Existence and asymptotics of traveling waves for nonlocal diffusion systems

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\textbf{Abstract}

In this paper, we deal with the existence and asymptotic behavior of traveling waves for nonlocal diffusion systems with delayed monostable reaction terms. We obtain the existence of traveling wave front by using upper-lower solutions method and Schauder's fixed point theorem for $c > c_\ast(\tau)$ and using a limiting argument for $c = c_\ast(\tau)$. Moreover, we find a priori asymptotic behavior of traveling waves with the help of Ikehara's Theorem by constructing a Laplace transform representation of a solution. Especially, the delay can slow the minimal wave speed for $\partial_2 f(0,0) > 0$ and the delay is independent of the minimal wave speed for $\partial_2 f(0,0) = 0$. © 2012 Elsevier Ltd. All rights reserved.

1. Introduction

In the early year, Fisher [9] proposed modeling the spatial spread of a mutant in a given population by the Laplacian reaction–diffusion equation

$$u_t = \Delta u + f(u),$$

where $f(u) = u(1 - u)$ and $u$ represents the gene fraction of the mutant. Since then, traveling wave fronts for reaction–diffusion systems have attracted much attention in biology, chemistry, epidemiology and physics, see [2,8,10,22,26]. There are many methods to deal with the existence of traveling wave. In pioneering works, the phase space analysis and the maximum principle were used in [19], and the Conley index and degree theory methods were developed in [22]. Due to the practical background and biological realism, many researchers have paid attention to traveling waves for reaction–diffusion equations with discrete and nonlocal delays reaction, see [7,11,12,14–16,18,19,22,23,26,27,29,31] and the references therein.

Despite the popularity of Laplacian diffusion models, diffusion has some drawbacks. One important shortcoming for ecological and epidemiological models is that Laplacian diffusion is a local operator where individuals in the population can only influence their immediate neighbors. With diffusion models there is some disconnect between experimentally collected data and a limited number of parameters that are available to fit that data. One method in overcoming these problems with the Laplacian operator is to describe these models concerning with the spatial migration by integral equation. Lee et al. [11] argued that, for processes where the spatial scale for movement is large in comparison with its temporal scale, non-local models using integro-differential equations may allow for better estimation of parameters from data and provide more insight into the biological system. The precise mathematical model is read as

$$u_t(t,x) = \int_{\mathbb{R}} J(x-y) u(t,y)dy - u(t,x) + f(u(t,x)).$$

The nonlocal model (1.2) with monostable nonlinearity have been widely investigated by authors (see [1,3–6,21]). More recently, authors in [17,18] showed the existence of traveling waves for (1.2) with delayed reaction
terms with the quasi-monotonicity and the exponential quasi-monotonicity, respectively. Following [17,18], authors [28,30] further investigated two componentwise nonlocal diffusion systems with the weak (exponential) quasi-monotonicity and the partial (exponential) quasi-monotonicity, respectively. The model (1.2) is closely related to traditional reaction–diffusion models. Taking the diffusion kernel

\[ f(x) = \delta(x) + \delta'(x), \quad (1.3) \]

where \( \delta \) is the Dirac delta, (1.2) reduces to the Laplacian reaction diffusion Eq. (1.1), (see, Medlock et al. [14]).

Motivated by the above works, we will consider the existence and asymptotic behavior of traveling waves for nonlocal diffusion systems with a discrete delay

\[ u_t(t,x) = d[J \ast u(t,x) - u(t,x)] + f(u(t,x), u(t - \tau, x)) \quad (1.4) \]

where \( d > 0, \tau > 0 \) are constants,

\[ J \ast u(t,x) = \int_R J(x-y)u(t,y)dy \]

and the functions \( J, f \) satisfy

(I) \( J \geq 0, J(x) = J(-x) \), \( \int_R J(y)dy = 1 \), and \( \int_R J(y)e^{-cy}dy < \infty \), \( \forall c > 0 \).

(F1) \( f(u,v) \in C^2([0,K]^2, \mathbb{R}), f(0,0) = f(K,K) = 0 \) for some \( K > 0 \) and \( f(u,v) > 0 \) for \( u \in (0,K) \), and \( \partial_d f(u,v) > 0 \) for \( (u,v) \in [0,K]^2 \), where \( K \) is a positive constant;

(F2) \( \partial_d f(0,0)u + \partial_d f(0,0)v \geq f(u,v) \) for any \( (u,v) \in [0,K]^2 \).

