Asymmetric wavelet reconstruction of particle hologram with an elliptical Gaussian beam illumination

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We propose an asymmetric wavelet method to reconstruct a particle from a hologram illuminated by an elliptical, astigmatic Gaussian beam. The particle can be reconstructed by a convolution of the asymmetric wavelet and hologram. The reconstructed images have the same size and resolution as the recorded hologram; therefore, the reconstructed 3D field is convenient for automatic particle locating and sizing. The asymmetric wavelet method is validated by both simulated holograms of spherical particles and experimental holograms of opaque, nonspherical coal particles. © 2013 Optical Society of America

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1. Introduction

Digital holography can simultaneously measure the 3D position [1–6] and size of a particle [1,4,7], and then retrieve the 3D motion or velocity field [7–13]. Thus it has shown great potential in particle field diagnostics [14]. Among the numerous methods for recording holograms, the Gabor inline configuration is widely used for its simplicity. The laser beam is collimated to a plane wave and illuminates the particle field, and the scattering light of the particle interferes with the undisturbed reference wave to form fringe patterns in the hologram, which is digitized by the CCD. Then the 3D optical field of particles can be reconstructed through a numerical diffraction of the hologram by the reference wave, and various reconstruction methods (typically, the Fresnel method [1], the convolution method [1], or the wavelet method [15,16]) have been proposed.

In order to minimize the effect of aberrations, the investigated object is usually placed in open air or enclosed by flat parallel windows. In practice, the test section containing the particle can be in a complex shape, such as cylinder pipes or tunnels [11,17–19]. The collimated laser beam will be transformed into an elliptical and astigmatic Gaussian beam when passing through the curvature surface [17]. Besides the concentric circular fringes, concentric elliptical fringes, hyperbolic fringes, or even parallel fringes have been observed in holograms when a particle is illuminated by an elliptical Gaussian beam [18–21]. Thus, the astigmatism will seriously affect the fringe patterns, and should be compensated for during the recording [22,23] or the reconstruction [18–21,24] process. The classical Fresnel method, convolution method, or wavelet method fails to reconstruct particle holograms with an elliptical Gaussian beam illumination, because the kernels of those methods are isotropic. To eliminate the quadratic phase term and reconstruct the object, De Nicola et al. [24] modified the chirp...
function of the Fresnel integral to compensate for aberrations, and demonstrated that the correct object image could be reconstructed in the presence of severe anamorphism, yielding to the different pixel sizes in the reconstructed image along the x and y directions. Numerical methods with astigmatism compensation have been proposed for Fresnel holograms to reconstruct both the intensity and phase distributions [25,26]. Besides, the fractional Fourier transform (FRFT) has been applied to reconstruct the hologram by adjusting the fractional order to neutralize the astigmatism [18–21]. The particle can be focused in the reconstructed optical field at the optional fractional orders, and usually a circular in-focus image can be obtained for a circular disk or spherical particle. The depth position of the particle can be determined through a prior relation between the fractional orders and the axial distance. However, due to the different fractional orders of the Fresnel diffraction in the transversal directions, the size and resolution of the reconstructed images will not only be different from the recorded hologram but will also vary with the depth positions. Therefore the reconstructed image should be scaled before postprocessing, for instance, for automatically detecting, locating, and sizing the particles.

When the particle is illuminated by an elliptical Gaussian beam and the hologram is recorded with the Gabor inline configuration, it is of interest to reconstruct the particle image with constant size and resolution along different depth positions in the image field, because the reconstructed particles can be detected and located without preprocessing the reconstructed images. In this work, based on the analysis of the formation of the particle hologram with an elliptical Gaussian beam illumination, we propose to reconstruct the particle hologram with the asymmetric wavelet method. Then both the simulated hologram of the spherical particle and the experimental hologram of the nonspherical coal particle are tested to validate the proposed algorithm.

