Optimization: A Journal of Mathematical Programming and Operations Research

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gopt20

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Version of record first published: 24 Jun 2011

To cite this article: Xian-Jun Long, Jian-Wen Peng & Soon-Yi Wu (2012): Generalized vector variational-like inequalities and nonsmooth vector optimization problems, Optimization: A Journal of Mathematical Programming and Operations Research, 61:9, 1075-1086

To link to this article: http://dx.doi.org/10.1080/02331934.2010.538056

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Generalized vector variational-like inequalities and nonsmooth vector optimization problems

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(Received 19 May 2010; final version received 30 October 2010)

In this article, we establish some relationships between a solution of generalized vector variational-like inequalities and an efficient solution or a weakly efficient solution to the nonsmooth vector optimization problem under the assumptions of pseudoinvexity or invariant pseudomonotonicity. Our results extend and improve the corresponding results in the literature.

Keywords: generalized vector variational-like inequality; nonsmooth vector optimization problem; pseudoinvexity; invariant pseudomonotonicity; locally Lipschitz function

AMS Subject Classifications: 58E35; 49J52; 90C29

1. Introduction

The concept of vector variational inequality (VVI) was first introduced and studied by Giannessi [7] in finite-dimensional spaces. Since then, VVIs have received much attention by many authors due to its potential application in Vector optimization problem (VOP), vector traffic equilibrium problem, economics and management science. A large number of results have appeared in the literature (see, e.g. [4,5,9,10,12,18,26] and the references therein).

In the recent years, VVIs have been used as a tool for studying VOPs (see, e.g. [1,2,8,13–15,19,20,22,23,25] and the references therein). In [8], Giannessi derived some relationships between a solution of a Minty VVI and an efficient solution or a weakly efficient solution to the VOP under the assumptions of convexity or monotonicity. Yang et al. [25] generalized the results of [8] to pseudoconvexity or pseudomonotonicity. On the other hand, the VVI has been extended to the vector variational-like inequality (VVLI). Recently, Ruiz-Garzón et al. [20] obtained some relations between Stampacchia VVLI and the VOP. Yang and Yang [23] gave some relations between a solution of a Minty VVLI and an efficient solution or a weakly efficient solution to the VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity. Very recently, Al-Homidan and Ansari [1] studied the...
relationship between the generalized Minty VVLI, generalized Stampacchia VVLI and VOPs involving nonsmooth invex functions. They also considered the generalized weak Minty VVLI and the generalized weak Stampacchia VVLI and obtained some relations between the solution of these problems and a weakly efficient solution of the nondifferentiable VOP.

Motivated by the work reported in [1,8,20,23,25], the purpose of this article is to establish a relation between a solution of the generalized VVLI and the nonsmooth VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity. This results extend and improve the corresponding results of [1,23].

The rest of this article is organized as follows. In Section 2, we recall some basic definitions and preliminary results. In Section 3, we investigate some relations between a solution of the generalized Minty or Stampacchia VVLI and an efficient solution of the nonsmooth VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity. In Section 4, we study the relation among the generalized weak Minty or weak Stampacchia VVLI and the nonsmooth VOP.

2. Preliminaries

Throughout this article, unless otherwise specified, we assume that $K$ is a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ is a given mapping, $J = \{1, 2, \ldots, p\}$. The interior of $K$ is denoted by $\text{int}K$.

Let $f = (f_1, f_2, \ldots, f_p) : K \to \mathbb{R}^p$ be a vector-valued mapping. In this article, we consider the following VOP:

$$\text{Min } f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$$

s.t. $x \in K$.

**Definition 2.1** A point $x \in K$ is said to be an efficient solution (respectively, a weakly efficient solution) of VOP iff

$$f(y) - f(x) = (f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \notin -R_+^p \setminus \{0\}$$

for all $y \in K$ (respectively, $f(y) - f(x) = (f_1(y) - f_1(x), \ldots, f_p(y) - f_p(x)) \notin -\text{int}R_+^p$ for all $y \in K$), where $R_+^p$ is the nonnegative orthant of $\mathbb{R}^p$.

