} \begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & a\xi \\
-a_{21} & 2i\omega_0 - a_{22} + Ke^{-2ir^{(i)}\omega_0} & -l_1\alpha - l_2\alpha^2 - l_4\alpha\beta \\
0 & -a_{32} & l_4\alpha\beta + l_9\alpha^2 \end{pmatrix}.
\]
where
\[
D_1 = \begin{vmatrix}
2i\omega_0 - a_{11} & -a_{12} & 0 \\
-a_{21} & 2i\omega_0 - a_{22} + Ke^{-2ir_0^0} & -a_{23} \\
0 & -a_{32} & 2i\omega_0
\end{vmatrix}.
\]

Similarly, substituting (3.18) and (3.22) into (3.20), we can get
\[
E_2 = 2 \left( \begin{array}{c}
2a\text{Re}\{z\} \\
-2l_1a\text{Re}\{x\} - 2l_2x\bar{x} - 2l_4\text{Re}\{x\bar{\beta}\} \\
2l_4\text{Re}\{\bar{\beta}\} + 2l_6x\bar{x} \\
-a_{11} & -a_{12} & 0 \\
-a_{21} & -a_{22} + K & -a_{23} \\
0 & -a_{32} & 2i\omega_0
\end{array} \right).
\]

thus, we have

\[
E^{(1)}_2 = \frac{2}{\partial_x} \begin{vmatrix}
2a\text{Re}\{z\} \\
-2l_1a\text{Re}\{x\} - 2l_2x\bar{x} - 2l_4\text{Re}\{x\bar{\beta}\} \\
2l_4\text{Re}\{\bar{\beta}\} + 2l_6x\bar{x} \\
-a_{11} & -a_{12} & 0 \\
-a_{21} & -a_{22} + K & -a_{23} \\
0 & -a_{32} & 2i\omega_0
\end{vmatrix},
\]

\[
E^{(2)}_2 = \frac{2}{\partial_y} \begin{vmatrix}
2a\text{Re}\{z\} \\
-2l_1a\text{Re}\{x\} - 2l_2x\bar{x} - 2l_4\text{Re}\{x\bar{\beta}\} \\
2l_4\text{Re}\{\bar{\beta}\} + 2l_6x\bar{x} \\
-a_{11} & -a_{12} & 0 \\
-a_{21} & -a_{22} + K & -a_{23} \\
0 & -a_{32} & 2i\omega_0
\end{vmatrix},
\]

\[
E^{(3)}_2 = \frac{2}{\partial_z} \begin{vmatrix}
2a\text{Re}\{z\} \\
-2l_1a\text{Re}\{x\} - 2l_2x\bar{x} - 2l_4\text{Re}\{x\bar{\beta}\} \\
2l_4\text{Re}\{\bar{\beta}\} + 2l_6x\bar{x} \\
-a_{11} & -a_{12} & 0 \\
-a_{21} & -a_{22} + K & -a_{23} \\
0 & -a_{32} & 2i\omega_0
\end{vmatrix}.
\]

Fig. 3. The trajectories and phase graphs of system (4.2) with \(\tau = 0.8, K = -4\), chaos vanishes, and a stable periodic solution bifurcate from the equilibrium \(E'\).
where

\[
D_2 = \begin{bmatrix}
-a_{11} & -a_{12} & 0 \\
-a_{21} & -a_{22} + K & -a_{23} \\
0 & -a_{32} & 0
\end{bmatrix}.
\]

Thus, we can determine \(W_{20}(0)\) and \(W_{11}(0)\). Furthermore, we can determine each \(g_{ij}\). Therefore, each \(g_{ij}\) is determined by the parameters and delay in \((1.3)\). Thus, we can compute the following values:

\[
c_1(0) = \frac{i}{2\sigma_0 \tau^0} \left( g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 + \frac{g_{21}}{2} \right), \quad \mu_2 = -\frac{\text{Re}\{c_1(0)\}}{\text{Re}\{\dot{\lambda}(\tau^0)\}}, \\
T_2 = -\frac{\text{Im}\{c_1(0)\} + \mu_2 \text{Im}\{\dot{\lambda}(\tau^0)\}}{\sigma_0 \tau^0}, \quad \beta_2 = 2\text{Re}\{c_1(0)\},
\]

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value \(\tau^0\), i.e., \(\mu_2\) determines the directions of the Hopf bifurcation: if \(\mu_2 > 0(<0)\), then the Hopf bifurcation is supercritical (subcritical) and the bifurcation exist for \(\tau > \tau_0(<\tau_0)\); \(\beta_2\) determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if \(\beta_2 < 0(>0)\); and \(T_2\) determines the period of the bifurcating periodic solutions: the period increase (decrease) if \(T_2 > 0(<0)\).

4. Application to control of chaos

In this section, we apply the results in the previous sections to system \((4.1)\) for the purpose of control of chaos. From Section 2, we know that under certain conditions, a family of periodic solutions bifurcate from the steady states of system \((4.1)\) at some critical values of \(\tau\) and the stability of the steady state maybe change along with increasing of \(\tau\). If the bifurcating periodic solution is orbitally asymptotically stable or some steady state become local stable, then chaos may vanish. Following this idea, we consider the following hybrid ratio-dependent three species food chain model proposed by Wang in [13].