A traveling wave of (1.4) is a solution of special form \( u(t,x) = U(x + ct) \), where the velocity \( c \) and the wave profile \( U \) satisfy the following functional differential equation

\[ cU'(\zeta) = d[U(\zeta) - U(\zeta)] + f(U(\zeta), U(\zeta - ct)) \quad (1.5) \]

with asymptotic boundary conditions

\[ \lim_{\zeta \to -\infty} U(\zeta) = 0, \quad \lim_{\zeta \to +\infty} U(\zeta) = K, \quad (1.6) \]

where \( 0 \) and \( K > 0 \) are equilibria of (1.4). A traveling wave \( U \) is called the traveling wave front if \( U \) is monotone. Now we formulate our main theorems as follows:

**Theorem 1.1 (Existence).** Assume that (J1), (F1) and (F2) hold. Then there exists a positive constant \( c_\tau(\tau) \) such that for each \( c \geq c_\tau(\tau) \), Eq. (1.4) admits a nondecreasing positive traveling wave front \( u(t,x) = U(x + ct) \) satisfying (1.5) and (1.6). Moreover, if \( c > c_\tau(\tau) \), then

\[ \lim_{\zeta \to -\infty} U(\zeta)e^{-\lambda\zeta} = 1, \quad \lim_{\zeta \to +\infty} U(\zeta)e^{-\lambda\zeta} = \lambda_1(c), \quad (1.7) \]

where \( \lambda = \lambda_\tau(c) > 0 \) is the smallest root of the equation

\[ \Delta_\tau(c, \lambda) = -c\lambda + d \int_R e^{-cty}dy - 1 + \partial_d f(0,0) + \partial_d f(0,0)e^{-ctc} = 0. \]

**Theorem 1.2 (Asymptotic behavior).** Assume that (J1), (F1) and (F2) hold and \( U(\zeta) \) is any positive traveling wave of (1.4) with speed \( c \geq c_\tau(\tau) \) satisfying (1.5) and (1.6). Then we can obtain the following assertions:

(i) If \( c > c_\tau(\tau) \), \( \lim_{\zeta \to -\infty} \tilde{U}(\zeta)e^{-\lambda\zeta} \) exists.

(ii) If \( c = c_\tau(\tau) \), there exists a positive constant \( \lambda_\tau \) such that \( \lim_{\zeta \to -\infty} \tilde{U}(\zeta)e^{-\lambda_\tau\zeta} \) exists.

(iii) If \( 0 < c < c_\tau(\tau) \), there is no traveling wave with speed \( c \) of (1.4) satisfying (1.5) and (1.6).

**Remark 1.1.** If \( \tau = 0 \) in (1.4), Theorems 1.1 and 1.2 reduce to some results in [3,6,21]. Moreover, the delay \( \tau > 0 \) can affect the minimal wave speed, (see Section 4).

This paper is organized as follows. Section 2 is devoted to the existence of traveling wave front for the nonlocal diffusion with the delayed monostable reaction by using the super-sub solution method and the Schauder’s fixed point theorem for \( c > c_\tau(\tau) \) and using a limiting argument for \( c = c_\tau(\tau) \). In Section 3, we find an a priori asymptotic behavior of traveling waves with the help of Ikehara’s Theorem by constructing a Laplace transform presentation of a solution for a class of the nonlocal diffusion with the monostable reaction. Last Section, there are some applications for main results. Especially, we apply our results to another version of the classical Logistic equation and the Nicholson’s blowflies with the delay and some results in [18] are improved and complemented. We also give the effect of the delay on minimal wave speed, that is, our results show that the delay will slow the minimal wave speed for \( \partial_d f(0,0) > 0 \) and the delay is independent of the minimal wave speed for \( \partial_d f(0,0) = 0 \).

### 2. Existence of traveling wave fronts

In this section, we consider existence of traveling wave fronts for the convolution diffusion systems with delayed monostable reaction terms by using the super-sub solution method, Schauder’s fixed point theorem and Theorem 3.2 in [18].

Let

\[ C_{[0,K]}(\mathbb{R}, \mathbb{R}) = \{ f \in C(\mathbb{R}, \mathbb{R}) \mid 0 \leq f(\xi) \leq K, \xi \in \mathbb{R} \}. \]

Define the operator \( F : C_{[0,K]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R}) \) by

\[ F[u](\xi) = \frac{1}{c}e^{-\frac{\xi}{c}} \int_{-\infty}^{\xi} e^{\frac{\xi}{c}}H[u](\eta)d\eta, \quad (2.1) \]

where

\[ H[u](\xi) = d^2[u(\xi) + (\beta - d)u(\xi) + f(u(\xi), u(\xi - ct))], \quad \xi \in \mathbb{R}. \]

(2.2)

We can easily see that the operator \( F \) is well defined and the problem is changed into investigating the fixed point of \( F \) which is a traveling wave front of (1.4) connecting 0 and \( K \).

In the following, we introduce the exponential decay norm. For \( 0 < \mu < \frac{1}{c} \), define

\[ B_{\mu}(\mathbb{R}, \mathbb{R}) = \{ f : f(\xi) \in C(\mathbb{R}, \mathbb{R}) \text{ and } \sup_{\xi \in \mathbb{R}}|f(\xi)|e^{-\mu|\xi|} < \infty \}. \]

It is easy to check that \( B_{\mu}(\mathbb{R}, \mathbb{R}) \) is a Banach space equipped with the decay norm \( \| \cdot \|_\mu \) defined by \( \| f \|_\mu = \sup_{\xi \in \mathbb{R}}|f(\xi)|e^{-\mu|\xi|} \) for \( \phi \in B_{\mu}(\mathbb{R}, \mathbb{R}) \).
Now we introduce the definition of an upper (or a lower) solution of (1.5).

