The topological structure of fuzzy sets with endograph metric

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Abstract

For a non-degenerate convex subset \(Y\) of the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\), let \(\mathcal{K}(Y)\) be the family of all fuzzy sets of \(\mathbb{R}^n\), which are upper-semicontinuous, fuzzy convex and normal with compact supports contained in \(Y\). We show that the space \(\mathcal{K}(Y)\) with the topology of endograph metric is homeomorphic to the Hilbert cube \(Q = [-1, 1]^n\) iff \(Y\) is compact; and the space \(\mathcal{K}(Y)\) is homeomorphic to \(\{ (x_n) \in Q : \sup |x_n| < 1 \}\) iff \(Y\) is non-compact and locally compact.

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1. Introduction and main results

Throughout this paper, \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean space, \(\| \cdot \|\) is used to denote the usual Euclidean norm in \(\mathbb{R}^n\), \(I\) is the unit interval and \(\mathbb{N} = \{1, 2, \ldots\}\) is the set of all natural numbers. For a metric space \((X, d)\), the box metric \(d\) on \(X \times I\) is defined by

\[
d((x, s), (y, s')) = \max\{d(x, y), |s - s'|\}.
\]

The term fuzzy set of the metric space \(X\) stands for a function from \(X\) to \(I\). Let \(f\) be a fuzzy set of \(X\). For every \(t \in (0, 1]\), the level set \(f_t\) is defined by \(f_t = \{ x \in X : f(x) \geq t \}\) and the support of \(f\), denoted by supp\(f\), is the closure \([x \in X : f(x) > 0]\) of the set \([x \in X : f(x) > 0]\) in \(X\). Let us denote by \(\mathcal{K}(\mathbb{R}^n)\) the family of all fuzzy sets of \(\mathbb{R}^n\) with the following properties:

1. \(f\) is normal, i.e., \(f(x_0) = 1\) for some \(x_0 \in \mathbb{R}^n\);
2. \(f\) is fuzzy convex, i.e., \(f(rx + (1 - r)y) \geq \min\{f(x), f(y)\}\) for all \(x, y \in \mathbb{R}^n\) and \(r \in I\), or equivalently, \(f_t\) is convex in \(\mathbb{R}^n\) for every \(t \in (0, 1]\);
3. \(f\) is upper semicontinuous and
4. supp \(f\) is compact.

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Remark 1. In the past, an element in $\mathcal{K}(\mathbb{R}^n)$ was called a fuzzy number. However, the name “fuzzy number” is debatable because the crisp version of this notion is a compact convex subset of $\mathbb{R}^n$, not a number. Even in case $n = 1$, such fuzzy sets also generalize closed intervals, not numbers. Recently, Dubois et al. in [7,8] put forward a new definition of a so-called gradual real number. It is a generalization of “fuzzy real numbers” proposed by fuzzy set mathematicians in the seventies like Hutton [13] and further actively studied in fuzzy topology ([20,12,15], especially).

For a non-degenerate convex subset $Y$ of $\mathbb{R}^n$, let

$$\mathcal{K}(Y) = \{ f \in \mathcal{K}(\mathbb{R}^n) : \text{supp } f \subseteq Y \}.$$ 

On $\mathcal{K}(Y)$, some natural metrics were defined such as sendograph metric, endograph metric and $L^p$ metrics, etc. It should be noted that endograph metric is much weaker than sendograph metric although their constructions are quite similar, see [9]. In this paper, we restrict our attention to the topological space $\mathcal{K}(Y)$ with endograph metric. We aim to give the topological structure of space $(\mathcal{K}(Y), D)$ for any non-degenerate locally compact convex subset $Y$ of $\mathbb{R}^n$.

To define endograph metric, we at first give the definition of Hausdorff metric in the family Cl(X) of non-empty closed subsets of a metric space $(X, d)$. For $A, B \in \text{Cld}(X)$, we may define their Hausdorff distance $d_H$ as follows:

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$ 

Then $0 \leq d_H(A, B) \leq +\infty$.

Next, to prove our main results, endograph metric is defined on a larger family than $\mathcal{K}(\mathbb{R}^n)$. Let

$$\text{USCC}(X) = \{ f : X \to I : f \text{ is upper semicontinuous and supp } f \text{ is compact in } X \}.$$ 

For $f \in \text{USCC}(X)$, the following set is called the hypograph of $f$:

$$\text{hypo}(f) = \{(x, t) \in X \times I : t \leq f(x) \text{ and } x \in X \}.$$ 

The endograph metric $D$ on $\text{USCC}(X)$ is defined by

$$D(f, g) = d_H(\text{hypo}(f), \text{hypo}(g)).$$ 

Note that $D(f, g) < +\infty$ for any $f, g \in \text{USCC}(X)$.

Identifying an upper semicontinuous fuzzy set with its hypograph is a standard method to represent the fuzzy set through a set. Lots of works on upper semicontinuous fuzzy sets and their hypographs have already been established, e.g., see Beer [2]. In particular, such an approach was previously utilized for upper semicontinuous fuzzy sets on a locally compact space $X$, see [11], where the so-called hit-or-miss topology was employed on the space of hypographs of fuzzy sets defined on $X$. It is well known that the topology induced by the endograph metric and the hit-or-miss topology on $\text{USCC}(X)$ are identical if $X$ is a compact metric space. Diamond and his coauthors in [6] proved that $\text{USCC}(X)$ with the endograph metric is a compact absolute retract and hence has the fixed point property if $X$ is a compact metric space. In [26–31], we studied the topological structure of the space $\text{USCC}(X)$ with the endograph metric and some of its subspaces.

