HORIZONTAL LAPLACE OPERATOR IN REAL FINSLER VECTOR BUNDLES*

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Abstract A vector bundle F over the tangent bundle TM of a manifold M is said to be a Finsler vector bundle if it is isomorphic to the pull-back π∗E of a vector bundle E over M([1]). In this article the authors study the h-Laplace operator in Finsler vector bundles. An h-Laplace operator is defined, first for functions and then for horizontal Finsler forms on E. Using the h-Laplace operator, the authors define the h-harmonic function and h-harmonic horizontal Finsler vector fields, and furthermore prove some integral formulas for the h-Laplace operator, horizontal Finsler vector fields, and scalar fields on E.

Key words h-Laplace operator, h-harmonic, Finsler vector bundle

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1 Introduction

It is well known that various kinds of Laplace operators play a very important role in differential geometry and physics, especially in the theory of harmonic integral and Bochner technique. Using Bochner technique, [2] proved some theorems on non-existence of certain types of vector fields (such as Killing vector fields and conformal Killing vector fields) on a compact Riemannian manifold whose Ricci curvature is positive or negative definite. Local expressions of any Laplace operators are called Weitzenböck formulas. These formulas primarily show the application of Bochner technique. Weitzenböck formulas of the Laplace operator for forms on Riemannian manifolds inspired intensive and important researches (see [3] and references therein). The key point in the theory of harmonic integrals and Bochner technique is to define a suitable Laplace operator, harmonic functions, and forms.

The purpose of this article is to generalize the Laplace operator in Riemannian manifolds to Finsler vector bundles as such bundles arise naturally in Finsler geometry [4–6]. We define h-Laplace operator first for functions and then for horizontal Finsler forms on E. We define the h-harmonic function and h-harmonic horizontal Finsler vector fields and using the h-Laplace

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operator, we prove some integral formulas of the scalar fields and horizontal Finsler vector fields on \( E \). In the end we obtain the Weitzenböck formula of the \( h \)-Laplace operator for the horizontal Finsler forms on \( E \).

2 Finsler Vector Bundles and Finsler Connections

In this section we introduce some notations and definitions \[1\]. Let us consider a compact manifold \( M, \dim M = m \), \( \{ U, (x^i) \} \) the coordinates in a local chart, and let \( \pi : E \to M \) be a vector bundle \( E \) of rank \( r \) over \( M \) with compact fibre \( \pi^{-1}(x) \).

**Definition 2.1** \[1\] A vector bundle \( F \) over the tangent bundle \( TM \) of \( M \) is said to be a Finsler vector bundle if it is isomorphic to the pull-back \( \pi^* E \).

**Definition 2.2** \[1\] A Finsler structure \( g \) on \( F \) is a smooth field of inner products in the fibres of \( F \).

Note that the natural projection, \( \pi \), defines on the vertical bundle, \( V = \{ X \in TE | d\pi(X) = 0 \} \) a structure of vector bundle of rank \( r \) over \( E \). We denote by \( X^V(E) \) the module of its sections, called vertical vector fields. The vertical vector fields form a subalgebra, \( X^V(E) \), of the Lie algebra, \( X(E) \), and is a finitely generated module over the ring of the smooth functions \( C^\infty(E) \) of \( E \).

A given supplementary subbundle \( H \) of \( V \), called horizontal bundle, that is, \( TE = H \oplus V \) (2.1)

defines a non-linear connection on \( E \), and we denote by \( X^H(E) \) the module of its sections, called horizontal Finsler vector fields. The horizontal Finsler vector fields on \( E \) form a finitely generated projective module \( X^H(E) \) over \( C^\infty(E) \). However, they do not, in general, form a subalgebra of the Lie algebra \( X(E) \).

It is a result of \[1\] that the horizontal bundle \( H \) and the vertical bundle \( V \) are Finsler vector bundles.

Let \( \{ U, (x^i, y^a) \} \) be the local coordinates on \( E \) and put \( \partial_i = \partial/\partial x^i, \dot{\partial}_a = \partial/\partial y^a \). In this article, the indices \( i, j, k, l, \cdots \) take the values \( 1, \cdots, m \), and \( a, b, c, d, \cdots \) take the values \( 1, \cdots, r \), and the usual summation convention for repeated indices is used. If \( N_a^i(x, y) \) are the coefficients of the non-linear connection, then the following vector fields, \( \{ \delta_i = \partial_i - N_a^i \dot{\partial}_a \}, \{ \dot{\partial}_a \} \),

are called the local adapted bases of \( H \) and \( V \), respectively. Their dual adapted bases are denoted by \( \{ dx^i \}, \{ dy^a = dy^a + N_a^i dx^i \} \).

For every vector field \( X \) on \( E \), there exists a unique decomposition \( X = X^H + X^V, X^H \in H, X^V \in V \).

\( X^H \) is called the horizontal part and \( X^V \) the vertical part of \( X \).

Let \( A(E) = \sum_{p=0}^{n+r} \Lambda^p(E) \) be the space of smooth forms on \( E \). We denote by \( H^* \) the dual vector bundle of \( H \). Then a horizontal 1-form or horizontal Finsler 1-form is a smooth section
of $\mathcal{H}^*$. If $\omega \in \wedge^1(E)$ is a Finsler $1$-form field on $E$, then it has a unique decomposition
\[
\omega = \omega^H + \omega^V.
\]
$\omega^H$ is called the horizontal component of $\omega$ and $\omega^V$ the vertical component of $\omega$. Note that $\omega^H(X^V) = \omega^V(X^H) = 0$. The horizontal Finsler forms are a graded subalgebra of $\mathcal{A}(E)$. This algebra is called the horizontal subalgebra, and is denoted by $\mathcal{A}_H(E)$.