**Definition 2.1.** A continuous function $\phi : \mathbb{R} \rightarrow [0, K]$ is called an upper solution of (1.5) if $\phi$ is almost everywhere continuously differentiable and

$$N_{c}[\phi](\xi) := -c\phi' + dJ_*(\phi' - \phi(\xi)) + f(\phi(\xi), \phi(\xi - c\tau)) \leq 0, \text{ a.e. on } \mathbb{R}.$$  

(2.3)

A lower solution of (1.5) is defined in a similar way by reversing the inequality in (2.3).

Let us define a function

$$\Delta_{c}(c, \lambda) := -c\lambda + d \int_{0}^{\infty} e^{-\lambda y} J(y) dy - 1 + \partial_{y} f(0, 0) \lambda e^{-\lambda \tau}.$$  

(2.4)

According to (F2), it is obvious that $\Delta_{c}(c, 0) = \partial_{y} f(0, 0) + \partial_{x} f(0, 0) > 0$ for all $c \geq 0$ and $\lim_{c \to \infty} \Delta_{c}(c, \lambda) = +\infty$ since

$$\Delta_{c}(c, \lambda) \geq -c\lambda + d \int_{0}^{\infty} \lambda^{2} y^{2} J(y) dy - \lambda^{2} f(0, 0) + \partial_{y} f(0, 0) \lambda e^{-\lambda \tau}.$$  

Differentiating (2.4) with respect to $c$, we obtain

$$\frac{\partial}{\partial c} \Delta_{c}(c, \lambda) = -\lambda - \partial_{y} f(0, 0) \lambda e^{-\lambda \tau} < 0, \text{ for all } \lambda > 0.$$  

Furthermore, according to (F1 and F2) and (J1), we have

$$\begin{align*}
\Delta_{c}(0, \lambda) &= d \int_{0}^{\infty} J(y) dy - 1 + \partial_{y} f(0, 0) + \partial_{y} f(0, 0) \\
&= d \int_{0}^{\infty} [e^{y} + e^{-y} - 2J(y) dy + \partial_{y} f(0, 0) + \partial_{y} f(0, 0) \\
&= 0
\end{align*}$$  

and

$$\lim_{c \to +\infty} \Delta_{c}(c, \lambda) = -\infty$$  

for each fixed $\lambda > 0$.

For fixed $c \geq 0$, $\Delta_{c}(c, \lambda)$ is a convex function with respect to $\lambda \in [0, +\infty)$. In fact, for any $\lambda_1, \lambda_2 \in [0, +\infty)$ and $\lambda_1 \neq \lambda_2$, we have

$$\begin{align*}
\frac{1}{2} \left[ \Delta_{c}(c, \lambda_1) + \Delta_{c}(c, \lambda_2) \right] &= -c\lambda_1 + c\lambda_2 + d \int_{0}^{\infty} e^{-\lambda_1 y} \frac{2}{2} J(y) dy - 1 + \partial_{y} f(0, 0) \\
&\quad + \partial_{y} f(0, 0) e^{-\lambda_1 \tau} + e^{-\lambda_2 \tau} \\
&\quad > -c\lambda_1 + c\lambda_2 + d \int_{0}^{\infty} e^{-\lambda_1 y} J(y) dy - 1 + \partial_{y} f(0, 0) \\
&\quad + \partial_{y} f(0, 0) \frac{e^{-\lambda_1 \tau} + e^{-\lambda_2 \tau}}{2} \\
&\quad = \Delta_{c}(c, \frac{\lambda_1 + \lambda_2}{2}).
\end{align*}$$  

According to the above results, it is obvious to see that the following lemma holds.

**Lemma 2.1.** Assume that (J1) and (F1) and (F2). Then there exists a unique $c_*(\tau) > 0$ such that

(i) if $c > c_*(\tau)$, then there exist two positive numbers $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) < \lambda_2(c)$ such that

$$\begin{align*}
\Delta_{c}(c, \lambda_1(c)) &= \Delta_{c}(c, \lambda_2(c)) = 0, \\
\frac{\partial}{\partial \lambda} \Delta_{c}(c, \lambda) &< 0 \text{ in } (\lambda_1(c), \lambda_2(c)), \\
\frac{\partial}{\partial \lambda} \Delta_{c}(c, \lambda) &> 0 \text{ in } [\lambda_1(c), \lambda_2(c)].
\end{align*}$$  

(ii) for $c < c_*(\tau)$, then $\Delta_{c}(c, \lambda) > 0$ for all $\lambda > 0$;

(iii) for $c = c_*(\tau)$, then $\lambda_1(c) = \lambda_2(c) = \lambda_*$, and if $c > c_*(\tau)$, then

$$\frac{\partial}{\partial \lambda} \Delta_{c}(c, \lambda) < 0 \text{ in } (\lambda_1(c), \lambda_2(c)), \\
\frac{\partial}{\partial \lambda} \Delta_{c}(c, \lambda) > 0 \text{ in } [\lambda_1(c), \lambda_2(c)]$$  

and $\frac{\partial}{\partial \lambda} \Delta_{c}(c, \lambda) < 0$.