2. Asymmetric Wavelet Reconstruction

When an elliptical Gaussian beam is incident on an opaque spherical particle, as illustrated in Fig. 1, the forward scattering of light by the particle interferes with the undisturbed reference wave to form the hologram. If the beam radius is much larger than the size of the particle, the forward scattered light by the particle is mostly composed of diffracted light. Thus the formation of the hologram can be approximately characterized by the scalar theory of diffraction. According to the Collins formula [27], the optical field in the hologram plane is

\[
U(u,v) = \int \int G_1(x,y)[1-T(x,y)]
\times \exp \left[ i \frac{\pi}{\lambda B_x} (A_{ux}u^2 - 2ux + D_{ux}x^2) \right]
\times \exp \left[ i \frac{\pi}{\lambda B_y} (A_{vy}v^2 - 2vy + D_{vy}y^2) \right] dx dy,
\]

where

\[
G_1(x,y) = \exp \left( -\frac{x^2}{\omega_x^2} - \frac{y^2}{\omega_y^2} \right) \exp \left[ -i \frac{\pi}{\lambda} \left( \frac{x^2}{R_{x1}} + \frac{y^2}{R_{y1}} \right) \right]
\]

is the wave front of the laser beam incident on the particle, and \(a_{x1y1}\) and \(R_{x1y1}\) are the beam radii and curvatures along the x and y directions. \(A_{xy}, B_{xy}, D_{xy}\) are the elements of the ABCD transform matrix from the particle to the CCD in the x and y directions. In this work, the laser beam propagates in the air with \(\left( \begin{array}{cc} A_{xy} & B_{xy} \\ C_{xy} & D_{xy} \end{array} \right) = \left( \begin{array}{cc} 1 & i \lambda \omega_0 x_0 \\ 0 & 1 \end{array} \right)\). \(T(x,y) = \sum_{k=1}^{10} A_k \exp[-B_k(x^2 + y^2)/r^2]\) denotes the opaque particle with a radius of \(r\) expressed as a complex Gaussian expansion [28]. It can provide analytical expressions of the hologram with clear physical meaning. This integral can be separated into two parts, the reference wave \(R\) and the object wave \(O\) [18]. The reference wave \(R\) is

\[
R(u,v) = \int \int G_1(x,y) \exp \left[ i \frac{\pi}{\lambda B_x} (A_{ux}u^2 - 2ux + D_{ux}x^2) \right]
\times \exp \left[ i \frac{\pi}{\lambda B_y} (A_{vy}v^2 - 2vy + D_{vy}y^2) \right] dx dy
\]

\[
= C_{ref} \exp \left( \frac{i \pi D_{x1} x^2}{\lambda B_x} + \frac{i \pi D_{y1} y^2}{\lambda B_y} \right) \cdot
\times \exp \left[ \frac{i \pi}{\lambda B_x} \left( \frac{B_{x1}}{\omega_{x1}^2} + \frac{i B_x}{R_{x1}} - i A_x \right) \right]
+ \frac{i \pi}{\lambda B_y} \left( \frac{B_{y1}}{\omega_{y1}^2} + \frac{i B_y}{R_{y1}} - i A_y \right).
\]

where the constant

\[
C_{ref} = \pi / \sqrt{(-i B_x / \pi \omega_{x1}^2) + (i B_x / R_{x1}) - i A_x}(-i B_y / \pi \omega_{y1}^2) + (i B_y / R_{y1}) - i A_y).
\]
The reference wave $R$ is still an elliptical Gaussian beam, and the objective wave $O$ is

$$
O(u, v) = \int \int G(x, y) T(x, y) \times \exp \left[ \frac{i \pi}{\lambda B_x} (A_x u^2 - 2ux + D_x x^2) \right] 
\times \exp \left[ \frac{i \pi}{\lambda B_y} (A_y v^2 - 2vy + D_y y^2) \right] \, dx \, dy
= C_{\text{obj}} \exp \left( \frac{i \pi D_x x^2}{\lambda B_x} + \frac{i \pi D_y y^2}{\lambda B_y} \right)
\times \sum_{k=1}^{10} \left\{ \exp \left[ \frac{i \pi^2 A_k}{\lambda B_x \left( \frac{b}{\sigma} \right)^2} \right] \right\}
\times \sum_{k=1}^{10} \left\{ \exp \left[ \frac{i \pi^2 A_k}{\lambda B_y \left( \frac{b}{\sigma} \right)^2} \right] \right\},
$$

where the constant $C_{\text{obj}} = \pi$.