Clearly, for every convex set $Y$ of $\mathbb{R}^n$, the space $\mathcal{K}(Y) = (\mathcal{K}(Y), D|_{\mathcal{K}(Y)} \times \mathcal{K}(Y))$ is a subspace of $(\text{USCC}(\mathbb{R}^n), D)$. In the following, for a metric space $X$ and the convex set $Y$ of $\mathbb{R}^n$, $\text{USCC}(X)$ and $\mathcal{K}(Y)$ are always used to denote the metric spaces endowed with endograph metric, respectively. It is easily verified that there exists a natural isometric embedding from $\mathcal{K}(Y)$ into $\text{USCC}(Y)$ for every convex set $Y$ of $\mathbb{R}^n$. Thus we may consider $\mathcal{K}(Y)$ as a subspace of $\text{USCC}(Y)$ for every convex set $Y$ of $\mathbb{R}^n$. Moreover, it should be noted that the other different form of endograph metric on $\mathcal{K}(Y)$ was given in [10], where endograph metric was first introduced, but the two are equivalent.

In [10], Goetschel and Voxman asked the following question: Does $\mathcal{K}([a, b])$ have the fixed point property? In [1], a positive answer was given to this question. In the present paper, we show the following:

**Theorem 1.** For non-degenerate convex subset $Y$ of $\mathbb{R}^n$, the space $(\mathcal{K}(Y), D)$ is homeomorphic to ($\cong$) the Hilbert cube $Q = [-1, 1]^n$ if and only if $(\mathcal{K}(Y), D)$ is compact if and only if $Y$ is compact.

**Corollary 1.** The space $\mathcal{K}(Y)$ has the fixed point property for every compact convex subset $Y$ of $\mathbb{R}^n$. 
In [10], the authors proved that \( \mathbb{K}(\mathbb{R}), D \) may be written as a union of increasing sequence of subspaces which are homeomorphic to \( Q \). Let
\[
\Sigma = \{ (x_n) \in Q : \sup |x_n| < 1 \}.
\]
As a subspace of \( Q \), \( \Sigma \) is called the radial-interior of the Hilbert cube \( Q \). Moreover, \( \Sigma = \bigcup_{k=2}^{\infty} [1 + 1/k, 1 - 1/k]^{\infty} \) may be written as a union of increasing sequence of subspaces which are homeomorphic to \( Q \). In the present paper, we prove the following theorem.

**Theorem 2.** For a non-degenerate convex subset \( Y \) of \( \mathbb{R}^n \), the space \( (\mathbb{K}(Y), D) \) is homeomorphic to \( \Sigma \) if and only if \( (\mathbb{K}(Y), D) \) is non-compact and \( \sigma \)-compact if and only if \( Y \) is non-compact and locally compact.

As a corollary of Theorem 2, we have

**Corollary 2.** The space \( (\mathbb{K}(\mathbb{R}^n), D) \) is homeomorphic to \( \Sigma \) for every \( n \).

**Remark 2.** (1) Here we hope to recall the crisp versions of the above theorems, that is, topological structures of hyperspaces. The most well-known result is the Curtis–Schori–West Hyperspace Theorem which states that \( \text{Cld}(X) \) is homeomorphic to the Hilbert cube \( Q = [-1, 1]^\infty \) if and only if \( X \) is a non-degenerate connected and locally connected compact metrizable space (see [5,25]; cf. [16, Theorem 8.4.5]). In [4], it was proved that the hyperspace \( \text{Comp}(X) \) of all non-empty compact subsets of a metric space \( X \) with the Hausdorff metric is homeomorphic to \( Q \setminus \{ 0 \} \) if and only if \( X \) is non-compact, connected, locally connected and locally compact. For a non-compact space \( X \), Sakai and Yang in [22] proved that \( \text{Cld}_f(X) \), the set \( \text{Cld}(X) \) with the hit-or-miss topology, is homeomorphic to \( Q \setminus \{ 0 \} \) if and only if \( X \) is locally compact, locally connected separable metric space without compact connected component. For a non-degenerate convex set \( Y \) of a locally convex real topological vector space, let \( \text{CC}(Y) \) and \( \text{Conv}(Y) \) be the hyperspaces consisting of all non-empty compact convex subsets and of all non-empty convex subsets of \( Y \) with the Hausdorff metric. In [19], the authors proved that \( \text{CC}(Y) \) is homeomorphic to \( Q \) if \( Y \) is compact and \( \dim Y \geq 2 \). Evidently, we may think that Theorem 1 is a fuzzy version of this result and they are very similar. Moreover, it was also proved in [19] that \( \text{CC}(\mathbb{R}^n) \approx Q \setminus \{ 0 \} \) for \( n \geq 2 \). When a fuzzy version of this result is referred to as Theorem 2, or rather, as Corollary 2, we find that, unlike the case above, the topological structures of the families of all non-empty crisp convex compact sets and of all non-empty fuzzy convex compact sets in \( \mathbb{R}^n \) are not the same. The authors in [23] pointed out that \( \text{Conv}(\mathbb{R}^n) \) is rather complicated. Hence they considered the hyperspace \( \text{Conv}_f(\mathbb{R}^n) \), the set \( \text{Conv}(\mathbb{R}^n) \) with the hit-or-miss topology, and showed that \( \text{Conv}_f(\mathbb{R}^n) \approx Q \times \mathbb{R}^n \) for \( n \geq 2 \). (2) From Theorems 1 and 2, it follows that the topological structure of space \( \mathbb{K}(Y) \) for a locally compact convex subset \( Y \) of \( \mathbb{R}^n \) is independent of the dimension of \( Y \). (3) Note that \( \Sigma \) is \( \sigma \)-compact but not locally compact. By Theorem 2, we have that, for a \( \sigma \)-compact non-locally-compact non-degenerate convex subset \( Y \) of \( \mathbb{R}^n \), \( \mathbb{K}(Y) \) is not \( \sigma \)-compact. For example let \( Y = \{ x \in \mathbb{R}^n : \|x\| < 1 \} \cup \{ (1, 0, \ldots, 0) \} \). Then \( \mathbb{K}(Y) \) is not \( \sigma \)-compact and we do not know what is \( \mathbb{K}(Y) \) homeomorphic to.