A Finsler connection on $E$ is a linear connection $\nabla$ on $E$, which is both horizontal and vertical metrical. The existence of such Finsler connection on $E$ was proved in [4]. If $\nabla$ is a Finsler connection on $E$, we have
\[
\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V, \forall X, Y \in \mathcal{X}(E),
\]
\[
\nabla_X \omega = (\nabla_X \omega^H)^H + (\nabla_X \omega^V)^V, \forall X \in \mathcal{X}(E), \omega \in \mathcal{A}^*(E).
\]
If we put
\[
\nabla^H_X Y = \nabla_X u Y \quad \text{and} \quad \nabla^V_X Y = \nabla_X V Y, \forall X, Y \in \mathcal{X}(E),
\]
then $\nabla = \nabla^H + \nabla^V$, where $\nabla^H$ is called the $h$-covariant derivative and $\nabla^V$ is called the $v$-covariant derivative of the Finsler connection.

Locally there exists a well-determined set of differentiable functions $\{F^i_{jk}(x, y), \mathcal{F}^a_{bk}(x, y), C^i_{ja}(y, x), C^a_{bc}(x, y)\}$ on $\pi^{-1}(U)$ such that
\[
\nabla\delta_i\delta_j = F^i_{jk}\delta_i, \nabla\delta_i\delta_j = \mathcal{F}^a_{bk}\delta_j, \nabla\delta_j\delta_i = C^i_{ja}\delta_i, \nabla\delta_i\delta_j = C^a_{bc}\delta_a.
\]

**Definition 2.3** A Finsler tensor field $T$ on $E$ is called a horizontal Finsler covariant tensor field of type $p$ if
\[
T(X_1, \cdots, X_p) = T(X^H_1, \cdots, X^H_p),
\]
where $X_i \in \mathcal{X}(E)$, and $X_i^H$ denote the horizontal part of $X_i$, etc.

A horizontal differential form of type $p$ or horizontal Finsler $p$-form over an open set $\pi^{-1}(U) \subset E$ is a differentiable section, $T : (x, y) \rightarrow [(x, y), T_{i_1\cdots i_p}(x, y)]$, of $\otimes^p \mathcal{H}^*$ over $\pi^{-1}(U)$ such that the fibre coordinates, $T_{i_1\cdots i_p}(x, y)$, are skew-symmetric with respect to $i_1, \cdots, i_p$.

Under the coordinate transformation on $E$, $T_{i_1\cdots i_p}(x, y)$ changes according to the law:
\[
T_{i_1\cdots i_p}(x', y') = T_{i_1\cdots i_p}(x, y) \frac{\partial x^{i_1}}{\partial x'^{i_1}} \cdots \frac{\partial x^{i_p}}{\partial x'^{i_p}}.
\]
Similarly, we can define the horizontal Finsler contravariant tensor field of type $p$ on $E$. If $T^{i_1\cdots i_p}(x, y)$ is a horizontal Finsler contravariant tensor, then it changes according to the law:
\[
T^{i_1\cdots i_p}(x', y') = T^{i_1\cdots i_p}(x, y) \frac{\partial x'^{i_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{i_p}}.
\]
In the following, we denote, by $\wedge^p_H(E)$, the space of smooth horizontal Finsler $p$-forms on $E$.

The decomposition (2.1) induces two projections, $p^*_H$ and $p^*_V$, of $df$ onto the horizontal Finsler forms and vertical Finsler forms, respectively.

Set
\[
d_H = p^*_H \circ d, \quad d_V = p^*_V \circ d.
\]
Here
\[ d_H f = \delta_i(f)dx^i \quad \text{and} \quad d_V f = \hat{\partial}_a(f)\delta y^a, \quad f \in C^\infty(E). \]
So that the differential operator, \( d \), can be decomposed as \( d = d_H + d_V \) when it acts on horizontal finsler forms. Notice that if
\[ \varphi = \frac{1}{p!} \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \wedge_p^H(E), \]
we have
\[ d_H \varphi = \frac{1}{p!} \delta_k(\varphi_{i_1 \cdots i_p}) dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \wedge_{p+1}^H(E). \]
Furthermore, we have
\[ d_H (\varphi \wedge \psi) = d_H \varphi \wedge \psi + (-1)^p \varphi \wedge d_H \psi, \varphi \in \wedge_p^H(E), \psi \in \wedge_q^H(E). \]
Therefore, \( d_H \) restricted to the horizontal Finsler forms is a type preserving differential operator.

In this article, we are mainly interested in the horizontal bundle, \( \mathcal{H} \), and the \( h \)-covariant derivative, \( \nabla^H \), of the metrical Finsler connection \( \nabla \). In the following, we restrict our discussion on horizontal bundle, \( \mathcal{H} \), its dual bundle, \( \mathcal{H}^* \), and their tensor bundles.

Now we shall investigate the \( h \)-covariant derivatives of tensors of arbitrary rank by means of the metrical connection.

For a horizontal Finsler contravariant tensor field \( T^{i_2 \cdots i_p}(x, y) \), its horizontal covariant derivative is
\[ \nabla_{\hat{\partial}_i} T^{i_2 \cdots i_p} = \delta_i(T^{i_1 \cdots i_p}) + \sum_{\sigma=1}^P T^{i_1 \cdots i_{\sigma-1} k i_{\sigma+1} \cdots i_p} F^i_{ki}. \tag{2.2} \]
For a horizontal Finsler covariant tensor of type \( T_{i_1 \cdots i_p}(x, y) \), the corresponding horizontal covariant derivative is given by
\[ \nabla_{\hat{\partial}_i} T_{i_2 \cdots i_p} = \delta_i(T_{i_1 \cdots i_p}) - \sum_{\sigma=1}^P T_{i_1 \cdots i_{\sigma-1} k i_{\sigma+1} \cdots i_p} F^k_{i_\sigma i}. \tag{2.3} \]
By means of the above formulas it is natural to extend the horizontal covariant derivatives of tensor of mixed covariant - and contra-variant valencies.

