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Nontrivial solutions for a class of quasilinear problems with jumping nonlinearities via a cohomological local splitting

Jing Zhang, Chong Li, Xiaoping Xue

1. Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \); we consider the quasilinear elliptic boundary value problem

\[
\begin{cases}
-\Delta_p u = au^{p-1}_+ - bu^{p-1}_- + f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega
\end{cases}
\]  

(1.1)

where \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian operator of \( u \) with \( 1 < p < \infty \), \( u_\pm = \max\{\pm u, 0\} \) and \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \). We can see that \( u \equiv 0 \) is the trivial solution of Eq. (1.1) when \( f(x, 0) \equiv 0 \). Therefore, in this case we seek the nontrivial solution of (1.1). Eq. (1.1) in the case of \( f \equiv 0 \) has been considered by Fucík [1] \((p = 2)\) and by other authors in [2–4]. The Fucik spectrum of the \( p \)-Laplacian on \( W^{1,p}_0(\Omega) \) is defined as the set \( \Sigma_p \) of those points \((a, b) \in \mathbb{R}^2\) for which the problem

\[
-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1}, \quad u \in W^{1,p}_0(\Omega)
\]

has nontrivial solutions (see [2]). Here \( u^+(x) = \max\{\pm u, 0\} \). In the case of \( a = b = \lambda \in \mathbb{R} \), the equation reads

\[
-\Delta_p u = \lambda|u|^{p-2}u.
\]

Hence \((\lambda, \lambda)\) belongs to \( \Sigma_p \) if and only if \( \lambda \) is an eigenvalue of \( -\Delta_p u \), i.e. there is a nonzero weak solution \( u \in W^{1,p}_0(\Omega) \) to \( -\Delta_p u = \lambda|u|^{p-2}u \). The set of all eigenvalues of \( -\Delta_p \) is denoted by \( \sigma(-\Delta_p) \). As shown in [5], the first eigenvalue \( \lambda_1 \) of \( -\Delta_p \) is positive, simple and admits a positive eigenfunction \( \varphi_1 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega) \) with

\[
\int_\Omega \varphi_1^2 \, dx = 1.
\]
The nonresonance case for Eq. (1.1), where \((a, b)\) is not in the set \(\Sigma_p\), was recently studied by Cuesta et al. [2] and Perera [6,7]. In the last few years, Drábek and Robinson [8] and Perera [7] have obtained a nontrivial solution of Eq. (1.1) in the case where \((a, b)\) lies in a neighborhood of a point of the form \((a_0, b_0)\) with \(a_0 \in X \setminus \sigma(-\Delta_p)\), but had difficulty with the lack of compactness of the associated variational functional. They overcame the difficulty by constructing a sequence of approximating nonresonance problems using linking and min–max arguments and passing to the limit. For \(a = b \in X \setminus \sigma(-\Delta_p)\) and \(q = 1\), the solvability of Eq. (1.1) was established by Drábek and Robinson [8]. The special case where \(a = b = \lambda_k\) was recently studied by Cuesta et al. [2] and Dancer and Perera [3]. Thus, in this paper we go further in this direction, establishing another existence result which has already been proved for Eq. (1.1) in [6,9,10]. The purpose of this paper is to study the general resonance case in the sense that \(\lambda_k < a, b < \lambda_{k+1}\) for two consecutive variational eigenvalues \(\lambda_k < \lambda_{k+1}\) of \(-\Delta_p\) on \(W_0^{1,p}(\Omega)\) (see the following for the definition of the variational spectrum), but using variational arguments and a cohomological local splitting.

In order to seek the nontrivial solution of Eq. (1.1), we need an additional condition on the function \(f\) near \(t = 0\) which involves the eigenvalues of the \(p\)-Laplacian problem.