**Definition 2.2** [6] Let $f : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$, the Clarke generalized directional derivative of $f$ at $x$ in the direction $v \in \mathbb{R}^n$ is defined by

$$f^\circ(x; v) := \limsup_{y \to x \atop t \downarrow 0} \frac{f(y + tv) - f(y)}{t}$$

and the Clarke generalized gradient of $f$ at $x$ is denoted by

$$\partial f(x) := \{\xi \in \mathbb{R}^n \mid f^\circ(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in \mathbb{R}^n\}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. 
Definition 2.3 [21] The set $K$ is said to be invex with respect to $\eta$ iff, for any $x, y \in K$ and $\lambda \in [0, 1]$, we have $y + \lambda \eta(x, y) \in K$.

Remark 2.1 If $\eta(x, y) = x - y$ for any $x, y \in K$, then Definition 2.3 reduces to the convexity of the set $K$.

Definition 2.4 [20] The vector-valued function $\eta: K \times K \rightarrow \mathbb{R}^n$ is said to be skew iff, for any $x, y \in K$, $\eta(x, y) + \eta(y, x) = 0$.

Definition 2.5 [3] Let $K$ be invex with respect to $\eta$ and let $x, y \in K$ be two arbitrary points in $K$. A set $P_{xz}$ is said to be a closed (respectively, open) $\eta$-path joining the points $x$ and $z = x + \eta(y, x)$ (contained in $K$) iff

$$P_{xz} := \{u = x + \lambda \eta(y, x) : \lambda \in [0, 1]\}$$

(respectively, $P^0_{xz} := \{u = x + \lambda \eta(y, x) : \lambda \in [0, 1]\}$).

Definition 2.6 Let $K$ be invex with respect to $\eta$ and $f: K \rightarrow \mathbb{R}$. $f$ is said to be

(i) prequasiinvex [16] with respect to $\eta$ on $K$ iff, for any $x, y \in K$ and $\lambda \in [0, 1]$,

$$f(y + \lambda \eta(x, y)) \leq \max\{f(x), f(y)\};$$

(ii) semi-strictly prequasiinvex [24] with respect to $\eta$ on $K$ iff, for any $x, y \in K$ and $\lambda \in [0, 1]$ with $f(x) \neq f(y)$,

$$f(y + \lambda \eta(x, y)) < \max\{f(x), f(y)\}.$$  

Definition 2.7 [11] Let $K$ be invex with respect to $\eta$ and $f: K \rightarrow \mathbb{R}$ be locally Lipschitz on $K$. $f$ is said to be

(i) pseudoinvex with respect to $\eta$ on $K$ iff, for any $x, y \in K$ and any $\xi \in \partial f(y)$,

$$\langle \xi, \eta(x, y) \rangle \geq 0 \Rightarrow f(x) \geq f(y);$$

(ii) quasiinvex with respect to $\eta$ on $K$ iff, for any $x, y \in K$ and any $\xi \in \partial f(y)$,

$$f(x) \leq f(y) \Rightarrow \langle \xi, \eta(x, y) \rangle \leq 0.$$  

Definition 2.8 [11] Let $K$ be invex with respect to $\eta$ and $f: K \rightarrow \mathbb{R}$ be locally Lipschitz on $K$. $\partial f$ is said to be invariant pseudomonotone with respect to $\eta$ on $K$ iff, for any $x, y \in K$ and any $\xi \in \partial f(x), \xi \in \partial f(y)$,

$$\langle \xi, \eta(y, x) \rangle \geq 0 \Rightarrow \langle \xi, \eta(y, x) \rangle \leq 0.$$

Mohan and Neogy [16] introduced Condition C defined as follows.

Condition C The vector-valued function $\eta: K \times K \rightarrow \mathbb{R}^n$ is said to satisfy Condition C iff, for any $x, y \in K$ and $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda \eta(x, y)) = -\lambda \eta(x, y),$$

$$\eta(x, y + \lambda \eta(x, y)) = (1 - \lambda) \eta(x, y).$$
The function \( \eta \) in Example 2.1 [17] satisfies Condition C but is not a skew function. The following example shows that there exists some function \( \eta \) which is a skew function but does not satisfy Condition C.