In the following, we assume that \( Y \) is a non-degenerate convex subset of \( \mathbb{R}^n \) and \( \mathbb{K}(Y) = (\mathbb{K}(Y), D) \). For a metric space \( (X, d) \) and \( A \subset X \), \( \overline{A} \) and \( \text{int}(A) \) are the closure and the interior of \( A \) in \( X \), respectively. For \( r > 0 \), let
\[
B_d(A, r) = \{ z \in X : d(a, z) < r \text{ for some } a \in A \}
\]
and \( B_d(x, r) = B_d(\{ x \}, r) \) for \( x \in X \). The subscript \( d \) may be omitted if no confusion arises. For two mappings \( f, g : A \rightarrow (X, d) \) from a set \( A \) to a metric space \( (X, d) \), define \( d(f, g) = \sup \{ d(f(x), g(x)) : x \in X \} \leq + \infty \).

2. Some lemmas

At first, we recall some necessary fundamental concepts and facts on infinite-dimensional topology. For more information on them, refer to [16,17].

A metric space \( X \) is called an **absolute retract** (AR) provided that for every metric space \( Z \) containing \( X \) as a closed subspace there exists a continuous map \( r : Z \rightarrow X \) such that \( r|_X = \text{id}_X \). We say that a metric space \( X \) has the **disjoint-cells property** provided that for every natural number \( n \), every continuous function \( f : I^n \times [0, 1] \rightarrow X \) can be approximated (arbitrarily closely) by continuous maps sending \( I^n \times \{ 0 \} \) and \( I^n \times \{ 1 \} \) to disjoint sets.
We use the following Toruńczyk’s Characterization Theorem to show Theorem 1.

**Lemma 1 (The Toruńczyk’s Characterization Theorem; Toruńczyk [24], cf. van Mill [16, Corollary 7.8.4]).** A space $X$ is homeomorphic to the Hilbert cube $Q$ if and only if it is a compact AR with the disjoint-cells property.

A closed subset $A$ of the space $X$ is said to be a $Z$-set (strong $Z$-set, respectively) in $X$ if for any continuous map $\varepsilon : X \to (0, 1)$ there exists a continuous map $f : X \to X$, such that $d(f(x), x) < \varepsilon(x)$ for every $x \in X$ and $f(X) \cap A = \emptyset$ ($f(X) \cap A = \emptyset$, respectively). Concerning it, we have the following trivial result:

**Lemma 2.** If $(X, d)$ is a compact metric space and for every $\varepsilon > 0$ there exists a continuous map $f : X \to X$ such that $f(X)$ is a $Z$-set in $X$ and $d(f(x), x) < \varepsilon$ for every $x \in X$, then $X$ has the disjoint-cells property.

We shall next show that $\mathbb{H}(Y)$ is an AR if $Y$ is locally compact. To this end, let

$$N(Y) = \{ f \in \text{USCC}(Y) : f \text{ is normal} \}.$$

The convex hull of $A \subset \mathbb{R}^n$ is denoted by $(A)$. It is well known that the convex hull of every compact subset in $\mathbb{R}^n$ is compact.

We define a map $r : N(Y) \to \mathbb{H}(Y)$ as follows:

$$(r(f))_z = (f_z),$$

where $z \in I$. The following two lemmas show that $r$ is well defined.

**Lemma 3 (Sakai and Yaguchi [21]).** For each $A, B \in \text{Cld}(\mathbb{R}^n)$, $d_H(\text{cl}(A), \text{cl}(B)) \leq d_H(A, B)$.

**Lemma 4 (Kim and Kim [14]).** (1) For $u \in N(\mathbb{R}^n)$, then:

(a) $u_x \in \text{Comp}(\mathbb{R}^n)$ for all $x \in I$;
(b) $u_x \subseteq u_\beta$ for all $0 \leq \beta < x \leq 1$;
(c) $\lim_{\beta \to x^-} d_H(u_\beta, u_x) = 0$ for all $x \in (0, 1)$ and
(d) $\lim_{\beta \to x^+} d_H(u_\beta, u_x) \exists$ for all $x \in [0, 1)$ and $\lim_{\beta \to 0^+} d_H(u_\beta, u_0) = 0$.