### 3 Horizontal Laplace Operator for Functions on \( E \)

In this section we assume that there is a Riemannian structure, \( G \), on the total space, \( E \), of a vector bundle, \( \pi : E \to M \). If \( \mathcal{V} \) is the vertical space, then the vectors orthogonal to \( \mathcal{V} \), with respect to \( G \), uniquely determine the horizontal space, \( \mathcal{H} \), and the map, \( N : (x, y) \in E \to \mathcal{H} \) defines a non-linear connection, \( N \), on the total space \( E \). A Finsler connection, \( \nabla \) on \( E \), which preserves by parallelism the distributions, \( \mathcal{H} \) and \( \mathcal{V} \), is called compatible with the Riemannian structure, \( G \), if \( \nabla_X G = 0, \forall X \in \mathcal{X}(E) \). By Proposition 4.1 and 4.2 in [4], there exist two positive symmetric Finsler tensor field, \( g_{ij}(x, y) \) and \( G_{ab}(x, y) \), defined on \( E \) such that in the adapted local frame \( \{ \delta_i, \hat{\partial}_a \} \),
\[ G(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + G_{ab}(x, y) dy^a \otimes dy^b. \tag{3.1} \]
In fact
\[ g_{ij}(x, y) = G(\delta_i, \delta_j), G_{ab}(x, y) = G(\hat{\partial}_a, \hat{\partial}_b), \]

**Example 3.1** Let \((M, F)\) be a real Finsler manifold. Then the tangent bundle, \(TM\), endowed with the Sasaki-type metric constructed from the given Finsler metric, \(F\), is a Riemannian vector bundle and we have \(g_{ij}(x, y) = G_{ij}(x, y)\).

In the above case, as in [7], we denote by \(C = y^i \frac{\partial}{\partial y^i}\) the Liouville vector field, \(E = g_{ij}(x)y^i y^j\) the absolute energy of \((M, F)\), and \(S = y^i \delta_i\) the geodesic spray of \((M, F)\).

If \(F\) comes from a Riemannian metric, namely, \(F(x, y) = g_{ij}(x)y^i y^j\), then the Riemannian structure on \(TM\) can be chosen to be the Sasaki metric[8]:

\[ G(x, y) = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j, N_j^i = \Gamma_m^i_j(x)y^m, \quad (3.2) \]

and the geometry of \(TM\) with this metric was investigated in [8].

In the case that \((M, F)\) is a Finsler manifold, we denote

\[ P^i_{jk} := \frac{\partial N^i_k}{\partial y^b} - F^i_{jk}. \quad (3.3) \]

If \(P^i_{jk} = 0\), then we call \((M, F)\) a Landsberg space [9]. Obviously, if \((M, F)\) is a Riemannian manifold then by (3.2) and (3.3) we have \(P^i_{jk} = 0\). The further meaning of the vanishing of \(P^i_{jk}\) was investigated in [10]. In the following, we will see that the horizontal Laplace operator has a simple form both for scalar fields and for horizontal Finsler forms in Landsberg spaces.

Now we denote by \(g^{ik}\) and \(G^{bc}\) the inverse matrix components of \((g_{ij})\) and \((G_{ab})\), respectively. Then the coefficients \(\left(P^i_{jk}, F^{i}_{jk}, C^i_{jk}, C^{i}_{ab}\right)\) of the metrical Finsler connection, \(\nabla\), determined by the structure, \(G\), are given by

\[ P^i_{jk}(x, y) = \frac{1}{2} g^{kh}(\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}), F^i_{jk}(x, y) = \frac{\partial N^i_k}{\partial y^b}, \quad (3.4) \]

\[ C^i_{jk}(x, y) = 0, \quad C^a_{bc} = \frac{1}{2} G^{ad}(\delta_c G_{db} + \delta_d G_{dc} - \delta_d G_{bc}). \]

Notice that the \(h(v)\)-metrical of the Finsler connection gives

\[ \nabla_{\delta_k} g_{jk} = \nabla_{\delta_k} g^{jk} = 0 \quad \text{and} \quad \nabla_{\delta_a} G_{bc} = \nabla_{\delta_a} G^{bc} = 0. \quad (3.5) \]

Since \(g_{ij}\) and \(G_{ab}\) are two symmetric Finsler tensors, it follows from (3.4) that

\[ P^i_{jk} = F^i_{kj} \quad \text{and} \quad C^c_{ab} = C^c_{ba}. \quad (3.6) \]

Now if we denote \(g = \det(g_{ij})\) and \(G = \det(G_{ab})\), then, like in Riemannian case, it is easy to check that

\[ F^i_{ik} = \frac{1}{\sqrt{g}} \delta_i(\sqrt{g}) = \delta_i(\ln \sqrt{g}) \quad \text{and} \quad C^b_{ab} = \frac{1}{\sqrt{g}} \delta_a(\sqrt{g}) = \delta_a(\ln \sqrt{g}). \quad (3.7) \]

In the following we will repeatedly use (3.5) and (3.7) without explicit statement. Now we define the \(h(v)\)-Laplace operator for a scalar field, \(f \in C^\infty(E)\). Notice that

\[ df = d_H(f)dx^i + d_V(f)\delta y^a \]
for any scalar field, \( f \in C^\infty(E) \), and furthermore, we have \( d_V(f) = 0 \) if and only if \( f \in C^\infty(E) \) or \( f \in \pi^*C^\infty(M) \). Thus, we are mainly interested in the horizontal derivative operator \( d_H \).