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(1.2)

So, we set up the Sobolev space \(W_0^{1,p}(\Omega)\) equipped with the norm

\[
\|u\|_W = \left( \int_{\Omega} |\nabla u|^p dx \right)^{1/p}
\]

and let

\[
T = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |u|^p dx = 1 \right\}.
\]

Denote by \(F\) the class of symmetric subsets of \(T\) and by \(i(M)\) the \(\mathbb{Z}_2\)-cohomological index of \(M \in F\) (see [11]). Then

\[
\lambda_k := \inf_{M \in F, i(M) \geq k} \sup_{u \in M} \int_{\Omega} |\nabla u|^p dx, \quad k \geq 1,
\]

is an increasing and unbounded sequence of eigenvalues of (1.2) (see [12]).

Setting \(F(x, u) = \int_0^1 f(x, s) ds\), let us assume relevant conditions on \(f(x, u)\):

\((F_1)\) there are \(R \geq 1\) and \(p < r < p^*\) such that for almost all \(x \in \Omega\) and all \(|t| \geq R\), \(0 < F(x, t) \leq \frac{1}{r} f(x, t)t\);

\((F_2)\) there are \(M_1 > 0, M_2 > 0\) and \(r \leq q < p^*, \text{where } p^* = \begin{cases} np \over n-p & \text{if } p < N, \\ +\infty & \text{otherwise,} \end{cases}\) such that for almost all \(x \in \Omega\) and all \(|t| \leq R\), \(|f(x, t)| \leq M_1 t^q + M_2 |t|^{r-1-}\);

\((F_3)\) there exists \(k \geq 1\) such that \(\lambda_k < \lambda_{k+1}\) and there are \(\lambda_k < \underline{\lambda} \leq \lambda < \lambda_{k+1}\) and \(\delta > 0\) such that for almost all \(x \in \Omega\) and all \(|t| \leq \delta\), \(\lambda/2p \cdot |t|^p \leq F(x, t) \leq \lambda/2p \cdot |t|^p\).

Then we have the main result of this paper:

**Theorem 1.1.** Assume that \((F_1)-(F_3)\) hold and \((a, b) \in [\frac{1}{2}, \frac{7}{2}]) \times [\frac{1}{2}, \frac{7}{2}]). Then we have a nontrivial solution of problem (1.1) in \(W_0^{1,p}(\Omega)\).

The proof of Theorem 1.1 is essentially based on variational methods, as Eq. (1.1) is the Euler–Lagrange equation of the functional

\[
J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{a}{p} \int_{\Omega} u^p dx - \frac{b}{p} \int_{\Omega} u^p dx - \int_{\Omega} F(x, u) dx.
\]

(1.3)

The weak solutions of Eq. (1.1) coincide with the critical points of the functional \(J\) defined on \(W_0^{1,p}(\Omega)\). In general, \(J\) is Fréchet differentiable along directions of the Sobolev space \(W_0^{1,p}(\Omega)\). Thus, the hypotheses of Theorem 1.1 allow us to use the notion of a cohomological local splitting introduced in Perera et al. (see [12]) applied to the functional \(J\) defined in the whole space \(W_0^{1,p}(\Omega)\). On the other hand, \((F_3)\) implies that 0 is a critical point of \(J\) in \(X\) with \(J(0) = 0\), so we want to show that \(J\) has another nontrivial critical point in \(X\).
In the last few years, an increasing interest has arisen in the study of problem (1.1). More recently, an abstract setting has been stated such that some existence results for solutions of Eq. (1.1) have been extended to the following boundary value problem (see [13,14]):

\[
\begin{cases}
-\text{div}\sigma(x, \nabla u) = au^{p-1} - bu^{p-1} + f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where \( A = A(x, \xi) \) is a Carathéodory function on \( \Omega \times \mathbb{R}^N \) such that the partial derivatives \( \sigma(x, \xi) = (\frac{\partial A}{\partial u_1}(x, \xi), \ldots, \frac{\partial A}{\partial u_n}(x, \xi)) \) exist for almost all \( x \in \Omega \) and all \( \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \) and are also Carathéodory functions, and \( f = f(x, t) \) is a Carathéodory function on \( \Omega \times \mathbb{R} \).