The light intensity in the hologram is

$$
I_{\text{holo}}(u, v) = U(u, v) \cdot \bar{U}(u, v)
= R \cdot \bar{R} + O \cdot \bar{O} + R \cdot \bar{O} + O \cdot R.
\tag{4}
$$

$R \cdot \bar{R}$ and $O \cdot \bar{O}$ are directly transmitted light and cannot be used to reconstruct the object. $R \cdot \bar{O}$ and $O \cdot \bar{R}$ will reconstruct the real and virtual images of the object, respectively. The quadratic phase of the interference between the reference and object waves is [18]

$$
\text{arg}(I_{\text{holo}}) = \text{arg}(O \cdot \bar{R} + R \cdot \bar{O})
= 3 \left\{ \frac{\pi x^2}{\lambda B_x \left( \frac{b}{\sigma} \right)^2} \right\}
+ \left( \frac{\pi y^2}{\lambda B_y \left( \frac{b}{\sigma} \right)^2} \right)
= \frac{\pi x^2}{\lambda B_x R_x} + \frac{\pi y^2}{\lambda B_y R_y}
= \frac{\pi x^2}{\lambda x_{\text{eq}}} + \frac{\pi y^2}{\lambda y_{\text{eq}}},
\tag{5}
$$

where $R_{x,y}$ is caused by the astigmatism, and $x_{\text{eq}}$ and $y_{\text{eq}}$ are the equivalent propagation distances in the $x$ and $y$ directions. The quadratic phase determines the linear chirp and the fringe patterns in the hologram.

To reconstruct the object from the hologram, the aberration should be compensated and the linear chirp should be removed. If $R_x = R_y$, the linear chirp is symmetric, and the fringe patterns are composed of concentric circular rings for spherical particles, such as holograms with a plane wave or circular Gaussian beam illumination. The object can be reconstructed by the classical methods, such as the Fresnel method, the convolution method, or the wavelet method. For holograms with particles illuminated by an elliptical Gaussian beam, $R_x \neq R_y$, and theoretical analysis has shown that the quadratic phase term is anisotropic due to the astigmatism. The classical reconstruction methods fail, because the kernels of those methods are isotropic. To eliminate the quadratic phase term and reconstruct the object, the FRFT has been applied to reconstruct the hologram [18, 20, 21]. The in-focus object was reconstructed using FRFT at the optimal fractional orders, with a prior relation between the fractional orders and the axial distance to evaluate the depth position. However, the reconstructed images in the fractional order domain are obtained by Fresnel diffraction of the hologram in the spatial domain using digital FRFT, and this results in magnifications of the size as well as the resolution of the reconstructed images. Both the size and resolution of the reconstructed image using the FRFT reconstruction vary with the fractional orders and depth position, which causes inconvenience because rescaling is necessary before postprocessing.

Although the classical wavelet method [15, 29] can reconstruct images with the same size and resolution as the recorded hologram, it fails to reconstruct the in-focus object image from holograms with elliptical Gaussian beam illumination, since the wavelet is isotropic in the lateral direction [15, 29]. If the wavelet is modified to be anisotropic, which compensates for the astigmatism of the elliptical Gaussian beam, the hologram could be reconstructed using the wavelet method. Similar to the classical symmetric wavelet reconstruction $\psi_{C_x} = \sin((x^2 + y^2/C_a^2) - M_x) \exp((-((x^2 + y^2)^2)/C_a^2))$ for holograms illuminated by the plane wave [15], we propose an asymmetric wavelet method to reconstruct the particle hologram illuminated by an elliptical Gaussian beam:

$$
\psi_{C_x, C_y}(x, y) = \left[ \sin \left( \frac{x^2}{C_a^2} + \frac{y^2}{C_b^2} \right) - M_x \right] \exp \left( -\left( (x^2 + y^2)^2/C_a^2 \right) \right),
\tag{6}
$$