Conversely, if $\{A_x\}_{x \in I}$ is a family of subsets of $\mathbb{R}^n$ satisfying conditions (a)–(d), then there exists a unique $u \in N(\mathbb{R}^n)$ such that $u_x = A_x$ for all $x \in I$.

(2) If $u \in \mathbb{H}(\mathbb{R}^n)$, then exactly similar statements with $\text{Comp}(\mathbb{R}^n)$ replaced by $\text{CC}(\mathbb{R}^n)$ hold.

**Lemma 5.** For each $f, g \in N(Y)$, $D(r(f), r(g)) \leq D(f, g)$.

**Proof.** Let $(y, x) \in \text{hypo}(r(f))$. By $y \in (r(f))_x = (f_x)$, we have that there exist $y_1, y_2, \ldots, y_k \in f_x$ such that $y = \sum_{i=1}^k \lambda_i y_i$, where $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$. Note that $(y_1, x), \ldots, (y_k, x) \in \text{hypo}(f)$. For $\varepsilon > 0$ and $i = 1, \ldots, k$, we can choose $(z_1, \gamma_1), \ldots, (z_k, \gamma_k) \in \text{hypo}(g)$ so that $d((y_i, x), (z_i, \gamma_i)) \leq D(f, g) + \varepsilon$. Let $\beta = \min\{\gamma_1, \ldots, \gamma_k\}$ and $z = \sum_{i=1}^k \lambda_i z_i$. Then $z_1, \ldots, z_k \in g_\beta$ and hence $z \in (g_\beta)$. Thus $(z, \beta) \in \text{hypo}(r(g))$. Moreover, we have that

$$\|y - z\| \leq \sum_{i=1}^k \lambda_i \|y_i - z_i\| \leq D(f, g) + \varepsilon,$$

$$|x - \beta| \leq D(f, g) + \varepsilon.$$

Hence

$$d((y, x), (z, \beta)) \leq D(f, g) + \varepsilon.$$

Similarly, for every $(z, \beta) \in \text{hypo}(r(g))$ and $\varepsilon > 0$, there exists $(y, x) \in \text{hypo}(r(f))$ such that

$$d((z, \beta), (y, x)) \leq D(f, g) + \varepsilon.$$

Thus $D(r(f), r(g)) \leq D(f, g)$. 

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Now a map \( m : \text{USCC}(Y) \setminus \{0\} \to N(Y) \) is defined by
\[
m(f)(y) = \frac{f(y)}{\max\{f(x) : x \in Y\}}
\]
for each \( y \in Y \). Then \( m \) is well defined. Let \( \overline{r} = r \circ m \).

**Lemma 6.** The map \( \overline{r} : \text{USCC}(Y) \setminus \{0\} \to \mathbb{K}(Y) \) is a retraction.

**Proof.** We first prove that \( m \) is continuous. It suffices to show that \( f \mapsto \max f \) is continuous, that is, if \( f_k \) is a sequence in \( \text{USCC}(\mathbb{R}^n) \) with the limit \( f \) then \( \lim_{k \to \infty} \max\{f_k(x) : x \in \mathbb{R}^n\} = \max\{f(x) : x \in \mathbb{R}^n\} \).

Conversely, there exists some \( \varepsilon_0 > 0 \) such that for each \( j \in \mathbb{N} \) there exists \( k_j > j \) satisfying \( \max\{f_{k_j}(x) : x \in \mathbb{R}^n\} - \max\{f(x) : x \in \mathbb{R}^n\} \geq \varepsilon_0 \).

This implies that \( D(f_{k_j}, f) \geq \varepsilon_0 \) for each \( j \in \mathbb{N} \), which is a contradiction with \( \lim_{n \to \infty} f_n = f \). Moreover, by Lemma 5, the map \( r \) is continuous. Hence \( \overline{r} \) is continuous. It is clear that \( \overline{r}(f) = f \) for every \( f \in \mathbb{K}(Y) \). Therefore, the map \( \overline{r} \) is a retraction.

**Remark 3.** Note that \( N(Y) \) is closed in \( \text{USCC}(Y) \) since, as mentioned in the proof of Lemma 6, \( f \mapsto \max f \) is continuous from \( \text{USCC}(Y) \) to \( I \). Moreover, \( m(f) = f \) if \( f \in N(Y) \). Hence \( D(\overline{r}(f), \overline{r}(g)) \leq D(f, g) \) for any \( f, g \in N(Y) \).

**Lemma 7.** (1) The space \( \text{USCC}(X) \) is homeomorphic to \( Q \) if and only if \( X \) is an infinite compact metric space [29].

(2) The space \( \text{USCC}(X) \) is homeomorphic to \( \Sigma \) if and only if \( X \) is a non-discrete non-compact locally compact separable metric space [31].

By applying Lemmas 6 and 7, we have the following.