For $c > c_*(\tau)$ and $\lambda_1(c), \lambda_2(c)$ defined as in Lemma 2.1, we define the continuous functions as follows:

$$\phi^+(\xi) := \min \{ K, e^{q\xi} J(\xi) + q e^{q\xi}, \xi \in \mathbb{R} \}$$  

(2.5)

and

$$\phi^-(\xi) := \max \{ 0, e^{q\xi} J(\xi) - q e^{q\xi}, \xi \in \mathbb{R} \}.$$  

(2.6)

We easily obtain that the following result holds.

**Lemma 2.2.** Assume that (J1) and (F1) and (F2) hold. Then for each $\gamma \in \left( 1, \min \left\{ \frac{\lambda_1(c)}{\lambda_2(c)}, \frac{\lambda_2(c)}{\lambda_1(c)} \right\} \right)$, and a large enough number $q$, the functions $\phi^+(\xi)$ and $\phi^-(\xi)$ are a pair of upper and lower solutions of (1.5).

Since $\partial_{y} f(u, v) \geq 0$ for $(u, v) \in [0, K]^2$, it is easy to see that the function $f(U(\xi), U(\xi - c\tau))$ satisfies the following quasi-monotone condition.

**Lemma 2.3.** Assume that $\partial_{y} f(u, v) \geq 0$ for $(u, v) \in [0, K]^2$. Then there is a positive constant $\beta > \max_{(u, v) \in [0, K]^2} |\partial_{y} f(u, v)| + d$ such that

$$f(\phi_2(\xi), \phi_2(\xi - c\tau)) - f(\phi_1(\xi), \phi_1(\xi - c\tau)) \geq \beta(\phi_2(\xi) - \phi_1(\xi)).$$  

where $\phi_2(\xi), \phi_1(\xi) \in C(\mathbb{R}, \mathbb{R})$ with $0 \leq \phi_1(\xi) \leq \phi_2(\xi) \leq K$ for $\xi \in \mathbb{R}$.

Let $\phi^-(\xi)$ and $\phi^+(\xi)$ be defined in Lemma 2.2 and define the set $I^+$ by

$$I^+ := \left\{ \phi \in C_{[0, K]}(\mathbb{R}, \mathbb{R}) \left| \begin{array}{l}
(1) \text{ } \phi(\xi) \text{ is nondecreasing on } \mathbb{R}; \\
(2) \text{ } \phi^-(\xi) \leq \phi(\xi) \leq \phi^+(\xi) \text{ for all } \xi \in \mathbb{R}.
\end{array} \right. \right\}$$  

For the set $I^+$, it is easily seen that the following lemma holds.

**Lemma 2.4.** The set $I^+$ is nonempty, convex, bounded and closed subset of $B_{\mu}(\mathbb{R}, \mathbb{R})$ with respect to the decay norm $\| \cdot \|_{\mu}$.

**Proof of Theorem 1.1.** For $c > c_*(\tau)$, according to Lemmas 2.1–2.4 and the proof of Theorem 3.2 in [18], it follows that $F$ has a fixed point $U(\xi)$ in $I^+$ and $U(\xi)$ is nondecreasing, that is,

$$U(\xi) = \frac{1}{c} e^{\frac{1}{c} \int_{-\infty}^{\xi} e^{\xi} H(U(y)) dy}.$$  

(2.7)
which implies \( \lim_{\xi \to -\infty} U(\xi) = K \) according to Hopital’s rule, the dominated convergence theorem and \( f(u,u) > 0, \ u \in (0,K) \). By the definition of \( \Gamma \),
\[ \text{max}\{0, \ e^{\epsilon} (\xi + c_0) - \frac{q(e^{\epsilon})(\xi + c_0)}{\epsilon} \} \leq U(\xi) \leq e^{\epsilon} (\xi + c_1) + q(e^{\epsilon})(\xi + c_1), \ \xi \in \mathbb{R} , \]
which implies that
\[ \lim_{\xi \to -\infty} |U(\xi) - e^{\epsilon} (\xi + c_1) - 1| \leq \lim_{\xi \to -\infty} q(e^{\epsilon})(\xi + c_1) = 0. \]
Hence we obtain
\[ \lim_{\xi \to -\infty} U(\xi) e^{\epsilon} (\xi + c_1) = 1 , \quad (2.8) \]
which implies that \( \lim_{\xi \to -\infty} U(\xi) = 0 \). By the above argument, we see easily that Eq. (1.4) admits a traveling wave front \( U(x + ct) \) connecting 0 and \( K \). Furthermore, we have
\[
\begin{align*}
\lim_{\xi \to -\infty} U(\xi) e^{\epsilon} (\xi + c_1) & = \lim_{\xi \to -\infty} \left[ \frac{1}{c_1} \int_{\xi}^{\xi + c_1} e^{\epsilon} d\theta + H[U(\theta)] \right] \\
& = \frac{1}{c_1} \left[ \int_{\xi}^{\xi + c_1} e^{\epsilon} d\theta + \frac{\sigma_1}{\epsilon} \right] \\
& = \lambda_1(\xi).
\end{align*}
\]
(2.9)
The remainder is to verify that (1.4) has a traveling wave front connecting 0 and \( K \) if \( \epsilon = c_1(\tau) \).