Note that problem (1.4) generalizes the analogous \( p \)-Laplacian problem (1.1). Thus, in order to state our main result, the function \( A = A(x, \xi) \) and its derivatives have to satisfy some assumptions which are trivially verified by the function \( A(x, \xi) = |\xi|^p/p \), which corresponds to the \( p \)-Laplacian operator. Assume:

1. there are \( p > 1 \) and \( A_1, A_2 > 0 \) such that for almost all \( x \in \Omega \) and all \( \xi \in \mathbb{R}^N \),
   \[ A_1|\xi|^p \leq A(x, \xi) \leq A_2|\xi|^p; \]
2. for almost all \( x \in \Omega \) and \( \xi \neq \xi^* \) in \( \mathbb{R}^N \),
   \[ (\sigma(x, \xi) - \sigma(x, \xi^*)) \cdot (\xi - \xi^*) > 0; \]
3. there is a \( c_1 > 0 \) such that for almost all \( x \in \Omega \) and all \( \xi \in \mathbb{R}^N \),
   \[ |\sigma(x, \xi)| \leq c_1(1 + |\xi|^{p-1}); \]
4. there are \( c_2, c_3 > 0 \) such that for almost all \( x \in \Omega \) and all \( \xi \in \mathbb{R}^N \),
   \[ c_2|\xi|^p \leq c_3|\sigma(x, \xi) \cdot \xi| \leq rA(x, \xi) - \sigma(x, \xi) \cdot \xi. \]

Then we have the following result:

**Theorem 1.2.** Assume that \((A_1)-(A_3)\) and \((F_1)-(F_2)\) hold. Moreover, suppose that a further condition is satisfied:

\[ \text{there exists a } k \geq 1 \text{ such that } \lambda_k < \lambda_{k+1} \text{ and there are } \lambda_k < \lambda < \lambda_{k+1} \text{ and } \delta > 0 \text{ such that for almost all } x \in \Omega \text{ and all } |t| \leq \delta, } \]
\[ A_2\lambda/2 \cdot |t|^p \leq F(x, t) \leq A_1\lambda/2 \cdot |t|^p. \]

Then for \( (a, b) \in [pA_2\lambda/2, pA_1\lambda/2] \times [pA_2\lambda/2, pA_1\lambda/2] \) we have a nontrivial solution of problem (1.4) in \( W_0^{1,p}(\Omega) \).

2. Preliminaries

Now we recall the notion of cohomological critical groups of an isolated critical point \( u \) of a \( C^1 \) functional \( J \) briefly. Let \( U \subset X \) be an isolated neighborhood of \( u \) such that there are no critical points of \( J \) in \( U \setminus \{u\} \), and \( X \) is a Banach space. The cohomological critical groups of \( J \) at an isolated critical point \( u \) are defined as

\[
C^q(J, u) = H^q(J^\ast \cap U, (J^\ast \setminus \{u\}) \cap U), \quad q = 0, 1, 2, \ldots
\]

where \( c = J(u) \) is the corresponding critical value, \( J^\ast = \{u \in X : J(f) \leq c\} \) is a level set of \( J \), and \( H^q(\cdot, \cdot) \) are the Alexander–Spanier cohomology groups with \( \mathbb{Z}_2 \)-coefficients, \( q = 0, 1, 2, \ldots \). They are independent of the choices of \( U \), and hence are well defined. For the details, we refer the reader to Chang [15]. Then, let us recall the notion of a cohomological local splitting introduced in [16].

**Definition 2.1.** Assume that zero is an isolated critical point of \( J \) with \( J(0) = 0 \). We say that \( J \) has a cohomological local splitting near zero in dimension \( k \) if there is a \( \rho > 0 \) and disjoint nonempty symmetric subsets \( D_0 \) and \( B_0 \) of the sphere \( S_\rho = \{u \in X : ||u||_X = \rho\} \) such that

\[ i(D_0) = i(S_\rho \setminus B_0) = k \]

and

\[
\begin{cases}
J(u) \leq 0, & \text{if } u \in D, \\
J(u) > 0, & \text{if } u \in B \setminus \{0\},
\end{cases}
\]

where \( D = \{tu : u \in D_0, 0 \leq t \leq 1\} \) and \( B = \{tu : u \in B_0, 0 \leq t \leq 1\} \).

The following lemma is proved in [12].