Let \( dV = \sqrt{g} \sqrt{\mathcal{G}} dx^1 \wedge \cdots \wedge dx^m \wedge \delta y^1 \wedge \cdots \wedge \delta y^r \) be the volume form associated to the Riemannian structure, \( G \), on \( E \) and \( \mathcal{L}_X \) be the Lie derivative with respect to \( X \in \mathcal{X}(E) \), then as in [7], the notations as gradient and divergent on \( E \) can be introduced. The divergence of \( X = X^i \delta_i + \dot{X}^a \dot{\delta}_a \in \mathcal{X}(E) \) is defined by

\[
\mathcal{L}_X dV = (\text{div}X) dV.
\] (3.8)

If we denote \( \text{div}_h X =: \text{div}(X^i \delta_i) \) and \( \text{div}_v X =: \text{div}(\dot{X}^a \dot{\delta}_a) \), then we have

**Proposition 3.1** Let \( X = X^i \delta_i + \dot{X}^a \dot{\delta}_a \in \mathcal{X}(E) \), then we have

\[
\text{div} X = \text{div}_h X + \text{div}_v X,
\] (3.9)

where

\[
\text{div}_h X = \nabla_{\delta_i} X^i - P_i X^i \quad \text{and} \quad \text{div}_v X = \nabla_{\dot{\delta}_a} \dot{X}^a + \dot{P}_a \dot{X}^a,
\] (3.10)

and

\[
P_i = \dot{\delta}_a (N^a_i) - \delta_i (\ln \sqrt{g}) \quad \text{and} \quad \dot{P}_a = \dot{\delta}_a (\ln \sqrt{g}).
\] (3.11)

**Proof** Notice that

\[
[\delta_i, \delta_j] = [\delta_j (N^a_i) - \delta_i (N^a_j)] \dot{\delta}_a, \quad [\delta_i, \dot{\delta}_a] = \dot{\delta}_a (N^b_i) \dot{\delta}_b \quad \text{and} \quad [\dot{\delta}_a, \dot{\delta}_b] = 0,
\]

and

\[
[X, \delta_i] = -\delta_i (X^j) \delta_j + X^j [\delta_i (N^a_j) - \delta_j (N^a_i)] \dot{\delta}_a - \delta_i (\dot{X}^a) \dot{\delta}_a - \dot{X}^a \dot{\delta}_a (N^i_a) \dot{\delta}_b,
\]

\[
[X, \dot{\delta}_a] = -\dot{\delta}_a (X^k) \delta_k + X^k \dot{\delta}_a (N^b_k) \dot{\delta}_b - \dot{\delta}_a (\dot{X}^b) \dot{\delta}_b.
\]

It follows from (3.8) that

\[
(\text{div}X) \sqrt{\mathcal{G}} \sqrt{\mathcal{G}} = (\mathcal{L}_X dV)(\delta_1, \cdots, \delta_m, \dot{\delta}_1, \cdots, \dot{\delta}_r)
= X(\text{div}(\delta_1, \cdots, \delta_m, \dot{\delta}_1, \cdots, \dot{\delta}_r))
- \sum_{i=1}^m dV(\delta_1, \cdots, \delta_{i-1}, [X, \delta_i], \delta_{i+1}, \cdots, \delta_m, \dot{\delta}_1, \cdots, \dot{\delta}_r)
- \sum_{a=1}^r dV(\delta_1, \cdots, \delta_m, \dot{\delta}_1, \cdots, \dot{\delta}_{a-1}, [X, \dot{\delta}_a], \dot{\delta}_{a+1}, \cdots, \dot{\delta}_r)
= X(\sqrt{\mathcal{G}} \sqrt{\mathcal{G}}) + \left\{ \delta_i (X^i) - [X^k \dot{\delta}_a (N^a_k) - \dot{\delta}_a (\dot{X}^a)] \right\} \sqrt{g} \sqrt{g}
\]

and we have

\[
\text{div} X = [X^i \delta_i (\ln \sqrt{g}) + \delta_i (X^i) - X^i [\dot{\delta}_a (N^a_i) - \delta_i (\ln \sqrt{g})] + [\dot{X}^a \dot{\delta}_a (\ln \sqrt{g}) + \dot{\delta}_a (\dot{X}^a)] + \dot{X}^a \dot{\delta}_a (\ln \sqrt{g})].
\]

This completes the proof.

Now if we define \( \text{grad} f \) by

\[
G(\text{grad} f, X) = X f, \quad \forall X \in \mathcal{X}(E),
\]
then in the adapted frame \(\{\delta_i, \dot{\alpha}_a\}\), we have

\[
\text{grad} f = \text{grad}_h f + \text{grad}_v f,
\]

where

\[
\text{grad}_h f = g^{ij}(\delta_j f)\delta_i = g^{ij}(\nabla_{\delta_j} f)\delta_i,
\]

\[
\text{grad}_v f = g^{ab}(\dot{\alpha}_b f)\dot{\alpha}_a = g^{ab}(\nabla_{\dot{\alpha}_b} f)\dot{\alpha}_a.
\]

The Laplace operator of a scalar field, \(f \in C^\infty(E)\), is then defined as

\[
\triangle f = \text{div} \circ \text{grad} f.
\]

Now the \(h(v)\)-Laplace operator on \(E\) is defined by

\[
\triangle_h := \text{div}_h \circ \text{grad}_h \text{ and } \triangle_v := \text{div}_v \circ \text{grad}_v.
\]