**Example 2.1** Let \( \alpha, \beta \) be two positive numbers and \( \eta(x, y) = \alpha(x^\beta - y^\beta) \), for any \( x, y \in R \). Then, it is easy to verify that \( \eta \) is a skew function. However, \( \eta \) does not satisfy Condition C. It is clear that this example contains Example 2.2 in [17] as a special case.

**Condition A** [24] Let \( K \) be invex with respect to \( \eta \), and let \( f : K \to R \). \( f \) is said to satisfy Condition A iff, for any \( x, y \in K \),

\[ f(y + \eta(x, y)) \leq f(x). \]

The following lemmas will be used in the sequel.

**Lemma 2.1** [19] Let \( K \) be invex with respect to \( \eta \) such that \( \eta \) satisfies Condition C, let \( f : K \to R \) be locally Lipschitz on \( K \). If \( f \) is pseudoinvex with respect to \( \eta \) on \( K \), then \( f \) is semi-strictly prequasiinvex with respect to \( \eta \) on \( K \).

**Lemma 2.2** [23] Let \( K \) be invex with respect to \( \eta \) such that \( \eta \) satisfies Condition C. If \( f : K \to R \) is lower semicontinuous and semi-strictly prequasiinvex with respect to \( \eta \) on \( K \), then \( f \) is prequasiinvex with respect to \( \eta \) on \( K \).

**Lemma 2.3** [19] Let \( K \) be invex with respect to \( \eta \) such that \( \eta \) satisfies Condition C, and \( f : K \to R \) be locally Lipschitz on \( K \).

(i) If \( f \) is quasiinvex with respect to \( \eta \) on \( K \), then \( f \) is prequasiinvex with respect to \( \eta \) on \( K \).

(ii) Conversely, if \( f \) is prequasiinvex with respect to \( \eta \) on \( K \) and \( \eta \) is continuous with respect to the second argument, then \( f \) is quasiinvex with respect to \( \eta \) on \( K \).

**Proposition 2.1** Let \( K \) be invex with respect to \( \eta \) such that \( \eta \) satisfies Condition C and is continuous with respect to the second argument, and let \( f : K \to R \) be locally Lipschitz on \( K \). If \( f \) is pseudoinvex with respect to \( \eta \) on \( K \), then \( \partial f \) is invariant pseudomonotone with respect to \( \eta \) on \( K \).

**Proof** Let \( x, y \in K \) with \( x \neq y \), be such that

\[ \langle \xi, \eta(x, y) \rangle \geq 0 \quad \text{for all } \xi \in \partial f(y). \]

Since \( f \) is pseudoinvex on \( K \),

\[ f(x) \geq f(y). \quad (2.1) \]

Now, we prove that

\[ \langle \xi, \eta(y, x) \rangle \leq 0 \quad \text{for all } \xi \in \partial f(x). \]

In fact, suppose by contradiction that there exists \( \xi^* \in \partial f(x) \) such that

\[ \langle \xi^*, \eta(y, x) \rangle > 0. \quad (2.2) \]

By the pseudoinvexity of \( f \) and Lemmas 2.1–2.3, \( f \) is a quasiinvex function. This fact together with (2.2) yields \( f(y) > f(x) \), which contradicts (2.1). Therefore, \( \partial f \) is invariant pseudomonotone with respect to \( \eta \) on \( K \). This completes the proof.
Remark 2.2 By Theorem 5.1 in [11] and Proposition 2.1, we know that if \( f \) and \( \eta \) satisfy Conditions A and C, respectively, then the invariant pseudomonotonicity of \( \partial f \) is equivalent to the psedoinvexity of \( f \) with respect to the same \( \eta \).

Theorem 2.1 [3] Let \( K \) be invex with respect to \( \eta \), let \( x, z \in K \) be two arbitrary points in \( K \) and let \( P_{xz} \) be an arbitrary \( \eta \)-path contained in \( \text{int} \ K \). Let \( g: K \to R \) be locally Lipschitz on \( K \). Then, for any \( y = x + \eta(z, x) \in K \), there exist \( w \in P_{xy}^0 \) and \( \xi \in \partial g(w) \) such that

\[
\langle \xi, \eta(z, x) \rangle = g(y) - g(x).
\]