**Lemma 2.2.** If \( J \) has a cohomological local splitting near zero in dimension \( k \), then \( C^k(J, 0) \neq 0 \).
In order to prove the existence of a critical point, we need to know that the homotopy type of sublevel sets can change only when crossing a critical level. So, we introduce Cerami’s variant of the Palais–Smale Condition (see [17]) and the Second Deformation Lemma (see [12]). It is well known that the Palais–Smale Condition and Cerami’s variant of the Palais–Smale Condition imply compactness of the critical set at each level \( c \in \mathbb{R} \). In the case where \( (a, b) \) is not in \( \Sigma_p, J \) satisfies the Palais–Smale Condition. To treat the case where \( (a, b) \in \Sigma_p \), we will use Cerami’s variant of the Palais–Smale Condition. Now we recall the definition of Cerami’s variant of the Palais–Smale Condition.

**Definition 2.3** (Cerami’s Variant of the Palais–Smale Condition). The functional \( J \) satisfies Cerami’s variant of the Palais–Smale Condition at level \( c \in \mathbb{R} \), for short (CPS)_c, if any sequence \( \{u_n\}_{n \in \mathbb{N}} \subset X \) such that

\[
\lim_{n \to +\infty} J(u_n) = c \quad \text{and} \quad \lim_{n \to +\infty} \|dj(u_n)\|_{X'}(1 + \|u_n\|_X) = 0,
\]

converges in \( X \) up to subsequences.

We say that \( J \) satisfies Cerami’s variant of the Palais–Smale Condition if \( J \) satisfies Cerami’s variant of the Palais–Smale Condition at any level \( c \in \mathbb{R} \).

**Lemma 2.4** (Second Deformation Theorem). If \( -\infty < a < b \leq +\infty \) and \( J \) has only a finite number of critical points at the level \( a \), has no critical values in \( (a, b) \), and satisfies (CPS)_c for all \( c \in [a, b] \), then \( J^a \) is a deformation retract of \( J \setminus K^b \), where \( K^b = \{ u \in X : dj(u) = 0, J(u) = b \} \) is the set of critical points of \( J \) at the level \( b \) if \( b < +\infty \), while \( J^b = X \) and \( K^b = \emptyset \) if \( b = +\infty \).

Now, let us consider the problem \((1.1)\) and let \( X = W_0^{1, p}(\Omega) \) be the Banach space with \( X' = W^{-1, p}(\Omega) \), the dual space of \( W_0^{1, p}(\Omega) \). Consider the functional

\[
J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{a}{p} \int_{\Omega} u^p dx - \frac{b}{p} \int_{\Omega} u^p dx - \int_{\Omega} F(x, u) dx, \quad u \in X.
\]

Then we point out some results related to the conditions of Theorems 1.1.

**Remark 2.5.** There exists a function \( M_3 \in L^\infty(\Omega) \), \( M_3(x) > 0 \) a.e. in \( \Omega \), and a constant \( M_4 \geq 0 \) satisfying

\[
F(x, t) \geq M_3(x)|t|^r - M_4 \quad \text{a.e. } x \in \Omega, \forall t \in \mathbb{R}.
\]

Now we show the compactness condition. The following lemma can be proved by the same arguments as in [13]. Here we give a proof for the reader’s convenience.

**Lemma 2.6.** Assume that \((F_1)\)–\((F_2)\) hold. Then \( J \) satisfies (CPS)_c in \( W_0^{1, p}(\Omega) \) for all \( c \in \mathbb{R} \).

**Proof.** Let \( \{u_n\}_{n \in \mathbb{N}} \subset X \) be a Cerami sequence for fixing \( c \in \mathbb{R} \), namely

\[
\lim_{n \to +\infty} J(u_n) = c, \quad \lim_{n \to +\infty} \|dj(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.
\]

(2.2)

Firstly, by a standard argument, we can prove the boundedness of \( \{u_n\} \) in \( X \).