We have

**Proposition 3.2** Let \(f \in C^\infty(E)\). Then

\[
\triangle f = \triangle_h f + \triangle_v f,
\]

where

\[
\triangle_h f = \frac{1}{\sqrt{g}} \delta_i [\sqrt{g} g^{ij} (\delta_j f)] - g^{ij} (\delta_j f) P_i = g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f - P_i g^{ij} \nabla_{\delta_j} f,
\]

\[
\triangle_v f = \frac{1}{\sqrt{g}} \dot{\alpha}_a [\sqrt{g} g^{ab} (\dot{\alpha}_b f)] + g^{ab} (\dot{\alpha}_b f) \dot{\alpha}_a = g^{ab} \nabla_{\dot{\alpha}_a} \nabla_{\dot{\alpha}_b} f + \dot{\alpha}_a g^{ab} \nabla_{\dot{\alpha}_b} f.
\]

**Proof** It follows from (3.10)–(3.13) directly.

We point out here that if the Riemannian structure, \(G\), on \(E = TM\) is induced by the Finsler metric \(F\) on \(M\), then, by \(g_{ij} = G_{ab}\) and (3.7), we have

\[
P_i = \dot{\alpha}_a (N^a_i) - \delta_i (\ln \sqrt{g}) = P^j_{ij}, \dot{\alpha}_a = \dot{\alpha}_a (\ln \sqrt{g}) = C^j_{ij}.
\]

**Proposition 3.3** If the Riemannian structure \(G\) on \(E = TM\) is the Sasaki-type metric, which is induced by the Finsler metric \(F\) on \(M\), then we have

\[
\text{div}_h \mathbf{C} = 0, \text{div}_v \mathbf{C} = n, \text{div}_h \mathbf{S} = \text{div}_v \mathbf{S} = 0, \triangle_h \mathbf{E} = 0, \triangle_v \mathbf{E} = 2n.
\]

**Proof** It follows from the proof of Theorem 1.1 in [7] and our definition of \(h(v)\)-divergence and \(h(v)\)-Laplace operator.

The expression of \(\triangle_h f\) is also obtained in [11–12]. Furthermore, if \(f \in C^\infty(E)\) or \(f \in \pi^*C^\infty(M)\), one obviously obtains that \(\triangle_v f = 0\). This motivates us to define the \(h\)-harmonicity on the total bundle \(E\): A scalar field \(f \in C^\infty(E)\) that satisfies \(\triangle_h f = 0\) is called a horizontal harmonic scalar field, or briefly, \(h\)-harmonic scalar field.

Thus the absolute energy of a Finsler space is \(h\)-harmonic. It is a main result of [7] that the absolute energy of a Finsler space \(\mathbf{E}\) cannot be harmonic in the usual sense.

Notice that if \((M, F)\) is a Landsberg space, we have \(P^i_{jk} = 0\), and by (3.7) and (3.11), \(P_i = P^j_{ji} = 0\). Thus we have
Theorem 3.4  If the Riemannian structure, $G$, on $TM$ comes from a Landsberg space $(M, F)$, then the $h$-Laplace operator, $\triangle_h$, has the following form:

$$\triangle_h f = \frac{1}{\sqrt{g}} \delta_i [\sqrt{g} g^{ij} (\delta_j f)] = g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f, f \in C^\infty(E).$$

Let $\pi : E \to M$ be a real vector bundle with a Riemannian structure, $G$. Notice that $(\text{div} X) dV = d[i(X) dV]$ for any vector field $X \in \mathcal{X}(E)$, and since any vector field, $X \in \mathcal{X}(E)$, can be decomposed into its horizontal and vertical part, we immediately have

$$(\text{div} X) dV = d[i(X) dV], X \in \mathcal{X}_H(E) \text{ or } X \in \mathcal{X}_V(E).$$

Thus we have

**Proposition 3.5** In a compact orientable Riemannian vector bundle, $E$, we have

$$\int_E (\nabla_{\delta_i} X^i - P_i X^i) dV = 0, \quad X = X^i \delta_i \in \mathcal{X}_H(E)$$

and

$$\int_E (\nabla_{\delta_a} Y^a - \dot{P}_a Y^a) dV = 0, \quad Y = Y^a \dot{\delta}_a \in \mathcal{X}_V(E).$$

**Theorem 3.6** In a compact orientable Riemannian vector bundle, $E$, we have

$$\int_E g^{ij} [\nabla_{\delta_i} \nabla_{\delta_j} f - P_i g^{ij} \nabla_{\delta_j} f] dV = 0$$

for any scalar field $f$ on $E$.

Since we have

$$\frac{1}{2} g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f^2 - P_i \nabla_{\delta_i} f^2 = f g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f + g^{ij} (\nabla_{\delta_i} f) (\nabla_{\delta_j} f) - f g^{ij} (\nabla_{\delta_i} f) P_j,$$

applying Theorem 3.6, we have

**Proposition 3.6** In a compact orientable Riemannian vector bundle, $E$, we have

$$\int_E [f g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f + g^{ij} (\nabla_{\delta_i} f) (\nabla_{\delta_j} f) - f g^{ij} (\nabla_{\delta_i} f) P_j] dV = 0$$

for any scalar field $f$ on $E$.

If a scalar field $f$ on $E$ satisfies

$$g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f - g^{ij} (\nabla_{\delta_i} f) P_j \geq 0$$

in the total space $E$, then $dV$ has a definite sign. Thus Theorem 3.6 gives

$$g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f = g^{ij} (\nabla_{\delta_i} f) P_j = 0$$

and consequently Theorem 3.7 gives

$$g^{ij} (\nabla_{\delta_i} f) (\nabla_{\delta_j} f) = 0,$$

from which we have

$$\nabla_{\delta_i} f = 0,$$
namely, $f$ is horizontally constant, since the form $g^{ij} (\nabla_{\delta_i} f)(\nabla_{\delta_j} f)$ is positive definite. Thus we have

**Theorem 3.8** If, in a compact orientable Riemannian vector bundle $E$, we have

$$g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f - g^{ij} (\nabla_{\delta_i} f) P_j \geq 0$$

for a scalar field $f$ on $E$, then $f$ is horizontally constant in the total space $E$, namely, $\delta f = 0$.