**Lemma 8.** The space \( \mathbb{K}(Y) \) is an AR if \( Y \) is locally compact.

**Proof.** It follows from Lemma 6 that \( \overline{r} : \text{USCC}(Y) \setminus \{0\} \to \mathbb{K}(Y) \) is a retraction. Therefore, it suffices to show that \( \text{USCC}(Y) \setminus \{0\} \) is an AR if \( Y \) is locally compact. By Lemma 7, in our assumptions, \( \text{USCC}(Y) \approx Q \) or \( \text{USCC}(Y) \approx \Sigma \). From the definition of absorber, see [16, p. 280], it follows that both \( \Sigma \) and \( \Sigma \setminus \{p\} \), for any \( p \), are absorbers of \( Q \) and hence, by [16, Theorem 6.5.2], \( \Sigma \setminus \{p\} \approx \Sigma \). Moreover, it is well known that both \( \Sigma \) and \( Q \setminus \{p\} \), for any \( p \), are AR’s. We are done. \( \square \)

At last, we verify the following easy lemma:

**Lemma 9.** The space \( \mathbb{K}(Y) \) is compact if and only if \( Y \) is compact.

**Proof.** We note that \( \mathbb{K}(Y) \) is closed in \( \text{USCC}(Y) \) since \( \mathbb{K}(Y) \) is a retract of the closed subset \( N(Y) \) of \( \text{USCC}(Y) \). It follows from Lemma 7(1) that \( \mathbb{K}(Y) \) is compact if \( Y \) is compact, that is, “if” part holds. “Only if” part is trivial since \( y \mapsto y_1 \), where \( y_1 \in \mathbb{K}(Y) \) maps \( y \) to 1 and the other points to 0, is a closed embedding from \( Y \) into \( \mathbb{K}(Y) \). \( \square \)

### 3. Proof of Theorem 1

To show Theorem 1, we need two technical lemmas. Note that \( Y \) is a non-degenerate convex subset of \( \mathbb{R}^n \). Fix a point \( y_0 \in Y \). For every \( \varepsilon \in (0, 1] \), define a map \( l_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \) as follows:
\[
l_\varepsilon(y) = (1 + \varepsilon)y - \varepsilon y_0.
\]
It is not hard to verify the following lemma:

**Lemma 10.** (1) The map \( l_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \) is homeomorphic.

(2) The map \( l_\varepsilon \) is affine, that is, for any \( y_1, y_2 \in \mathbb{R}^n \) and \( t \in I \), we have
\[
l_\varepsilon(ty_1 + (1 - t)y_2) = tl_\varepsilon(y_1) + (1 - t)l_\varepsilon(y_2).
\]
(3) $I_2(Y) \supseteq Y$.
(4) $I_2(Y) \neq Y$ if $Y$ is bounded.

Moreover, we have the following lemma:

**Lemma 11.** For every $f \in \mathbb{K}(Y)$ and $y \in \mathbb{R}^n$, let

$$\varphi_e(f)(y) = f(l_e(y)).$$

Then $\varphi_e(f) \in \mathbb{K}(Y)$ and $\varphi_e : \mathbb{K}(Y) \to \mathbb{K}(Y)$ is continuous.

Furthermore, $D(\varphi_e(f), f) \leq \varepsilon \text{diam } Y$ for every $f \in \mathbb{K}(Y)$ and $\varphi_e(\mathbb{K}(Y))$ is a $\mathbb{Z}$-set in $\mathbb{K}(Y)$ if $Y$ is compact, where $\text{diam } Y$ is the diameter of $Y$.

**Proof.** From (1) in Lemma 10, it follows that $\varphi_e(f)$ is normal and $\text{supp } \varphi_e(f) = \iota_e^{-1}(\text{supp } f)$ is compact. (1) and (2) in Lemma 10 imply that $\varphi_e(f)$ is upper semicontinuous and fuzzy convex, respectively. Thus $\varphi_e(f) \in \mathbb{K}(Y)$. It is not hard to verify that

$$D(\varphi_e(f), \varphi_e(g)) \leq D(f, g)$$

for any $f, g \in \mathbb{K}(Y)$. Hence $\varphi_e : \mathbb{K}(Y) \to \mathbb{K}(Y)$ is continuous.

Now suppose that $Y$ is compact. By (4) in Lemma 10, there exists $y_1 \in Y$ such that $\varphi_e(f)(y_1) = 0$ for every $f \in \mathbb{K}(Y)$. This shows that $\varphi_e(\mathbb{K}(Y))$ is a $\mathbb{Z}$-set in $\mathbb{K}(Y)$. At last, to verify that $D(\varphi_e(f), f) \leq \varepsilon \text{diam } Y$ for every $f \in \mathbb{K}(Y)$, it suffices to note that for any $y \in \mathbb{R}^n$ and $t \in I$, $(y, t) \in \text{hypo}(\varphi_e(f))$ if and only if $(l_e(y), t) \in \text{hypo}(f)$ and

$$d((y, t), (l_e(y), t)) = \|y - l_e(y)\| = \varepsilon \|y - y_0\| \leq \varepsilon \text{diam } Y.$$

**Proof of Theorem 1.** It is trivial that $\mathbb{K}(Y) \approx Q$ implies that $\mathbb{K}(Y)$ is compact. It follows directly from Lemma 9 that $Y$ is compact if $\mathbb{K}(Y)$ is compact. At last, suppose that $Y$ is compact. By Lemmas 8 and 9, $\mathbb{K}(Y)$ is a compact AR. By Lemmas 2 and 11, $\mathbb{K}(Y)$ has the disjoint-cells property. Thus, we have $\mathbb{K}(Y) \approx Q$ by Lemma 1. □

**4. Proof of Theorem 2**

At first, we give the following two lemmas. Their proofs are similar to ones of corresponding results in the proof of [28, Theorem 2] and hence we omit them.