In other words, for any \( y = x + \eta(z, x) \in K \), there exist \( \lambda \in ]0, 1[ \), \( w := x + \lambda \eta(z, x) \) and \( \xi \in \partial g(w) \) such that

\[
\langle \xi, \eta(z, x) \rangle = g(x + \eta(z, x)) - g(x).
\]

3. Generalized VVLIls

Let \( K \) be a nonempty subset of \( R^n \) and let \( \eta: K \times K \to R^n \) be a given mapping. Let \( f = (f_1, f_2, \ldots, f_p): K \to R^n \) be a vector-valued mapping. In this section, we consider the following three types of generalized VVLI problems:

(I) type 1 generalized Minty vector variational-like inequality problem (GMVVLIP)_1: find \( x \in K \) such that, for any \( y \in K \) and any \( \xi_i \in \partial f_i(y) \), \( i = 1, 2, \ldots, p \),

\[
(\langle \xi_1, \eta(y, x) \rangle, \ldots, \langle \xi_p, \eta(y, x) \rangle) \notin -R_+^p \setminus \{0\}.
\]

(II) type 2 generalized Minty vector variational-like inequality problem (GMVVLIP)_2: find \( x \in K \) such that, for any \( y \in K \) and any \( \xi_i \in \partial f_i(y) \), \( i = 1, 2, \ldots, p \),

\[
(\langle \xi_1, \eta(x, y) \rangle, \ldots, \langle \xi_p, \eta(x, y) \rangle) \notin R_+^p \setminus \{0\}.
\]

(III) generalized Stampacchia vector variational-like inequality problem (GSVVLIP): find \( x \in K \) such that, for any \( y \in K \), there exists \( \zeta_i \in \partial f_i(x) \), \( i = 1, 2, \ldots, p \),

\[
(\langle \xi_1, \eta(y, x) \rangle, \ldots, \langle \xi_p, \eta(y, x) \rangle) \notin -R_+^p \setminus \{0\}.
\]

It is worth noting that (GMVVLIP)_1 and GSVVLIP are introduced and studied by Al-Homidan and Ansari [1]. When \( \eta(y, x) = y - x \), (GMVVLIP)_1 reduces to the generalized Minty VVI problem considered by Lee [13]. When \( f_1, f_2, \ldots, f_p \) are differentiable, (GMVVLIP)_1 and (GMVVLIP)_2, respectively, become the type 1 Minty vector variational-like inequality (MVVI)_1 and type 2 Minty vector variational-like inequality (MVVI)_2 researched in [23].

Now, we investigate the relation between a solution of generalized vector variational-like inequalities and an efficient solution to the nonsmooth VOP under the suitable conditions.

Theorem 3.1 Let \( K \) be a nonempty invex set with respect to \( \eta \) such that any \( \eta \)-path is contained in \( K \) and \( \eta \) satisfies Condition C. For each \( i \in J \), let \( \partial f_i \) be invariant pseudomonotone with respect to \( \eta \) on \( K \) and \( f_i \) satisfy Condition A. If \( x_0 \in K \) is a solution of (GMVVLIP)_1, then \( x_0 \) is an efficient solution to the VOP.
*Proof* Let $x_0 \in K$ be a solution of $(\text{GMVVLIP})_1$. Suppose by contradiction that $x_0$ is not an efficient solution of VOP. Then, there exists $y \in K$ such that
\begin{equation}
(f_1(y) - f_1(x_0), \ldots, f_p(y) - f_p(x_0)) \in -R^p_+ \setminus \{0\}.
\end{equation}
(3.1)

Let $y(\lambda) = x_0 + \lambda y$, for any $\lambda \in [0, 1]$. Since $K$ is invex, $y(\lambda) \in K$, for any $\lambda \in [0, 1]$. From Condition A and (3.1), we have
\begin{equation}
(f_1(y(1)) - f_1(y(0)), \ldots, f_p(y(1)) - f_p(y(0))) \in -R^p_+ \setminus \{0\}.
\end{equation}
(3.2)

By the mean-value Theorem 2.1, there exist $\lambda_i \in ]0, 1[, i = 1, 2, \ldots, p$ and $\xi_i \in \partial f_i(y(\lambda_i))$, where $y(\lambda_i) = x_0 + \lambda_i \eta(y, x_0)$, such that
\begin{equation}
\langle \xi_i, \eta(y, x_0) \rangle = f_i(x_0 + \eta(y, x_0)) - f_i(x_0), \quad i = 1, 2, \ldots, p.
\end{equation}
(3.3)