Let \( c_\tau(\tau), n \in \mathbb{N} \), be a sequence with \( c_\tau(\tau) > c_1(\tau) \) and \( c_\tau(\tau) \to c_1(\tau), n \to \infty \). Then by the above argument there exists a function \( U(t) \) satisfying (2.7) with \( \epsilon = c_\tau(\tau) \). Without loss of generality, we may assume that \( u_0 = \frac{\epsilon}{2} \).

We easily see that \( U(t) \), \( n \in \mathbb{N} \) are uniformly bounded and equicontinuous since \( U(t) \), \( n \in \mathbb{N} \) are also uniformly bounded. By the Ascoli-Arzelà lemma, there exists a \( U(\xi) \) such that \( U(t) \), \( n \in \mathbb{N} \) uniformly converge to \( U(\xi) \) on any compact of \( \mathbb{R} \) as \( n \to \infty \). Since \( U(t) \), \( n \in \mathbb{N} \) are nondecreasing on \( \mathbb{R} \), then \( U(t) \) is also nondecreasing with \( U(0) = \frac{\epsilon}{2} \). So the dominated convergence theorem implies that
\[
U(\xi) = \frac{1}{c_1(\tau)} \int_{\xi}^{\xi + c_1(\tau)} e^{\epsilon} d\theta + H[U(\theta)], \quad (2.10)
\]
Furthermore, \( \lim_{\xi \to -\infty} U(\xi) = 0 \) and \( \lim_{\xi \to +\infty} U(\xi) = K \). Thus \( U(x + c_1(\tau) t) \) is a traveling wave front of (1.4).

For \( c \geq c_1(\tau) \), we can obtain \( U(\xi) > 0 \) for \( \xi \in \mathbb{R} \). Indeed, note that \( 0 \leq U(\xi) \leq K \) for \( \xi \in \mathbb{R} \). If \( U(\xi) = 0, \xi < c_0 \) since \( U(\xi) \) is nondecreasing. By (1.5), for \( 0 < c_0 \), we have \( U(\xi) = U(\xi) > 0 \), which is a contradiction. This completes the proof. □

3. Asymptotic behavior of traveling waves

In this section, we will find a priori asymptotic behavior of traveling waves with the help of Ikehara’s Theorem. Our method is to establish the asymptotic behavior of the profile \( U(\xi) \) as \( \xi \to -\infty \) by using the approach developed by Carr and Chmaj [3].

We recall a version of Ikehara’s Theorem.

Lemma 3.1 [3], Proposition 2.3. Let \( l(\lambda) = \int_{0}^{\infty} u(x) e^{-i\lambda x} dx \), with \( u \) being a positive decreasing function. Assume that \( l(\lambda) \) has the representation

\[ l(\lambda) = \frac{e(\lambda)}{(\lambda + \alpha)^{1/2}}, \]

where \( k > -1 \) and \( e \) is analytic in the strip \( -\alpha \leq Re \lambda < \alpha \). Then
\[
\lim_{\lambda \to \infty} u(\lambda) e^{\lambda} = \frac{e(-\alpha)}{\alpha^{1/2}}.
\]

In what follows, we assume that \( U(x + ct) \) is any traveling wave of (1.4) satisfying (1.5) and (1.6) and \( 0 < U(\xi) \leq K \). Then we can obtain the following results.

Lemma 3.2. For \( \xi < 0 \), \( \int_{-\infty}^{\xi} U(\theta) d\theta \to \infty \).

Proof. Let \( \sigma_1 = \partial_1 f(0,0) + \partial_2 f(0,0) > 0 \) and \( \sigma_2 = \partial_1 f(0,0) - \partial_2 f(0,0) \). Since \( \lim_{\xi \to -\infty} U(\xi) = 0 \), there exists \( \xi' < 0 \) such that for any \( \xi < \xi' \),
\[
\frac{\sigma_1}{4} (U(\xi) + (U(\xi - c t))) > M(U^2(\xi) + 2 U(\xi) U(\xi - c t) + U^2(\xi - c t)),
\]
where \( M = \max_{u \in [0,K]^2} \{ \min \{ |\partial_1 f(u,v)|, |\partial_2 f(u,v)|, |\partial_{21} f(u,v)|, |\partial_{22} f(u,v)| \} \) . Then for any \( \xi < \xi' \), we have
\[
cU(\xi) = d(f U(\xi) - U(\xi)) + f U(\xi) - \frac{\sigma_1}{4} (U(\xi) + (U(\xi - c t)))
\]
(3.1)

Note that
\[
\int_{-\infty}^{\xi} U(\theta) d\theta = -c \int_{0}^{\xi} U(\xi - c t) d\theta
\]
(3.2)
and
\[
\int_{-\infty}^{\xi} (f U(\theta) - U(\theta)) d\theta = \int_{-\infty}^{\xi} \left( \int_{0}^{\xi} f(\theta) U(\theta - c t) d\theta \right) d\theta
\]
(3.3)
as \( \eta \to -\infty \) by Fubini’s Theorem and Lebesgue’s Dominated Convergence Theorem.