**Lemma 12.** For every compact $K$ of $\mathbb{K}(Y)$, and $\varepsilon > 0$, the set

$$A = \bigcup \{f^{-1}([\varepsilon, 1]) : f \in K\}$$

is compact in $Y$.

**Lemma 13.** If $\mathbb{K}(Y)$ is $\sigma$-compact, then $Y$ is locally compact.

In order to prove Theorem 2, we also need to verify the following.

**Lemma 14.** If $Y$ is non-compact, then every compact subset of $\mathbb{K}(Y)$ is a strong $\mathbb{Z}$-set in $\mathbb{K}(Y)$.

**Proof.** Suppose that $K$ is a compact subset of $\mathbb{K}(Y)$ and $\varepsilon : \mathbb{K}(Y) \to (0, 1)$ is a continuous map. Trivially, there exists $\gamma \in (0, 1)$ such that $\varepsilon(f) > 4\gamma$ for every $f \in B_{D}(K, 8\gamma)$. By Lemma 12, the set $A = \bigcup \{f^{-1}([\gamma, 1]) : f \in K\}$ is a compact subset of $Y$.

Since $Y$ is not compact, we may choose $x_0 \in Y \setminus A$ and define $\phi : \mathbb{K}(Y) \to \mathbb{K}(Y)$ as follows:

$$\phi(f) = r \left( f \lor \left( \frac{\varepsilon(f)}{2} x_0 \right) \right).$$
where \( r : N(Y) \to \mathbb{K}(Y) \) is defined in Section 2 and \( \varepsilon(f)/2x_0 \in \text{USCC}(\mathbb{R}^n) \) maps \( x_0 \) to \( \varepsilon(f)/2 \) and the other points to 0.

Trivially, the map \( \phi \) is continuous and \( D(f, \phi(f)) \leq \varepsilon(f)/2 < \varepsilon(f) \). Moreover, let \( \delta = \min\{d(x_0, A), \gamma\} \). Then

\[
\phi(\mathbb{K}(Y)) \cap B_{D}(K, \delta) = \emptyset.
\]

To show this fact, it suffices to verify that \( \phi(f) \notin B_{D}(K, \delta) \) for every \( f \in \mathbb{K}(Y) \). If \( \varepsilon(f) \geq 4\gamma \), then \( \phi(f)(x_0) \geq \varepsilon(f)/2 \geq 2\gamma \). But \( g(x) < \gamma \) for every \( x \in Y \setminus A \) and \( g \in K \). Therefore, \( D(\phi(f), g) \geq \delta \) for every \( g \in K \). Hence \( \phi(f) \notin B_{D}(K, \delta) \). If \( \varepsilon(f) < 4\gamma \), then \( f \notin B_{D}(K, 8\gamma) \). It follows from \( D(f, \phi(f)) < \varepsilon(f) \) that \( \phi(f) \notin B_{D}(K, 4\gamma) \). Hence \( \phi(f) \notin B_{D}(K, \delta) \).

As a conclusion, we have shown that there exists a continuous map \( \phi : \mathbb{K}(Y) \to \mathbb{K}(Y) \) such that \( D(f, \phi(f)) < \varepsilon(f) \) for every \( f \in \mathbb{K}(Y) \) and there exists an open neighborhood \( U \) of \( K \) in \( \mathbb{K}(Y) \) such that \( \phi(\mathbb{K}(Y)) \cap U = \emptyset \). Therefore, the subset \( K \) is a strong \( Z \)-set in \( \mathbb{K}(Y) \).

The following lemma is easy and its proof is omitted.

**Lemma 15.** If \( Y \) is locally compact, then there exists a sequence \( (Y_k)_{k=1}^\infty \) of non-degenerate convex compact subsets of \( Y \) such that \( Y_k \subset \text{int}(Y_{{k+1}}) \), where \( \text{int}(A) \) is the interior of \( A \) in \( Y \), for every \( k \) and every compact subset of \( Y \) is contained in some \( Y_k \).

The following lemma is the key to the proof of Theorem 2.

**Lemma 16.** If \( Y \) is locally compact and non-compact, then \( \mathbb{K}(Y) \) is strongly universal for compacta, that is, for every compact metric space \( A \) and its compact subspace \( B \), every continuous map \( \phi : A \to \mathbb{K}(Y) \) whose restriction \( \phi|B \) is an embedding and every \( \varepsilon > 0 \), there exists an embedding \( \psi : A \to \mathbb{K}(Y) \) such that \( \psi|B = \phi|B \) and \( D(\phi, \psi) < \varepsilon \).