Combining (3.2) and (3.3) yields
\begin{equation}
(\langle \xi_1, \eta(y, x_0) \rangle, \ldots, \langle \xi_p, \eta(y, x_0) \rangle) \in -R^p_+ \setminus \{0\}.
\end{equation}

It follows that
\begin{equation}
\langle \xi_i, \eta(y, x_0) \rangle \leq 0, \quad i = 1, 2, \ldots, p,
\end{equation}
(3.4)

where strict inequality holds for some $i \in J$.

By Condition C,
\begin{equation}
\eta(y(\lambda_i), x_0) = \eta(x_0 + \lambda_i \eta(y, x_0), x_0) = \lambda_i \eta(y, x_0), \quad i = 1, 2, \ldots, p,
\end{equation}
which together with (3.4) yields
\begin{equation}
\langle \xi_i, \eta(y(\lambda_i), x_0) \rangle \leq 0, \quad i = 1, 2, \ldots, p,
\end{equation}
(3.5)

where strict inequality holds for some $i \in J$. Without loss of generality, we can assume that the strict inequality holds for $i = 2$.

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all equal. Then, it follows from (3.5) that
\begin{equation}
(\langle \xi_1, \eta(y(\lambda_1), x_0) \rangle, \ldots, \langle \xi_p, \eta(y(\lambda_p), x_0) \rangle) \in -R^p_+ \setminus \{0\},
\end{equation}
which contradicts the fact that $x_0$ is a solution of $(\text{GMVVLIP})_1$.

Consider the case when $\lambda_1, \lambda_2, \ldots, \lambda_p$ are not equal. Let $\lambda_1 \neq \lambda_2$. From Condition C, we have
\begin{equation}
\eta(y(\lambda_1), y(\lambda_2)) = \frac{\lambda_1 - \lambda_2}{\lambda_1} \eta(y(\lambda_1), x_0) = \frac{\lambda_1 - \lambda_2}{\lambda_2} \eta(y(\lambda_2), x_0),
\end{equation}
(3.6)

\begin{equation}
\eta(y(\lambda_2), y(\lambda_1)) = \frac{\lambda_2 - \lambda_1}{\lambda_1} \eta(y(\lambda_1), x_0) = \frac{\lambda_2 - \lambda_1}{\lambda_2} \eta(y(\lambda_2), x_0).
\end{equation}
(3.7)

If $\lambda_1 > \lambda_2$, then from (3.5) and (3.7), we know that there exists $\xi_1 \in \partial f_1(y(\lambda_1))$ such that
\begin{equation}
\langle \xi_1, \eta(y(\lambda_2), y(\lambda_1)) \rangle \geq 0.
\end{equation}

Since $\partial f_1$ is invariant pseudomonotone, we know that for all $\xi_1 \in \partial f_1(y(\lambda_2))$,
\begin{equation}
\langle \xi_1, \eta(y(\lambda_1), y(\lambda_2)) \rangle \leq 0.
\end{equation}
This fact together with (3.6) yields
\[
\langle \xi_1, \eta(y(\lambda_2), x_0) \rangle \leq 0, \quad \text{for all } \xi_1 \in \partial f_1(y(\lambda_2)).
\]
If \( \lambda_1 < \lambda_2 \), then from (3.5) and (3.6), we know that there exists \( \xi_2 \in \partial f_2(y(\lambda_2)) \) such that
\[
\langle \xi_2, \eta(y(\lambda_1), y(\lambda_2)) \rangle > 0.
\] (3.8)
By the invariant pseudomonotonicity of \( \partial f_2 \) and (3.8), we know that for all \( \xi_2 \in \partial f_2(y(\lambda_1)) \),
\[
\langle \xi_2, \eta(y(\lambda_2), y(\lambda_1)) \rangle < 0.
\]
It follows from (3.7) that for all \( \xi_2 \in \partial f_2(y(\lambda_1)) \),
\[
\langle \xi_2, \eta(y(\lambda_1), x_0) \rangle < 0.
\]
Therefore, for the case \( \lambda_1 \neq \lambda_2 \), let \( \bar{\lambda} = \min\{\lambda_1, \lambda_2\} \). There exists \( \bar{\xi}_i \in \partial f_i(y(\bar{\lambda})) \) such that
\[
\langle \bar{\xi}_i, \eta(y(\bar{\lambda}), x_0) \rangle \leq 0, \quad i = 1, 2,
\]
where strict inequality holds for \( i = 2 \). By continuing this process, we can find \( \lambda^* \in ]0, 1[ \) and \( \xi_i^* \in \partial f_i(y(\lambda^*)) \) such that \( \lambda^* = \min\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \) and
\[
\langle \xi_i^*, \eta(y(\lambda^*), x_0) \rangle \leq 0, \quad i = 1, 2, \ldots, p,
\]
where strict inequality holds for some \( i \in J \). This contradicts the fact that \( x_0 \) is a solution to the (GMVVLIP)_1. This completes the proof.