In fact, let us remark that, in particular, in the following, we use \( \{\varepsilon_n\} \) for any infinitesimal sequence depending only on \( \{u_n\} \). (2.2) implies

\[
\lim_{n \to +\infty} \|dj(u_n)\|_{X'}\|u_n\|_X = 0,
\]

\[
J(u_n) = \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx - \frac{a}{p} \int_{\Omega} u_n^p dx - \frac{b}{p} \int_{\Omega} u_n^p dx - \int_{\Omega} F(x, u_n) dx = c + \varepsilon_n.
\]

(2.3)

Therefore, we have

\[
\varepsilon_n = (dj(u_n), u_n) = \int_{\Omega} |\nabla u_n|^p dx - a \int_{\Omega} u_n^p dx - b \int_{\Omega} u_n^p dx - \int_{\Omega} f(x, u_n) u_n dx.
\]

(2.4)

Thus, taking \( r > p \) as in \((F_1), (2.3)\) and \((2.4)\) imply

\[
rc + (r - 1)\varepsilon_n = rf(u_n) - (dj(u_n), u_n)
\]

\[
= \left( \frac{r}{p} - 1 \right) \int_{\Omega} |\nabla u_n|^p dx - \left( \frac{r}{p} - 1 \right) a \int_{\Omega} u_n^p dx - \left( \frac{r}{p} - 1 \right) b \int_{\Omega} u_n^p dx
\]

\[
+ \int_{\Omega} (f(x, u_n) u_n - rF(x, u_n)) dx.
\]
hence, there exists a constant $a_1 > 0$ such that
\[
\left| \frac{r}{p} - 1 \right| \int_{\Omega \setminus \Omega_{n,R}} |\nabla u_n|^p \, dx \leq a_1.
\]

On the other hand, it is clear that from (F2), we obtain for a.e. $x \in \Omega$ and all $t \in R$,\n\[
|F(x, t)| \leq M_1 |t| + \frac{M_2}{q} |t|^q.
\]

So, there exists $a_2 > 0$, and
\[
\int_{\Omega \setminus \Omega_{n,R}} (f(x, u_n)u_n - rF(x, u_n)) \, dx \leq a_2,
\]
while (F1) implies
\[
\int_{\Omega_{n,R}} (f(x, u_n)u_n - rF(x, u_n)) \, dx \geq 0.
\]

Hence, from all the previous estimates, we have
\[
rc + (r - 1)\varepsilon_n \geq -a_1 + \left( \frac{r}{p} - 1 \right) \int_{\Omega \setminus \Omega_{n,R}} |\nabla u_n|^p \, dx - \left( \frac{r}{p} - 1 \right) a \int_{\Omega} u_{n,a}^p \, dx - \left( \frac{r}{p} - 1 \right) b \int_{\Omega} u_{n,b}^p \, dx - a_2
\]
\[
\geq -a_1 + \left( \frac{r}{p} - 1 \right) \|u_n\|_X^p - \left( \frac{r}{p} - 1 \right) \int_{\Omega \setminus \Omega_{a,b,R}} |\nabla u_n|^p \, dx - \left( \frac{r}{p} - 1 \right) \lambda_{k+1} \int_{\Omega} u_{n,a}^p \, dx - a_2
\]
\[
\geq -a_3 + \left( \frac{r}{p} - 1 \right) \|u_n\|_X^p - \left( \frac{r}{p} - 1 \right) \lambda_{k+1} \|u_n\|_P^p
\]

where $a_3$ is a constant and independent of $n$, and we see that the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. Then, from the boundedness of $\{u_n\}$ in $X$, we can see that if $u \in X$ exists such that, up to subsequences if necessary, we get $u_n \rightharpoonup u$ weakly in $X$, $u_n \rightarrow u$ strongly in $L^s(\Omega)$ for each $s \in [1, p^*)$, and $u_n \rightarrow u$ a.e. in $\Omega$. Furthermore, we have $u \in L^\infty(\Omega)$.