**Proof.** By [3, Lemma 1.1] and Lemmas 8 and 14, without loss of generality, we may assume that \( \phi(B) \cap \phi(A \setminus B) = \emptyset \) and \( 0 < \varepsilon < 1 \). Define a sequence \( U_0, U_1, U_2, \ldots \) of open subsets of \( A \) such that

\[
\emptyset = U_0 \subset U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \cdots
\]

and

\[
A \setminus B = \bigcup_{i=1}^\infty U_i = \bigcup_{i=1}^\infty \overline{U_i}.
\]

Inductively define

\[
ev_0 = \frac{1}{4} \min\{\varepsilon, \inf\{D(\phi(a), \phi(b)) : a \in \overline{U_1}, b \in B\}\}
\]

and, for every \( i \geq 1 \),

\[
ev_i = \frac{1}{4} \min\{2e_{i-1}, \inf\{D(\phi(a), \phi(b)) : a \in \overline{U_{i+1}}, b \in B\}\}.
\]

Then, by the compactness of \( A \) and Lemmas 12 and 15, there exists a sequence \( (k_i)_i \in \mathbb{N} \) such that

\[
\phi(a)(x) < e_i
\]

for all \( a \in A \) and \( x \in Y \setminus Y_k \), where \( \{Y_k : k = 1, 2, \ldots\} \) is a sequence of subsets of \( Y \) satisfying the conditions in Lemma 15. For simplicity sake, we assume that \( k_i = i \) for every \( i \). Now we define a map \( h_i : A \to N(Y) \) for every \( i = 1, 2, \ldots \) as follows:

\[
h_i(a)(x) = \begin{cases} 
\phi(a)(x) & x \in \text{int}(Y_i), \\
\max\{e_i, \phi(a)(x)\} & x = Y_i \setminus \text{int}(Y_i), \\
0 & x \in Y \setminus Y_i.
\end{cases}
\]
Then $h_i$ is well defined and continuous. Let 
$$ g_i = r \circ h_i : A \to \mathbb{K}(Y_i), $$
where $r$ is defined in Section 2. Then we have
\begin{align*}
(\ast) & \quad D(g_i, \phi) \leq \varepsilon_i \quad \text{and} \\
(\ast\ast) & \quad D(g_i, g_{i+1}) \leq \varepsilon_i.
\end{align*}
We may construct inductively a sequence $(\psi_j : A \to \mathbb{K}(Y_{j+1}))$ of continuous maps such that
\begin{align*}
(1) & \quad \psi_0 = g_1, \\
(2) & \quad \psi_j|B = g_j|B, \\
(3) & \quad \psi_j|U_i : U_i \to \mathbb{K}(Y_{j+1}) \text{ is an embedding}, \\
(4) & \quad \psi_j|U_{i-1} = \psi_{j-1}|U_{i-1}, \\
(5) & \quad D(\psi_j(a), g_j(a)) \leq \varepsilon_i \quad \text{for any } a \in A \setminus U_i \quad \text{and} \\
(6) & \quad D(\psi_j(a), g_{j+1}(a)) \leq 3\varepsilon_{j-1} \quad \text{for any } a \in A \setminus U_{j-1}.
\end{align*}
Now we assume $\psi_k : A \to \mathbb{K}(Y_{k+1})$ to be constructed and satisfy (1)–(6). We shall construct $\psi_{k+1}$. Note that, for $k = 0$, the construction is also valid except a few differences shown below. For every integer $k \geq 0$, take a continuous map $\pi : A \to I$ satisfying
$$ \pi(A \setminus U_{k+1}) \subset \{0\} \quad \text{and} \quad \pi(A \setminus U_{k+1}) \subset \{1\}, $$
where, especially, the map $\pi$ maps every point of $A$ to 1 for $k = 0$. Using it, we may connect $\psi_k|U_k$ and $g_{k+1}|A \setminus U_{k+1}$ through $\max\{\psi_k, g_{k+1}\}$ by a map $l : A \to N(Y_{k+1})$ defined by the following: if $0 \leq \pi(a) \leq \frac{1}{2}$, let
$$ l(a)(x) = (1 - 2\pi(a))\psi_k(a)(x) + 2\pi(a) \max\{\psi_k(a)(x), g_{k+1}(a)(x)\} $$
and if $\frac{1}{2} \leq \pi(a) \leq 1$, let
$$ l(a)(x) = 2(1 - \pi(a)) \max\{\psi_k(a)(x), g_{k+1}(a)(x)\} + (2\pi(a) - 1))g_{k+1}(a)(x). $$
Then $l$ is well defined, continuous and has the following property (see [29, Lemma 6]):
$$ D(l(a), g_{k+1}(a)) \leq D(\psi_k(a), g_{k+1}(a)) \quad \text{for any } a \in A. $$
Thus we may define a continuous map $\psi'_{k+1} : A \to \mathbb{K}(Y_{k+1})$ as $l$ followed by $r$. Then we have:
\begin{align*}
(8) & \quad \psi'_{k+1}|U_k = \psi_k|U_k, \\
(9) & \quad \psi'_{k+1}|A \setminus U_{k+1} = g_{k+1}|A \setminus U_{k+1} \\
(10) & \quad D(\psi'_{k+1}(a), g_{k+1}(a)) \leq D(\psi_k(a), g_{k+1}(a)) \quad \text{for any } a \in A.
\end{align*}
Trivially, $\psi'_{1} = l = g_1 = \psi_0$ in the case that $k = 0$. Since $\mathbb{K}(Y_{k+1})$ is a Z-set of $\mathbb{K}(Y_2)$, $\psi'_{k+1}(A)$ is a Z-set in $\mathbb{K}(Y_{k+2})$. It follows from $\mathbb{K}(Y_{k+2}) \approx Q$ and [17, Corollary 5.3.12] that there exists a continuous map $\psi_{k+1} : A \to \mathbb{K}(Y_{k+2})$ such that:
\begin{align*}
(11) & \quad D(\psi_{k+1}, \psi'_{k+1}) < \varepsilon_{k+1}, \\
(12) & \quad \psi_{k+1}|B \cup U_k = \psi'_{k+1}|(B \cup U_k) \quad \text{and} \\
(13) & \quad \psi_{k+1}|U_{k+1} : U_{k+1} \to \mathbb{K}(Y_{k+2}) \text{ is an embedding}.
\end{align*}
Then $\psi_{k+1}$ satisfies (1)–(6) for $i = k + 1$. In fact, (2) follows (9) and (12), (13) implies (3). By (8) and (12), we have that (4) holds. By (11) and (9), we know that (5) also holds. At last, we verify (6). For every $a \in A \setminus U_k$, by (11), we have
$$ D(\psi_{k+1}(a), \psi'_{k+1}(a)) < \varepsilon_{k+1}. $$
Moreover, for $k \geq 1$, we have
$$ D(\psi'_{k+1}(a), g_{k+1}(a)) \leq D(\psi_k(a), g_{k+1}(a)) \quad \text{by (10)}. $$
For $k = 0$, the above formula also holds since $\psi'_1 = g_1$. Hence
\[
D(\psi_{k+1}(a), \phi(a)) \leq D(\psi_{k+1}(a), \psi'_{k+1}(a)) + D(\psi'_{k+1}(a), g_{k+1}(a)) + D(g_{k+1}(a), \phi(a)) \\
\leq e_k + e_{k+1} + 2e_k + e_{k+1} \\
\leq 3e_k.
\]
That is, (6) holds. This completes the inductive construction.