By Theorem 3.1 and Proposition 2.1, we have the following corollary.

**Corollary 3.1** Let \( K \) be a nonempty invex set with respect to \( \eta \) such that any \( \eta \)-path is contained in \( K \), \( \eta \) satisfies Condition C and is continuous with respect to the second argument. For each \( i \in J \), let \( f_i \) be locally Lipschitz, pseudoinvex function with respect to the same \( \eta \) on \( K \) and \( f_i \) satisfy Condition A. If \( x_0 \in K \) is a solution of \((GMVVLIP)_1\), then \( x_0 \) is an efficient solution to the VOP.

**Remark 3.1**

(i) Corollary 3.1 generalizes and improves Theorem 3.1 of Al-Homidan and Ansari [1] since the invexity of \( f_i \) has been weakened by pseudoinvexity of \( f_i \) and the condition \( \eta \) is skew has been removed.

(ii) It is clear that Corollary 3.1 also generalizes and extends Proposition 1 of Giannessi [9], Theorem 2.1 of Lee [13], Theorem 3.1 of Yang et al. [25], Theorem 3.1(i) of Yang and Yang [23].

**Theorem 3.2** Let \( K \) be a nonempty invex set with respect to \( \eta \) such that \( \eta \) satisfies Condition C and is continuous with respect to the second argument. For each \( i \in J \), let \( f_i \) be locally Lipschitz, pseudoinvex function with respect to the same \( \eta \) on \( K \). If \( x_0 \in K \) is an efficient solution to the VOP, then \( x_0 \) is a solution of \((GMVVLIP)_2\).
Proof Let \( x_0 \in K \) be an efficient solution to the VOP. If \( x_0 \) is not a solution of (GMVVLIP)\(_2\), then there exist \( y \in K, \xi_i \in \partial f_i(y) \) such that

\[
\langle \xi_i, \eta(x_0, y) \rangle, \ldots, \langle \xi_p, \eta(x_0, y) \rangle \in R_+^p \setminus \{0\},
\]

or equivalently,

\[
\langle \xi_i, \eta(x_0, y) \rangle \geq 0, \quad i = 1, 2, \ldots, p,
\]

where strict inequality holds for some \( i \in J \). Without loss of generality, we can assume that the strict inequality holds for \( i = 1 \). For \( i = 1 \), by the pseudoinvexity of \( f_1 \) and Lemmas 2.1–2.3, \( f_1 \) is quasiinvex. It follows that

\[
f_1(x_0) > f_1(y).
\]

For \( i = 2, \ldots, p \), by the pseudoinvexity of \( f_i \),

\[
f_i(x_0) \geq f_i(y).
\]

Therefore,

\[
(f_1(y) - f_1(x_0), \ldots, f_p(y) - f_p(x_0)) \in -R_+^p \setminus \{0\},
\]

which contradicts that \( x_0 \) is an efficient solution of VOP. This completes the proof.

\[\square\]

**Theorem 3.3** Let \( K \) be a nonempty invex set with respect to \( \eta \). For each \( i \in J \), let \( \partial f_i \) be invariant pseudomonotone with respect to \( \eta \) on \( K \). If \( x_0 \in K \) is a solution of GSVVLIP, then \( x_0 \) is a solution of (GMVVLIP)\(_2\).