Finally, testing $dI(u_n)$ on $u_n - u$, by (F2) and the previous properties of $\{u_n - u\}$, direct estimates imply
\[
\int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla (u_n - u) \, dx \rightarrow 0.
\]

Indeed, from the Hölder inequality and [18, Lemma 5]
\[
0 = \lim_{n \rightarrow +\infty} \int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla (u_n - u) \, dx
\]
\[
\geq \lim_{n \rightarrow +\infty} (\|u_n\|_X^{p-1} - \|u\|_X^{p-1})(\|u_n\|_X - \|u\|_X) \geq 0.
\]

Therefore, we have $\lim_{n \rightarrow +\infty} \|u_n\|_X = \|u\|_X$. This implies that $\lim_{n \rightarrow +\infty} \|u_n - u\|_X = 0$. □

It is known that Eq. (1.4) is the Euler–Lagrange equation of the functional

\[
I(u) = \int_{\Omega} A(x, \nabla u) \, dx - \frac{a}{p} \int_{\Omega} u_{n,a}^p \, dx - \frac{b}{p} \int_{\Omega} u_{n,b}^p \, dx - \int_{\Omega} F(x, u) \, dx, \quad u \in X.
\]

We remark that when $A = A(x, \xi)$ is independent of $t$, $I$ is Fréchet differentiable in $X$.

**Lemma 2.7.** Assume that $A = A(x, \xi)$ is independent of $t$ and conditions (F1) and (A1), (A3) hold. Then the functional $I$ is $C^1$ in the whole Banach space $X = W^{1,p}_0(\Omega)$ with
\[
dl : u \in X \mapsto dl(u) \in X'.
\]
the differential operator defined as
\[
\langle dI(u), v \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v dx - a \int_\Omega u^{p-2} u_+ v dx - b \int_\Omega u^{p-2} u_- v dx - \int_\Omega f(x, u) v dx
\]
for all \( v \in X \). (For the proof, see [13].)

Furthermore, also Lemma 2.6 can be improved.

**Lemma 2.8.** Assume that \((A_1)-(A_4)\) and \((F_1)-(F_2)\) hold. Then \( J \) satisfies (CPS) in \( W^{1,p}_0(\Omega) \) for all \( c \in \mathbb{R} \).

### 3. The proof of the main results

Let \( J \) be the functional given by (1.3).

**Lemma 3.1.** If the origin is the only critical point of \( J \) in \( X \), then for any \( c < 0 \), we have
\[
C^q(J, 0) \approx H^q(X, J^c), \quad q = 0, 1, 2, \ldots
\]

**Proof.** Taking \( U = X \) in (2.1), we then have
\[
C^q(J, 0) = H^q(J^0, J^0 \setminus \{0\}), \quad q = 0, 1, 2, \ldots
\]

By Lemmas 2.4 and 2.6, we can obtain that the sublevel \( f^0 \) is a deformation retract of \( X \) while \( f^c \) is a deformation retract of \( f^0 \) for any \( c < 0 \). So, we have the conclusion. \( \Box \)

**Lemma 3.2.** Assume that \((F_1)-(F_2)\) hold. Then there is a \( c_0 < 0 \) such that for all \( c < c_0, f^c \) is homotopic to \( S_1 \), the unit sphere in \( X \), and hence contractible.