Now we define an extension $\psi : A \rightarrow \mathbb{K}(Y)$ of $\phi|B$ as follows:
\[
\psi(a) = \begin{cases} 
\phi(a) & \text{if } a \in B, \\
\psi_j(a) & \text{if } a \in \overline{U_j}.
\end{cases}
\]
Then $\psi$ is well defined and is as required. In fact, since $\psi|U_i = \psi_j|U_i$ and $U_i$ is open, $\psi$ is continuous in $U_i$ for every $i$ and hence $\psi$ is continuous in $A \setminus B$. Moreover, for every $\delta > 0$, choose $i$ large enough such that $3e_i < \delta$. Then for every $a \in A \setminus (\overline{U_i} \cup B)$, there exists $j > i$ such that $a \in U_j \setminus U_{j-1}$. Thus, by (6),
\[
D(\psi(a), \phi(a)) = D(\psi_j(a), \phi(a)) \\
\leq 3e_{j-1} < 3e_i < \delta.
\]
Therefore, we know that $\psi$ is continuous in $B$ since $\phi$ is continuous in $B$.

Next, we show that $\psi$ is one-to-one, that is, for any $a, b \in A$, we have that $\psi(a) \neq \psi(b)$ if $a \neq b$. Trivially, we only consider the case that $a \in U_i \setminus U_{i-1}$ for some $i$ and $b \in B$. Then $\psi(a) = \psi_j(a)$. Thus, by (6), we have
\[
D(\psi(a), \phi(a)) \leq 3e_{i-1}.
\]
It follows from the definition of $e_{i-1}$ that $\psi(a) \neq \phi(b) = \psi(b)$.

At last, it directly follows from (6) that $D(\phi, \psi) < \varepsilon$. □

To prove Theorem 2, we need also the following result:

**Lemma 17** (Mosgiski [18]). Let $X$ be an AR. Then $X$ is homeomorphic to $\Sigma$ if and only if $X$ is $\sigma$-compact and strongly universal for compacta and every compact subset $A$ of $X$ is a strong $Z$-set in $X$.

**Proof of Theorem 2.** Trivially, if $\mathbb{K}(Y) \approx \Sigma$, then $\mathbb{K}(Y)$ is non-compact and $\sigma$-compact. Next, if $\mathbb{K}(Y)$ is $\sigma$-compact and non-compact, then Lemmas 13 and 9 imply that $Y$ is locally compact and non-compact. At last, we assume that $Y$ is locally compact and non-compact. In Section 2, we have shown that $\mathbb{K}(Y)$ is an AR. Choose a sequence $(Y_i)$ satisfying the conditions of Lemma 15 for $Y$. Since $\mathbb{K}(Y) = \bigcup_{i=1}^{\infty} \mathbb{K}(Y_i)$ and, by Theorem 1, $\mathbb{K}(Y_i) \approx Q$ for every $i \in \mathbb{N}$, $\mathbb{K}(Y)$ is $\sigma$-compact. By Lemma 16, the space $\mathbb{K}(Y)$ is strongly universal for compacta. And Lemma 14 implies that every compact subset of $\mathbb{K}(Y)$ is a strong $Z$-set in $\mathbb{K}(Y)$. It follows from Lemma 17 that $\mathbb{K}(Y) \approx \Sigma$. □

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**References**