**Proof** Let \( x_0 \in K \) be a solution of GSVVLIP. Suppose by contradiction that \( x_0 \) is not a solution of (GMVVLIP)\(_2\). Then, there exist \( y \in K, \xi_i \in \partial f_i(y) \) such that

\[
\langle \xi_i, \eta(x, y) \rangle, \ldots, \langle \xi_p, \eta(x, y) \rangle \in R_+^p \setminus \{0\},
\]

or equivalently,

\[
\langle \xi_i, \eta(x, y) \rangle \geq 0, \quad i = 1, 2, \ldots, p,
\]

where strict inequality holds for some \( i_0 \in J \). By the invariant pseudomonotone of \( f_i, i = 1, 2, \ldots, p \),

\[
\langle \xi_i, \eta(y, x) \rangle \leq 0, \quad i = 1, 2, \ldots, p, \quad \xi_i \in \partial f_i(x),
\]

where strict inequality holds for the above mentioned \( i_0 \in J \). It follows that

\[
\langle \xi_1, \eta(y, x) \rangle, \ldots, \langle \xi_p, \eta(y, x) \rangle \in -R_+^p \setminus \{0\}.
\]

This contradicts that \( x_0 \) is a solution of GSVVLIP. This completes the proof. \[\square\]

**Remark 3.2** Theorems 3.2 and 3.3, respectively, generalize and extend Theorem 3.1(ii) and Theorems 3.3 of Yang and Yang [23] from smooth case to the nonsmooth case.
4. Generalized weak vector variational-like inequalities

Let $K$ be a nonempty subset of $\mathbb{R}^n$ and let $\eta: K \times K \to \mathbb{R}^n$ be a given mapping. Let $f=(f_1,f_2,\ldots,f_p): K \to \mathbb{R}^n$ be a vector-valued mapping. In this section, we consider the following generalized weak vector variational-like inequality problems:

(I) generalized weak Minty vector variational-like inequality problem (GWMVVLIP): find $x \in K$ such that, for any $y \in K$ and any $\xi_i \in \partial f_i(y)$, $i = 1, 2, \ldots, p$,

$$((\xi_1, \eta(x,y)), \ldots, (\xi_p, \eta(x,y))) \notin \text{int} \mathbb{R}_+^p.$$ 

(II) generalized weak Stampacchia vector variational-like inequality problem (GWSVVLIP): find $x \in K$ such that, for any $y \in K$, there exists $\xi_i \in \partial f_i(x)$, $i = 1, 2, \ldots, p$,

$$((\xi_1, \eta(y,x)), \ldots, (\xi_p, \eta(y,x))) \notin -\text{int} \mathbb{R}_+^p.$$ 

It is worth noting that GWMVVLIP and GWSVVLIP are introduced and studied by Al-Homidan and Ansari [1].

Now, we present some results which show the relationship among the solutions of GWMVVLIP, GWSVVLIP and a weakly efficient solution of VOP.

**Theorem 4.1** Let $K$ be a nonempty invex set with respect to $\eta$. For each $i \in J$, let $\partial f_i$ be invariant pseudomonotone with respect to $\eta$ on $K$. If $x_0 \in K$ is a solution of GWSVVLIP, then $x_0$ is a solution to the GWMVVLIP.

**Proof** Let $x_0 \in K$ be a solution of GWSVVLIP. If $x_0$ is not a solution of GWMVVLIP, then there exist $y \in K$ and $\xi_i \in \partial f_i(x)$, $i = 1, 2, \ldots, p$, such that

$$((\xi_1, \eta(x_0,y)), \ldots, (\xi_p, \eta(x_0,y))) \in \text{int} \mathbb{R}_+^p.$$ 

It follows that

$$\langle \xi_i, \eta(x_0,y) \rangle > 0, \ i = 1, 2, \ldots, p.$$ 

By the invariant pseudomonotonicity of $\partial f_i$, $i = 1, 2, \ldots, p$,

$$\langle \xi_i, \eta(y,x_0) \rangle < 0, \ \text{for all } \xi_i \in \partial f_i(x_0),$$

which contradicts the fact that $x_0$ is a solution of GWSVVLIP. This completes the proof.