Integrating (3.1) from \( -\infty \) to \( \xi \), according to (3.2) and (3.3), then for any \( \xi < \xi' \).
Lemma 3.3. There exists a positive constant $q$ such that $U(\xi) = O(e^{q\xi})$ as $\xi \to -\infty$. Moreover, $\sup_{t \in \mathbb{R}} U(\xi)e^{-q\xi} < \infty$.

Proof. Letting $V(\xi) = \int_0^\xi \tilde{U}(\theta)d\theta$, it is easily seen that $V(\xi)$ is nondecreasing and $\lim_{\xi \to -\infty} V(\xi) = 0$. We first prove that $V(\xi)$ is integrable on $(-\infty, \xi]$ for $\xi > \xi'$. It is obvious that $U(\xi)$ is integrable on $[\xi', \xi]$ for $\xi < 0$. This completes the proof. □

Remark 3.1. Lemma 3.3 implies that $\int_0^\xi \tilde{U}(\theta)e^{-q\theta}d\theta < \infty$ for any $0 < Re\lambda < \varrho$.

Proof of Theorem 1.2. For any $\lambda$ with $0 < Re\lambda < \varrho$ and using Remark 3.1, we can now define a two-sided Laplace transform of $\tilde{U}$ by

$L(\lambda) = \int_\mathbb{R} \tilde{U}(\theta)e^{-\lambda\theta}d\theta$.

Note that

$$\int_\mathbb{R} e^{-\lambda\theta}d\theta = \frac{1}{\lambda}$$

which implies that

$$\int_\mathbb{R} e^{-\lambda\theta}e^{-j\theta}d\theta = \frac{e^{-\lambda\theta}e^{-j\theta}}{-\lambda} + \frac{1}{\lambda}.$$
Using (3.11), it follows that
\[ d \tau f(0,0) \tilde{U}(\xi) + \partial f(0,0) \tilde{U}(\xi - \epsilon \tau) - f \left( \tilde{U}(\xi), \tilde{U}(\xi - \epsilon \tau) \right) = 0, \quad (3.11) \]

Since
\[ \partial f(0,0) \tilde{U}(\xi) + \partial f(0,0) \tilde{U}(\xi - \epsilon \tau) - f \left( \tilde{U}(\xi), \tilde{U}(\xi - \epsilon \tau) \right) = O \left( \tilde{U}^2(\xi) + \tilde{U}^2(\xi - \epsilon \tau) \right) \]

as \( \xi \to \infty \), the right side in (3.11) is defined for \( \lambda \) with \( 0 < \Re \lambda < 2 \). We now use a property of Laplace transform ([25, p. 58]). Since \( U(\lambda) \) is not analytic for \( 0 < \Re \lambda < \kappa \) and \( L(\lambda) \) has a singularity at \( \lambda = \kappa \). Hence, \( c \geq c_1(\tau) \), \( L(\lambda) \) is defined for \( 0 < \Re \lambda < \lambda_1(c) \).

We first prove (iii) of Theorem 1.2. We argue by contradiction, that is, for \( 0 < c < c_1(\tau) \), there exists a traveling wave front \( \tilde{U}(\xi) \) connecting 0 and \( K \). Since \( \Delta_2(\lambda, c) \) has no real zeros, \( L(\lambda) \) is defined for all \( \lambda \) such that \( \Re \lambda > 0 \).

Using (3.11), it follows that
\[ \int_{-\infty}^{\infty} e^{-i \theta} \left[ \Delta_1(\lambda, c) \tilde{U}(0) - \partial f(0,0) \tilde{U}(0) - \partial f(0,0) \tilde{U}(0 - \epsilon \tau) - f \left( \tilde{U}(\xi), \tilde{U}(\xi - \epsilon \tau) \right) \right] \, d\theta = 0 \]

which implies a contradiction since \( \Delta_1(\lambda, c) \to +\infty \) as \( \lambda \to \infty \).