The scale parameters $C_a$ and $C_b$ of the daughter wavelet are related with the transfer matrix (recording distance):

$$
C_a^2 = \frac{\lambda B_x R_x}{\pi}, \quad C_b^2 = \frac{\lambda B_y R_y}{\pi} = \frac{\lambda z_{\text{eq}}}{\pi}.
\tag{7}
$$
Thus the astigmatism caused by the different equivalent propagation distances along the x and y directions is compensated. $M_w(\sigma_a, \sigma_b)$ is used to adjust a zero mean value of the wavelet according to the following equation:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{c_a=1, c_b=1}(x, y) \, dx \, dy = 0. \quad (8)$$

We can obtain $M_w$:

$$M_w(\sigma_a, \sigma_b) = \frac{i}{2\sigma_a \sigma_b} \left( \frac{1}{\sqrt{1 + i \sigma_a^2}} \frac{1}{\sqrt{1 + i \sigma_b^2}} - \frac{1}{\sqrt{1 - i \sigma_a^2}} \frac{1}{\sqrt{1 - i \sigma_b^2}} \right). \quad (9)$$

where $\sigma_a$ and $\sigma_b$ are the width parameters of the envelope function $\exp(-(x^2/C_{\sigma_a}^2) - (y^2/C_{\sigma_b}^2))$, which is the windowing function and has a low-pass property for local analysis. The values of variables $\sigma_a$ and $\sigma_b$ should be properly chosen to avoid the undersampling effect of the wavelet, and can be determined as in [29]. For plane wave illumination, $R_x = R_y = 1$, $\sigma_a = \sigma_b = \sigma$, and then $M_w(\sigma) = \sigma^2/(1 + \sigma^4)$; therefore the asymmetric wavelet is reduced to symmetrical, which is the same as in [29].

The particle can be reconstructed by a convolution of the wavelet and hologram:

$$I_{re}(x, y) = I_{holo}(x, y) \otimes \psi_{c_a, c_b}(x, y) = F^{-1}[F(I_{holo}) \cdot F(\psi_{c_a, c_b})] \quad (10)$$

where $F$ and $F^{-1}$ denote the forward and inverse Fourier transforms, respectively. The product of the Fourier spectra of the wavelet function and the hologram is transformed back into the spatial domain through an inverse Fourier transform. Consequently, the reconstructed image has the same size and resolution as the original hologram. Compared with FRFT, the wavelet reconstruction method has the advantage that the transverse position of each pixel in the reconstructed images at different depth positions does not change; therefore the reconstructed 3D optical field can be easily extended for automatically detecting, locating, and sizing the reconstructed particle.

3. Validation

The asymmetric wavelet method was validated first by the simulated hologram of spherical particles and then by the experimental hologram of nonspherical coal particles. For the simulated hologram, the scattering light of an elliptical Gaussian beam by a particle was computed using the generalized Lorenz–Mie theory [30,31]. Figure 2(a) shows a simulated hologram of a particle with diameter 140 µm illuminated by an elliptical Gaussian beam. The wave radii of the beam waist are $\omega_x = 2 \mu m$ and $\omega_y = 3 mm$ in the x and y directions, respectively. The particle is located at $z_p = 2.7 cm$, and the CCD is placed 2.3 cm after the particle to record the particle hologram. According to Eq. (5), the $R_x$ is 1.8519, and the $R_y$ equals 1. Thus the equivalent propagation distances are 4.26 cm ($z_{x, eq}$) and 2.3 cm ($z_{y, eq}$) in the x and y directions, respectively. Elliptical fringes rather than circular rings are observed in the holograms. Figure 2(b) compares the fringes in the x and y directions and the particle hologram fringes with the same configurations as in Fig. 2(a) except for the plane wave illumination. The intensity of the fringe for elliptical Gaussian beam illumination decreases away from the center. The fringes’ frequency in the y direction of Fig. 2(a) (in green in Fig. 2(b)) equals that with plane illumination [in black in Fig. 2(b)]. Meanwhile, the frequency of the horizontal fringe [in blue in Fig. 2(b)] is quite different, since the equivalent propagation distance is different. Figures 2(c) and 2(d) show the reconstructed images at the depth positions of 2.3 and 4.26 cm, respectively, and the reconstructed images are focalized in the y and x directions, respectively. But a clear particle image cannot be observed since only one direction is in focus without compensating for astigmatism.