If a scalar field $f$, not horizontally constant on $E$, satisfies

$$g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f - g^{ij} (\nabla_{\delta_i} f) P_j = cf$$

with $c$ being a constant and cannot be zero. Substituting this in (3.15), we find

$$\int_E \left[ cf^2 + g^{ij} (\nabla_{\delta_i} f)(\nabla_{\delta_j} f) \right] dV = 0.$$

Thus, if $c > 0$, then we should have $f = 0$, and we have

**Theorem 3.9** If, in a compact orientable Riemannian vector bundle $E$, a scalar field, $f$, which is not horizontal constant satisfies

$$g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} f - g^{ij} (\nabla_{\delta_i} f) P_j = cf,$$

with $c$ being a constant, then the constant $c$ is negative.

Now if we apply the horizontal Laplacian, $\triangle_h$, to the square of the length of a horizontal Finsler vector field, $X^k(x,y)$, and notice that $\nabla_{\delta_i} g^{jk} = 0$, we have

$$\frac{1}{2} \left\{ g^{ij} \nabla_{\delta_i} \nabla_{\delta_j} (X^k X_k) - g^{ij} [\nabla_{\delta_i} (X^k X_k)] P_j \right\}$$

$$= g^{ij} (\nabla_{\delta_i} \nabla_{\delta_j} X^k) X_k + g^{ij} (\nabla_{\delta_i} X_k)(\nabla_{\delta_j} X^k) - g^{ij} [\nabla_{\delta_i} X^k] X_k P_j.$$

Thus, applying Theorem 3.6, we have

**Theorem 3.10** In a compact orientable Riemannian vector bundle, $E$, we have

$$\int_E \left\{ g^{ij} (\nabla_{\delta_i} \nabla_{\delta_j} X^k) X_k + g^{ij} (\nabla_{\delta_i} X_k)(\nabla_{\delta_j} X^k) - g^{ij} [\nabla_{\delta_i} X^k] X_k P_j \right\} dV = 0$$

(3.20)

for any horizontal Finsler vector field, $X^k(x,y)$.

If the second covariant derivative, $\nabla_{\delta_i} \nabla_{\delta_j} X^k$, of the horizontal Finsler vector, $X^k$, vanishes and $P_j = 0$, then $g^{ij} (\nabla_{\delta_i} X_k)(\nabla_{\delta_j} X^k)$ is positive definite. We have from (3.20) that

$$\nabla_{\delta_i} X^k = 0,$$

that is, the first horizontal covariant derivative vanishes.

**Theorem 3.11** If, in a compact orientable Riemannian vector bundle $E$, the second covariant derivative of a horizontal vector vanishes and $P_j = 0$, then the first horizontal covariant derivative vanishes too.

If a horizontal Finsler vector field $X^k(x,y)$, which is not zero, satisfies

$$g^{ij} (\nabla_{\delta_i} \nabla_{\delta_j} X^k) - g^{ij} \nabla_{\delta_i} X^k P_j = c X^k$$

(3.21)
with \( c \) being a constant, substituting (3.21) into (3.20), we have
\[
\int_E [cX^kX_k + g^{ij}(\nabla_{\delta_i}X_k)(\nabla_{\delta_j}X^k)]dV = 0.
\]

Thus, \( X^k \) is different from zero and \( c \) cannot be positive. Hence, we have

**Theorem 3.12** If, in a compact orientable Riemannian vector bundle \( E \), a horizontal Finsler vector field, \( X^k(x,y) \), other than zero vector satisfies
\[
g^{ij}(\nabla_{\delta_i}X^k) - g^{ij}\nabla_{\delta_i}X^k]P_j = cX^k,
\]
with \( c \) being a constant, then \( c \) is negative.

Consider a horizontal Finsler vector field, \( X^i(x,y) \), in a compact orientable Riemannian vector bundle \( E \) and calculate the divergence of the vector field
\[
X^i(\nabla_{\delta_i}X^j) - X^j(\nabla_{\delta_i}X^i).
\]
Then we obtain
\[
\nabla_{\delta_j}[X^i(\nabla_{\delta_i}X^j) - X^j(\nabla_{\delta_i}X^i)] = X^i(\nabla_{\delta_i}\nabla_{\delta_j} - \nabla_{\delta_j}\nabla_{\delta_i})X^i + (\nabla_{\delta_j}X^i)(\nabla_{\delta_i}X^j) - (\nabla_{\delta_i}X^j)^2
\]
\[= X^i[\nabla_{[\delta_i,\delta_j]} + \Omega(\delta_i,\delta_j)]X^i + (\nabla_{\delta_j}X^i)(\nabla_{\delta_i}X^j) - (\nabla_{\delta_i}X^j)^2.
\]

**Theorem 3.13** In a compact orientable Riemannian vector bundle \( E \), we have
\[
\int_E [X^i[\nabla_{[\delta_i,\delta_j]} + \Omega(\delta_i,\delta_j)]X^i + (\nabla_{\delta_j}X^i)(\nabla_{\delta_i}X^j) - (\nabla_{\delta_i}X^j)^2]dV = 0
\]
for any horizontal Finsler vector field, \( X^i(x,y) \).