**Theorem 4.2** Let $K$ be a nonempty invex set with respect to $\eta$. For each $i \in J$, let $f_i$ be locally Lipschitz, pseudoinvex function with respect to the same $\eta$ on $K$. If $x_0 \in K$ is a solution of GWSVVLIP, then $x_0$ is a weakly efficient solution to the VOP.

**Proof** Suppose that $x_0 \in K$ is a solution of GWSVVLIP, but $x_0$ is not a weakly efficient solution to VOP. Then, there exists $y \in K$ such that

$$(f_1(y) - f_1(x_0), \ldots, f_p(y) - f_p(x_0)) \in -\text{int} \mathbb{R}_+^p,$$

or equivalently,

$$f_i(y) < f_i(x_0), \ i = 1, 2, \ldots, p.$$
By the pseudoinvexity of $f_i$, $i = 1, 2, \ldots, p$

$$\langle \xi_i, \eta(y, x_0) \rangle < 0, \quad \text{for all } \xi_i \in \partial f_i(x_0).$$

It follows that

$$\left(\langle \xi_1, \eta(y, x_0) \rangle, \ldots, \langle \xi_p, \eta(y, x_0) \rangle\right) \in -\text{int}\mathbb{R}_+^p,$$

which contradicts that $x_0$ is a solution of GWSVVLIP. This completes the proof.

**Theorem 4.3** Let $K$ be a nonempty invex set with respect to $\eta$ such that $\eta$ satisfies Condition C and is continuous with respect to the second argument. For each $i \in J$, let $f_i$ be locally Lipschitz, pseudoinvex function with respect to the same $\eta$ on $K$. If $x_0 \in K$ is a weakly efficient solution of VOP, then $x_0$ is a solution to the GWMVVLIP.

**Proof** Let $x_0 \in K$ be a weakly efficient solution of GWMVVLIP. Suppose by contradiction that $x_0$ is not a solution of GWMVVLIP. Then, there exist $y \in K$ and $\xi_i \in \partial f_i(y)$ such that

$$\left(\langle \xi_1, \eta(x_0, y) \rangle, \ldots, \langle \xi_p, \eta(x_0, y) \rangle\right) \in \text{int}\mathbb{R}_+^p,$$

that is

$$\langle \xi_i, \eta(x_0, y) \rangle > 0, \quad i = 1, 2, \ldots, p.$$

By the pseudoinvexity of $f_i$, $i \in J$ and Lemmas 2.1–2.3, $f_i$ is quasiinvex. It follows that

$$f_i(x_0) > f_i(y), \quad i = 1, 2, \ldots, p.$$

Therefore

$$\left(\langle f_1(y) - f_1(x_0), \ldots, f_p(y) - f_p(x_0) \rangle\right) \in -\text{int}\mathbb{R}_+^p,$$

which contradicts that $x_0$ is a weakly efficient solution of VOP. This completes the proof.

**Remark 4.1**

(i) Theorems 4.1–4.3, respectively, generalize and improve Propositions 4.1, 4.3 and 4.4 of Al-Homidan and Ansari [1] since the invexity of $f_i$ has been weakened by pseudoinvexity of $f_i$ or invariant pseudomonotonicity of $\partial f_i$.

(ii) Theorems 4.1 and 4.2, respectively, generalize and extend Theorems 4.1 and 4.3 of Yang and Yang [23] from smooth case to nonsmooth case.

5. Conclusions

In this article, we establish some relationships between a solution of generalized vector variational-like inequalities and an efficient solution or a weakly efficient solution to the nonsmooth VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity. Our results extend and improve some corresponding results of Al-Homidan and Ansari [1] and Yang and Yang [23] to more general cases. However, it will be interesting to establish the relation between VOPs and VVIs defined by bifunctions. This may be the topic of some of our forthcoming papers.
Acknowledgements
This work was supported by the National Natural Science Foundation of China (11001287), the Education Committee Project Research Foundation of Chongqing (No. KJ100711), the Natural Science Foundation Project of Chongqing (CSTC 2010BB9254 and CSTC 2009BB8240) and the Research Fund of Chongqing Technology and Business University (09-56-06).

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