Next we prove (i) and (ii) of Theorem 1.2. From now on, we consider the case \( c \geq c_1(\tau) \). In order to apply Lemma 3.1, we rewrite (3.11) as
\[ \int_{-\infty}^{\infty} \tilde{U}(\theta) e^{-i \theta} d\theta = \int_{-\infty}^{\infty} \left[ \partial f(0,0) \tilde{U}(0) + \partial f(0,0) \tilde{U}(0 - \epsilon \tau) - f \left( \tilde{U}(\xi), \tilde{U}(\xi - \epsilon \tau) \right) \right] d\theta / \Delta_1(\lambda, c) \]

Note that \( \int_{-\infty}^{\infty} \tilde{U}(\theta) e^{-i \theta} d\theta \) is analytic for \( \Re \lambda > 0 \). Also, \( \Delta_1(\lambda, c) = 0 \) does not have any zero with \( \Re \lambda = \lambda_1(c) \) other than \( \lambda = \lambda_1(c) \). In fact, letting \( \lambda = \lambda_1(c) + i \beta \), then \( \Delta_1(\tau, c) = 0 \) implies that
\[ c \lambda_1(c) = d \left[ \int_{-\infty}^{\infty} e^{-i \theta} \partial f(y) \sin \beta y d\theta - 1 \right] + \partial f(0,0) \]

and
\[ c \beta = d \int_{-\infty}^{\infty} e^{-i \theta} \sin \beta y d\theta + \partial f(0,0) e^{-i \epsilon \tau} \sin \beta \tau. \quad (3.12) \]

According to (3.12) and \( \Delta_1(\lambda_1(c), c) = 0 \), we can obtain
\[ d \int_{-\infty}^{\infty} e^{-i \theta} \sin \beta y d\theta + \partial f(0,0) e^{-i \epsilon \tau} \sin \beta \tau = 0. \quad (3.13) \]

If \( \partial f(0,0) = 0 \), (3.13) can imply \( \sin \beta y = 0 \) and it is easily seen that \( \beta = 0 \) by (3.12). If \( \partial f(0,0) > 0 \), according to (3.13), then we have \( \sin \beta y = 0 \) and \( \sin \frac{\beta \tau}{2} = 0 \) which imply that \( \beta = 0 \) by (3.12).

Now we consider two cases:

Case 1: \( \tilde{U}(\xi) \) is increasing for large \( -\xi > 0 \);

Case 2: \( \tilde{U}(\xi) \) is not monotone for large \( -\xi > 0 \).

If Case 1 holds, then we can choose a translation of \( \tilde{U} \) that is increasing for \( \xi < 0 \). Letting \( \tilde{U}(\xi) = \tilde{U}(\xi - \epsilon \tau) \), it is clear that \( \tilde{U}(\xi) \) is decreasing for \( \xi > 0 \) which implies that (i) and (ii) of Theorem 1.2 hold.

If Case 2 holds, let
\[ p = \frac{1}{c} \chi \quad \text{and} \quad \tilde{U}(\xi) = \tilde{U}(\xi) e^{\chi \xi}, \]

where \( L = \max_{u \in \mathbb{R}, u \in (K]} \left\{ \left| \partial f(u, v) \right| \right\} \). Then for large enough \( -\epsilon > 0 \), we have
\[ \tilde{U}(\xi) = d f \tilde{U}(\xi) e^{\chi \xi} + \left( 1 + L \tilde{U}(\xi) + f(\tilde{U}(\xi), \tilde{U}(\xi - \epsilon \tau)) \right) e^{\chi \xi} > 0. \]

Then we can choose a translation of \( \tilde{U} \) that is increasing for \( \xi < 0 \). Letting \( \tilde{U}(\xi) = \tilde{U}(\xi - \epsilon \tau) \), it is clear that \( \tilde{U}(\xi) \) is decreasing for \( \xi > 0 \). Letting \( L(\lambda) = \int_{-\infty}^{\infty} e^{-i \theta} \tilde{U}(\xi) d\theta \), noting that \( L(\lambda) = L(\lambda - \epsilon \tau) \), we apply Lemma 3.1 to \( \tilde{U}(\xi) \) to get the existence of
\[ \lim_{\xi \to -\infty} \frac{\tilde{U}(\xi)}{e^{i \epsilon \tau}} = \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{-i \epsilon \tau} \]

for \( c > c_1(\tau) \) and
\[ \lim_{\xi \to -\infty} \frac{\tilde{U}(\xi)}{e^{i \epsilon \tau}} = - \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{-i \epsilon \tau} \]

for \( c = c_1(\tau) \). This completes the proof of Theorem 1.2. \( \square \)

4. Applications and the effect of delay on the minimal wave speed

We first consider another version of the classic Logistic equation. More precisely, we investigate the existence and asymptotic behavior of traveling wave fronts for the equation
\[ u_t(t, x) = d [f(u, v) - u(t, x)] + ru(t - \tau, x)(1 - u(t, x)). \quad (4.1) \]

which was proposed by Pan et al. [18], where \( r \) is a positive constant. Since \( f(u, v) = r(1-u)v \) satisfies assumptions (F1)-(F2), we can obtain