**Proof.** Firstly, note that from Remark 2.5, for \( u \in S_1 \) and for all \( s > 0 \), we have
\[
f(su) \leq \frac{s^p}{p} - \frac{as^p}{p} \int_\Omega u^p dx - \frac{bs^p}{p} \int_\Omega u^p dx - s^p \int_\Omega M_3(x)|u|^p dx + M_4|\Omega|;
\]
then as \( r > p \) and \( M_3(x) > 0 \) a.e. in \( \Omega \), we get
\[
f(su) \rightarrow -\infty \quad \text{as} \quad s \rightarrow +\infty.
\]
From \((F_1)-(F_2)\), there exists \( M_5 > 0 \) such that for almost all \( x \in \Omega \) and all \( t \in \mathbb{R} \),
\[
f(x, t) t \geq rF(x, t) - M_5.
\]
So for \( u \in S_1 \) and for all \( s > 0 \),
\[
\frac{d}{ds}(J(su)) = s^{p-1} \int_\Omega |\nabla u|^p dx - as^{p-1} \int_\Omega u^p dx - bs^{p-1} \int_\Omega u^p dx - \int_\Omega f(x, su) u dx
\]
\[
\leq \frac{1}{s} \left[ \int_\Omega |\nabla u|^p dx - a \int_\Omega (su_+)^p dx - b \int_\Omega (su_-)^p dx - c \int_\Omega f(x, su) dx + M_5|\Omega| \right]
\]
\[
\leq \frac{r}{s} \frac{1}{r} \left[ \int_\Omega |\nabla u|^p dx - a \int_\Omega (su_+)^p dx - b \int_\Omega (su_-)^p dx - \int_\Omega f(x, su) dx + M_5 \right]
\]
\[
= \frac{r}{s} (J(su) - c_0),
\]
where we can take \( c_0 = -\frac{M_5}{r}|\Omega| \). So, we take \( c < c_0 \) for \( u \in S_1, s > 0 \), and then we can get \( J(su) \leq c \). This implies that \( \frac{d}{ds}(J(su)) < 0 \). Moreover, by \( J(0) = 0 > c \), there is a unique \( s_0 = s_0(u) > 0 \) such that \( J(s_0 u) = c \) and for all \( 0 \leq s < s_0(u) \) we have \( J(su) > c \), while for all \( s > s_0(u) \) we have \( J(su) < c \). Thus by the Implicit Function Theorem, the map \( s_0 : S_1 \rightarrow (0, +\infty) \) is \( C^1 \) and allows us to describe the sublevel \( f^c = \{ su : u \in S_1, s \geq s_0(u) \} \). Then, we get the conclusion: \( f^c \) is homotopic to \( S_1 \). \( \Box \)

For the proof of a cohomological local splitting near zero, let \( S_\rho \) be the sphere of radius \( \rho \) in \( X \) where \( \rho > 0 \), and consider the eigenvalues \( \{ \lambda_k \} \) of \( -\Delta_p \); we prepare the following lemma.

**Lemma 3.3.** Assuming that \( \lambda_k < \lambda \leq \lambda_{k+1} \) and
\[
\forall_{\lambda} = \left\{ u \in S_\rho : \int_\Omega |\nabla u|^p dx < \lambda \int_\Omega |u|^p dx \right\},
\]
then we have \( i(\forall_{\lambda}) = k \).
Proof. Set
\[ U_\lambda = \left\{ u \in X : \int_{\Omega} |\nabla u|^p dx < \lambda \int_{\Omega} |u|^p dx \right\}. \]

From [12], we know that \( i(U_\lambda) = k \). Moreover, the inclusion \( V_\lambda \hookrightarrow U_\lambda \) and the odd continuous radial projection \( u \in U_\lambda \mapsto pu/\|u\|_X \in V_\lambda \) hold, so we get \( i(V_\lambda) = i(U_\lambda) = k \) by the monotonicity of the index. □

Proof of Theorem 1.1. We shall show this by contradiction. Suppose that the origin is the only critical point of \( J \) in \( X \). Then, Lemmas 3.1 and 3.2 yield that
\[ C^q(J, 0) = 0, \quad q = 0, 1, 2, \ldots. \]

Hence, it is enough to show that there exists a \( k \geq 1 \) such that the corresponding critical group \( C^q(J, 0) \) is nontrivial. By Lemma 2.2, we can see that we need only to prove that \( J \) has a cohomological local splitting near zero.

Firstly, take \( k \geq 1 \) satisfying the condition \( (F_3) \). Choose an arbitrary \( \lambda_k < \lambda < \lambda_1 \); let
\[ D_0 = \left\{ u \in S_\rho : \int_{\Omega} |\nabla u|^p dx < \lambda \int_{\Omega} |u|^p dx \right\} \]
and
\[ B_0 = \left\{ u \in S_\rho : \int_{\Omega} |\nabla u|^p dx \geq \lambda_{k+1} \int_{\Omega} |u|^p dx \right\}. \]

Then from Lemma 3.3, we have
\[ i(D_0) = i(S_\rho \setminus B_0) = k. \]