**4 Horizontal Laplace Operator for Finsler Forms on \( E \)**

In this section, we are concerned with the Laplace operator, \( \Delta \), on horizontal Finsler forms on \( E \). We are mainly interested in the horizontal Laplace operator, \( \Delta_H \), for horizontal Finsler forms. We first give a Riemannian inner product in the space of the horizontal Finsler forms, then we give the local expression of the adjoint operator of \( d_H \) with respect to this inner product and finally, we define \( \Delta_H \) for horizontal Finsler forms. At the end of this section, we give the expression of \( \Delta_H \) in terms of the horizontal covariant derivatives of the metric Finsler connection \( \nabla \).

Now, let \( \varphi(x,y) \in \Lambda^p_H(E) \) and \( \psi(x,y) \in \Lambda^{p+1}_H(E) \). Locally,
\[
\varphi = \frac{1}{p!}\varphi_{i_1\cdots i_p}(x,y)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad \text{and} \quad \psi = \frac{1}{p!}\psi_{i_1\cdots i_p}(x,y)dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\]
Set
\[
\langle \varphi, \psi \rangle = \frac{1}{p!} \varphi_{i_1\cdots i_p} \psi_{i_1\cdots i_p} = \sum_{i_1 < \cdots < i_p} \varphi_{i_1\cdots i_p} \psi_{i_1\cdots i_p},
\]
where
\[
\varphi_{i_1\cdots i_p} = \varphi_{j_1\cdots j_p} g^{j_1 i_1} \cdots g^{j_p i_p}.
\]
Since $\varphi, \psi$ are horizontal Finsler tensor fields, the above inner product is independent of the local coordinates on $E$, thus $\langle \varphi, \psi \rangle$ is a global inner product on $E$. Especially,

$$|\varphi|^2 = \langle \varphi, \varphi \rangle = \frac{1}{p!} \varphi^{i_1 \cdots i_p} \varphi_{i_1 \cdots i_p} \geq 0.$$ 

Now we define the global inner product

$$(\varphi, \psi) = \int_E \langle \varphi, \psi \rangle dV \quad \text{and} \quad \|\varphi\|^2 = \int_E \langle \varphi, \varphi \rangle dV. \quad (4.1)$$

Consider a covariant tensor of rank $p$, which has $\varphi_{i_1 \cdots i_p}$ as components on $E$, and its horizontal covariant derivative is a new horizontal Finsler covariant tensor of rank $p + 1$ which has

$$\nabla_{\delta_i} \varphi_{i_1 \cdots i_p} = \delta_i (\varphi_{i_1 \cdots i_p}) - \sum_{\nu=1}^p \varphi_{i_2 \cdots i_{\nu-1} a i_{\nu+1} \cdots i_p} F^a_{i_{\nu} i} \quad (4.2)$$
as its components, where $F^a_{i_{\nu} i}$ are the horizontal connection coefficients of the metric Finsler connection.

If we denote by $k_1 \cdots \widehat{k}_\nu \cdots k_{p+1}$ the sequence got from $k_1 \cdots k_{p+1}$ by suppressing $k_\nu$, we have

$$(d_H \varphi)_{k_1 \cdots k_{p+1}} = \sum_{\nu=1}^{p+1} (-1)^{\nu-1} \delta_{k_\nu} (\varphi_{k_1 \cdots \widehat{k}_\nu \cdots k_{p+1}}). \quad (4.3)$$

Noting the symmetry of the horizontal connection coefficients, $F^i_{jk} = F^i_{kj}$, we can replace the horizontal derivatives by the horizontal covariant derivatives and obtain

$$(d_H \varphi)_{k_1 \cdots k_{p+1}} = \sum_{\nu=1}^{p+1} (-1)^{\nu-1} \nabla_{\delta_{k_\nu}} \varphi_{k_1 \cdots \widehat{k}_\nu \cdots k_{p+1}}. \quad (4.4)$$

To establish the expression for $\delta H$, namely, the codifferential of $d_H$, let $\varphi$ and $\psi$ be two horizontal Finsler forms of rank $p$ and $p + 1$, respectively, on $E$, then by (3.17) and the compactness of $E$, we have

$$\int_E \left[ \nabla_{\delta_i} (\varphi_{i_1 \cdots i_p} \psi^{i_1 \cdots i_p}) - \varphi_{i_1 \cdots i_p} \psi^{i_1 \cdots i_p} P_i \right] dV = 0.$$ 

By (3.5), the horizontal covariant derivatives of $g_{ij}$ and $g^{ij}$ are zero, and we have

$$\int_E (\nabla_{\delta_i} \varphi_{i_1 \cdots i_p}) \psi^{i_1 \cdots i_p} dV = - \int_E \left( \nabla_{\delta_i} \psi^{i_1 \cdots i_p} - \psi^{i_1 \cdots i_p} P_i \right) \varphi_{i_1 \cdots i_p} dV$$

$$= - \int_E \left[ g^{i_1 j_1} \cdots g^{i_p j_p} g^{ij} \left( \nabla_{\delta_i} \psi^{j_1 \cdots j_p} - \psi^{j_1 \cdots j_p} P_i \right) \right] \varphi_{i_1 \cdots i_p} dV.$$ 

Now, the term on the left hand side reduces to $p!(d_H \varphi, \psi)$, and by putting

$$(\delta_H \psi)_{i_1 \cdots i_p} = -g^{k_1 j_1} \cdots g^{k_p j_p} \left( \nabla_{\delta_k} \psi^{i_1 \cdots i_p} - \psi^{i_1 \cdots i_p} P_k \right), \quad (4.5)$$

the right-hand side term reduces to $p!(\varphi, \delta_H \psi)$. So we have $(d_H \varphi, \psi) = (\varphi, \delta_H \psi)$ for any horizontal Finsler form $\varphi \in \wedge^p_H(E)$ and $\psi \in \wedge^{p+1}_H(E)$.