**Theorem 4.1.** Assume that (F1) holds. Then there exists a positive constant \( c_1(\tau) \) such that for each \( c \geq c_1(\tau) \), (4.1) admits a nondecreasing positive traveling wave front \( u(t, x) = u(x + c t) \) connecting 0 and 1. Moreover, if \( c > c_1(\tau) \), then
\[ \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{-i \epsilon \tau} = 1, \quad \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{i \epsilon \tau} = \lambda_1(c), \]

where \( \lambda_1(c) > 0 \) is the smallest root of the equation
\[ \Delta(0, \lambda) = -c \lambda + d \int_{-\infty}^{\infty} e^{-i \theta} \sin \beta y d\theta - 1 + r e^{-i \epsilon \tau} = 0. \quad (4.3) \]

**Theorem 4.2.** Assume that (F1) holds and \( \tilde{U}(\xi) \) is a traveling wave of (4.1) with speed \( c \geq c_1(\tau) \) connecting 0 and 1 and \( 0 < \tilde{U}(\xi) < 1 \). Then we have the following conclusions

(i) If every \( c > c_1(\tau) \), \( \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{-i \epsilon \tau} \) exists.
(ii) If \( c = c_1(\tau) \), there exists a constant \( \lambda_1 > 0 \) such that
\[ \lim_{\xi \to -\infty} \tilde{U}(\xi) e^{-i \epsilon \tau} = \lambda_1(c) \]

(iii) If \( 0 < c < c_1(\tau) \), there is no traveling wave front connecting 0 and 1.
Remark 4.1. In [18], authors showed that (4.1) admits a traveling wave front \( U(x + ct) \) which connects 0 and 1 for \( c \geq c_s(\tau) \) and \( \lim_{t \to \infty} U(\xi) e^{-\lambda t} = 1 \) for \( c > c_s(\tau) \). As far as we know, the asymptotic behaviors for any traveling wave fronts are not still reported. In Theorem 4.4, we may obtain more results about the asymptotic behaviors of any traveling wave fronts. Thus, we improve and complement results in [18].

Next, consider the following diffusive Nicholson’s blowflies equation with a discrete delay
\[
 u_{t}(x,t) = d f(u(x, t)) - ru + rpu(t - x)e^{-e^{x}t} - x,
\]
which has been investigated in [7,12,13,20,24] and the references cited therein when \( J_s(0,0) = J_r(0,0) \) is in place of \( J_s(u(x, t) - u(x, t)) \), where \( r > 0 \) and \( \tau > 0 \). When \( 1 < p < e \), \( f(u, t) = -ru + rpe^{-e^{x}t} \) satisfies assumptions (F1) and (F2). Therefore, we have the following theorems.

Theorem 4.3. Assume that (J1) holds. Then there exists a positive constant \( c_s(\tau) \) such that for each \( c \geq c_s(\tau) \), (4.4) admits a non-decreasing positive traveling wave front \( u(x + ct) = U(x + ct) \) connecting 0 and \( ln p \). Moreover, if \( c > c_s(\tau) \), then
\[
 \lim_{t \to -\infty} U(\xi) e^{-\lambda t} = 1, \quad \lim_{t \to -\infty} U(\xi) e^{-\lambda t} = \lambda_1(c),
\]
where \( \lambda_1(c) > 0 \) is the smallest root of the equation
\[
 \Delta t (c, \lambda) = -c^2 + \int_{R} e^{-\lambda t} f(y) dy - 1 + r + rpe^{-e^{x}t} = 0.
\]

Theorem 4.4. Assume that (J1) holds and \( U(\xi) \) is a traveling wave of (4.4) with speed \( c \geq c_s(\tau) \) connecting 0 and \( ln p \), and \( 0 < U(\xi) \leq ln p \). Then we have the following conclusions

(i) If every \( c > c_s(\tau) \), \( \lim_{t \to -\infty} U(\xi) e^{-\lambda t} \) exists.

(ii) If \( c = c_s(\tau) \), there exists a constant \( \lambda > 0 \) such that \( \lim_{t \to -\infty} U(\xi) e^{-\lambda t} \) exists.

(iii) If \( 0 < c < c_s(\tau) \), there is no traveling wave front connecting 0 and \( ln p \).

Finally, we consider the effects of the delay in (1.4) on the minimal speed. By Lemma 2.1, it is easily seen that
\[
 \Delta t (\lambda_1(c), c(\tau)) = 0, \quad \frac{\partial}{\partial \tau} \Delta t (\lambda_1(c), c(\tau))|_{\tau = -} = 0
\]
which implies that \( c_s(\tau) \) and \( \lambda_1(c) \) are differential functions with respects to \( \tau \). Moreover, it is easy to see that
\[
 \frac{dc_s(\tau)}{d\tau} = -c_s(\tau) \frac{\partial f(0,0)}{\partial \xi} e^{-\lambda_1(c)\tau} \frac{\partial f(0,0)}{\partial \xi} - \lambda_1(c) \leq 0,
\]
which implies that the delay can slow the minimal wave speed for \( \partial f(0,0) > 0 \) and the delay is independent of the minimal wave speed for \( \partial f(0,0) = 0 \). Especially, the delay can slow the minimal wave speed in (4.1) and (4.4) since \( \partial f(0,0) > 0 \) and \( \partial f(0,0) = rp > 0 \), respectively.

References