Furthermore, from \((F_2), (F_3)\), there exists \( M_6 > 0 \) such that for almost all \( x \in \Omega \) and all \( t \in \mathbb{R} \),
\[ \frac{\lambda}{p} t^p - M_6 |t|^q \leq \frac{a}{p} t^p + \frac{b}{p} t^p + F(x, t) \leq \frac{a}{p} t^p + M_6 |t|^q. \]

Hence, we have for all \( u \in X \),
\[ J(u) \leq \frac{1}{p} \int_{\Omega} (|\nabla u|^p - \frac{\lambda}{p} |u|^p) dx + M_6 \int_{\Omega} |u|^q dx, \]
\[ J(u) \geq \frac{1}{p} \int_{\Omega} (|\nabla u|^p - \frac{\lambda}{p} |u|^p) dx - M_6 \int_{\Omega} |u|^q dx. \]

If we choose \( \rho \) small enough, it follows that
\[ J(u) \leq -\frac{\lambda}{p} \left( \frac{\lambda}{\lambda_1} - 1 + o(1) \right) \|u\|_{X}^p \leq 0 \quad \text{if } u \in D, \]
\[ J(u) \geq \frac{\lambda_{k+1}}{p} \left( 1 - \frac{\lambda}{\lambda_{k+1}} + o(1) \right) \|u\|_{X}^p > 0 \quad \text{if } u \in B \setminus \{0\}, \]
where \( D \) and \( B \) are as in Definition 2.1. Then by the definition of the cohomological local splitting, \( J \) has a cohomological local splitting near zero in dimension \( k \), and by Lemma 2.2, \( C^q(J, 0) \neq 0 \). This produces a contradiction and the conclusion holds. □

Finally, we consider the problem (1.4), and suppose that the conditions of Theorem 1.2 are satisfied. Thus, the corresponding functional \( I \) is \( C^1 \) in the whole Banach space \( X = W^{1,p}_0(\Omega) \) (see Lemma 2.7).

Proof of Theorem 1.2. Also arguing by contradiction, suppose that the origin is the only critical point of \( I \) in \( X \). Then, Lemmas 3.1 and 3.2 yield that
\[ C^q(J, 0) = 0, \quad q = 0, 1, 2, \ldots. \]

Let \( k \geq 1 \) be such that condition \( (F_1) \) holds. Choose an arbitrary \( \lambda_k < \lambda < \lambda_1 \) and let \( D_0, B_0 \) be the same as in the proof of Theorem 1.1. Then we also have
\[ i(D_0) = i(S_\rho \setminus B_0) = k. \]

On the other hand, from \((F_2), (F_3)\), there exists \( M_7 > 0 \) such that for almost all \( x \in \Omega \) and all \( t \in \mathbb{R} \),
\[ A_2 \lambda \cdot |t|^p - M_7 |t|^q \leq \frac{a}{p} t^p + \frac{b}{p} t^p + F(x, t) \leq A_1 \lambda \cdot |t|^p + M_7 |t|^q. \]
Hence, we have, for all $u \in X$,
\[
I(u) \leq A_2 \int_{\Omega} (|\nabla u|^p - \lambda |u|^p) \, dx + M_2 \int_{\Omega} |u|^q \, dx, \\
I(u) \geq A_1 \int_{\Omega} (|\nabla u|^p - \lambda |u|^p) \, dx - M_2 \int_{\Omega} |u|^q \, dx.
\]
If we choose $\rho$ small enough, it follows that
\[
I(u) \leq -A_2 \lambda \left( \frac{\lambda^p}{\lambda} - 1 + O(1) \right) \|u\|_X^p \leq 0 \quad \text{if } u \in D, \\
I(u) \geq A_1 \lambda \quad \left( 1 - \frac{T}{\lambda_{k+1}} + 0(1) \right) \|u\|_X^p > 0 \quad \text{if } u \in B \setminus \{0\},
\]
where $D$ and $B$ are as in Definition 2.1. Then by the definition of cohomological local splitting, $I$ has a cohomological local splitting near zero in dimension $k$, and by Lemma 2.2, $C^1(I, 0) \neq 0$. This produces a contradiction and the conclusion holds. \(\square\)

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References