Fig. 4. The trajectories and phase graphs of system \((4.2)\) with \(\tau = 2.5, K = -4, E^r\) become local stable.
From Routh–Hurwitz criterion, it's easy to know that the positive equilibrium $s$ of system (4.2) has a positive equilibrium $E$ become unstable, and a stable periodic solution bifurcate from the equilibrium $E^*$. When $u = 0$ or $k > 0$, system (4.2) becomes

$$\begin{align*}
\dot{x}(t) &= 0.2 - 0.6x(t) + x(t)y(t), \\
\dot{y}(t) &= 1.2y(t) - 0.6x(t)y^2(t) - \frac{6y(t)}{1+my(t)}, \\
\dot{z}(t) &= 0.8z(t),
\end{align*}$$

(4.1)

which is chaotic when $m = 3.6$. For control of chaos, we add a time-delayed force $K(y(t) - y(t - \tau))$ to system (4.1), that is the following delayed feedback control system

$$\begin{align*}
\dot{x}(t) &= 0.2 - 0.6x(t) + x(t)y(t), \\
\dot{y}(t) &= 1.2y(t) - 0.6x(t)y^2(t) - \frac{6y(t)}{1+my(t)} + K(y(t) - y(t - \tau)), \\
\dot{z}(t) &= 0.8z(t).
\end{align*}$$

(4.2)

System (4.2) has a positive equilibrium $E^*(0.5821,0.2564,0.3559)$, the linearized system of system (4.2) at $E^*$ is

$$\dot{u}(t) = \begin{pmatrix} -0.3436 & 0.5821 & 0 \\ -0.0394 & K + 0.4435 & -0.8 \\ 0 & 0.5774 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -K & 0 \\ 0 & 0 & 0 \end{pmatrix} u(t - \tau),$$

(4.3)

where $u(t) = (x(t),y(t),z(t))^T$. The characteristic equation of system (4.3) is

$$\lambda^3 - (K + 0.0999)\lambda^2 + (0.3324 - 0.3436K)\lambda + 0.1587 + (K\lambda^2 + 0.3436K) e^{-\lambda\tau} = 0.$$  

(4.4)

When $\tau = 0$ or $K = 0$, Eq. (4.4) becomes

$$\lambda^3 - 0.0999\lambda^2 + 0.3324\lambda + 0.1587 = 0.$$  

From Routh–Hurwitz criterion, it's easy to know that the positive equilibrium $E^*$ is unstable, and using Matlab 7.0, we can get chaotic phenomena (see Fig. 1)
From the arithmetic in Section 3, when $\tau = \tau_1^{(0)}$, we compute $\text{Re}c_1(0) = -68.6350 - 4.3100i$, thus, we have $\mu_2 > 0$ and $\beta_2 < 0$, which means the bifurcating periodic solutions from the equilibrium $E'$ is supercritical and asymptotically stable. when $\tau = \tau_2^{(0)}$, we compute $\text{Re}c_1(0) = -407213 - 27.8648i$, thus, we have $\mu_2 < 0$ and $\beta_2 < 0$, which means the bifurcating periodic solutions from the equilibrium $E'$ is subcritical and asymptotically stable. Thus, we have the following conclusion about the stability and Hopf bifurcations of the positive equilibrium $E'$ of system (4.2).
5. Conclusion

Suppose that \( s_j(k) \), \( k = 1, 2, \ldots \), is defined by (4.7).

(i) When \( \tau \in [0, \tau_1(0)) \cup (\tau_1(1), +\infty) \), the positive equilibrium \( E^\ast \) is unstable.

(ii) When \( \tau \in (\tau_2(0), \tau_1(1)) \), the positive equilibrium \( E^\ast \) is stable.

(iii) System (4.2) undergoes a Hopf bifurcation at the positive equilibrium \( E^\ast \) when \( \tau = \tau_j(k) \), for \( k = 1, 2; j = 0, 1, \ldots \). Especially, when \( \tau \in (\tau_1(0), \tau_1(1)) \) (or \( \tau \in (\tau_1(1), \tau_2(1)) \)), there is a stable periodic solution bifurcating from \( E^\ast \).

Using Matlab 7.0, we can get the phase graphs of system (4.2). Fig. 2 shows the positive equilibrium \( E^\ast \) is still chaotic when \( \tau = 0.1 \) (See Fig. 2). When \( \tau = 0.8 \), the positive equilibrium \( E^\ast \) is unstable but a stable periodic orbit bifurcate from \( E^\ast \) (See Fig. 3). When \( \tau \) pass through \( \tau_2(0) \), \( E^\ast \) becomes stable (See Fig. 4). When \( \tau \) pass through \( \tau_1(1) \), the positive equilibrium \( E^\ast \) is unstable and a stable periodic orbit bifurcate from \( E^\ast \) (See Fig. 5). With the increasing of \( \tau \) continuously, the stability of \( E^\ast \) is unchanged (See Fig. 6).

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References