Now, we define

$$\Delta_H = d_H \circ \delta_H + \delta_H \circ d_H, \quad \wedge^p_H(E) \rightarrow \wedge^p_H(E). \quad (4.6)$$
\(\triangle_H\) is a differential operator, called horizontal Laplace operator for horizontal Finsler forms on \(E\). A horizontal Finsler vector field, \(\varphi = \varphi dx^i \in \wedge^1_H(E)\), satisfying \(\triangle_H \varphi = 0\) is called \(h\)-harmonic Finsler 1-form on \(E\).

Now, we have the following Weitzenböck formula for \(\triangle_H\) on horizontal Finsler forms on \(E\).

**Theorem 4.1** The \(h\)-Laplace operator, \(\triangle_H\), for horizontal Finsler forms on \(E\) has the following form:

\[
(\triangle_H \varphi)_{i_1 \cdots i_p} = -g^{ik} \left[ \nabla_{\delta_k} \circ \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} + \sum_{\nu=1}^{p} (-1)^{\nu-1} (\nabla_{\delta_k} \nabla_{\delta_i} \varphi_{i_1 \cdots i_p}) \right] \\
+ g^{ik} \left[ \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} + \sum_{\nu=1}^{p} \nabla_{\delta_i} \varphi_{i_{1 \cdots \nu} \cdots i_p} \right] P_k, \varphi \in \wedge^p_H(E).
\] (4.7)

**Proof** Since by (4.5), we have

\[
(\delta_H \varphi)_{i_1 \cdots i_p} = -g^{ik} [\nabla_{\delta_k} \varphi_{i_1 \cdots i_p - 1} - \varphi_{i_1 \cdots i_p - 1} P_k],
\]

and thus by (4.4), we get

\[
(d_H \circ \delta_H \varphi)_{i_1 \cdots i_p} = \sum_{\nu=1}^{p} (-1)^{\nu} g^{ik} \nabla_{\delta_i} (\nabla_{\delta_k} \varphi_{i_1 \cdots i_p}) - \varphi_{i_1 \cdots i_p} P_k \\
= \sum_{\nu=1}^{p} (-1)^{\nu} g^{ik} \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} - \sum_{\nu=1}^{p} (-1)^{\nu} g^{ik} \nabla_{\delta_i} (\varphi_{i_{1 \cdots \nu}} P_k).
\] (4.8)

On the other hand, we have

\[
(\delta_H \circ d_H \varphi)_{i_1 \cdots i_p} = -g^{ik} \nabla_{\delta_k} \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} - \sum_{\nu=1}^{p} (-1)^{\nu} g^{ik} \nabla_{\delta_k} \nabla_{\delta_i} \varphi_{i_{1 \cdots \nu}} P_k \\
+ g^{ik} \left( \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} + \sum_{\nu=1}^{p} (-1)^{\nu} \nabla_{\delta_i} \varphi_{i_{1 \cdots \nu} \cdots i_p} \right) P_k.
\] (4.9)

By (4.8) and (4.9), we complete the proof.

In particular, for a form of degree zero, namely, a scalar field \(f \in C^\infty(E)\), we have \(d_H \circ \delta_H f = 0\) and

\[
\triangle_H f = \delta_H \circ d_H f = -g^{ik} \nabla_{\delta_k} \nabla_{\delta_i} f + g^{ik} (\nabla_{\delta_i} f) P_k.
\] (4.10)

Thus, we have the similar relationship between \(h\)-Laplace operator, \(\triangle_h\), for scalar fields and \(h\)-Laplace operator, \(\triangle_H\), for horizontal Finsler forms on \(E\):

\[
\triangle_H f = -\triangle_h f.
\] (4.11)

Similarly, if \((M, F)\) is a Landsberg space, then \(E = TM\) with the Sasaki-type metric \(G\) has the property \(P^i_j = 0\) and by (3.7) and (3.11), we have \(P^i_j = P^j_i = 0\). In this case we have the following Weitzenböck formula for \(\triangle_H\) on horizontal Finsler forms on \(E = TM\).

**Corollary 4.2** If the Riemannian structure \(G\) on \(TM\) comes from a Landsberg space \((M, F)\), then the \(h\)-Laplace operator, \(\triangle_H\), for horizontal Finsler forms on \(TM\) has the following form:

\[
(\triangle_H \varphi)_{i_1 \cdots i_p} = -g^{ik} \left[ \nabla_{\delta_k} \circ \nabla_{\delta_i} \varphi_{i_1 \cdots i_p} + \sum_{\nu=1}^{p} (-1)^{\nu-1} (\nabla_{\delta_k} \nabla_{\delta_i} \varphi_{i_1 \cdots i_p}) \right].
\] (4.12)
From (4.12), we can see that the Weitzenböck formula of horizontal Laplace operator $\Delta_H$ for horizontal Finsler forms on $E = TM$ is very similar to that of the usual Laplace operator for forms in Riemannian manifolds [13–14].

**Remark 4.1** If $E = TM$ is endowed with a Sasaki metric induced by the given Riemannian metric on $M$, then the non-linear connection on the bundle $E$ coincides with the Levi-Civita connection on $M$. If, furthermore, we restrict $\varphi$ to be the pull-back of forms on $M$, then the horizontal covariant derivative in this case coincide with the usual covariant derivative of the Levi-Civita connection. Thus, one can look Theorem 4.1 and Corollary 4.2 as the natural generalization of the Laplace operator from Riemannian manifold [13–14] to Finsler vector bundles.

**References**