Figure 3 shows the results of the reconstructed in-focus image of the hologram in Fig. 2(a) using asymmetric wavelet transform with astigmatism.
compensation, and its comparison with those images reconstructed by modified Fresnel approximation [24] and fractional order transform [18,20,21]. The particle is focused at the depth position $z = 2.3$ cm in asymmetric wavelet reconstruction, $z_x = 4.26$ cm and $z_y = 2.3$ cm in modified Fresnel approximation, and $\alpha_x = 0.601\pi/2$ and $\alpha_y = 0.408\pi/2$ in FRFT. A clear in-focus object image with sharp contrast to the background is observed in all reconstructed images.

Both the shape of the particles (in pixel) and the size of the images are different with asymmetric wavelet reconstruction, modified Fresnel approximation reconstruction, and FRFT. For modified Fresnel approximation reconstruction in Fig. 3(a), the particle image is a circular disk (in pixel). The shape of the in-focus image of a spherical particle reconstructed by the asymmetric wavelet method is elliptical, as shown in Fig. 3(b). For FRFT reconstruction in Fig. 3(c), the particle image looks like an ellipse (in pixel) with a smaller size than that in Fig. 3(b). This is caused by the different magnifications of the pixel size in the $x$ and $y$ directions. For modified Fresnel approximation reconstruction, the reconstructed image displays a field with pixel sizes $\delta_{x,\text{Fresnel}} = \lambda z_x/\sqrt{N\sigma_x}$, $\delta_{y,\text{Fresnel}} = \lambda z_y/\sqrt{N\sigma_y}$, and thus a circular object will be elliptical as a result of the modified Fresnel approximation reconstruction in the presence of astigmatism with $z_{x,\text{eq}} \neq z_{y,\text{eq}}$. As shown in Fig. 3(b), the particle in the ROI (region of interest) region of Fig. 3(a) is rescaled into an ellipse. The relation between the pixel size $\delta_{x,y,f}$ of the reconstructed image in the fractional order domain and that $\delta_{x,y}$ of the recorded hologram in the spatial domain is $\delta_{x,y,f} = \delta_x/\cos(\alpha_x)$, $\delta_{x,y,f} = \delta_y/\cos(\alpha_y)$. By taking the magnification of the numerical FRFT into account, the reconstructed image in the fractional order domain can be rescaled with its pixel size equaling that of the recorded hologram by 2D nearest neighbor interpolation. Figure 3(d) shows the ROI region [the region of the recorded hologram, in green in Fig. 3(c)] of the rescaled image field. Comparison between the rescaled ROI region of Figs. 3(a), 3(b), and 3(d) reveals that the rescaled particle image with modified Fresnel approximation and FRFT reconstruction is exactly the same as that with asymmetric wavelet reconstruction. Note that the particle is spherical, while its reconstructed image is elliptical, as shown in Figs. 3(b) and 3(d). The elliptical cross section in the reconstructed image is caused by the difference in the transverse magnifications in the $x$ and $y$ directions during the propagation of the elliptical Gaussian beam from the particle to the CCD. The magnification can be evaluated from the beam waist radius ratio at the particle and CCD plane [21]: $M_x = \alpha_xZ_{\text{CCD}}/\alpha_xZ_p$, $M_y = \alpha_yZ_{\text{CCD}}/\alpha_yZ_p$. The beam radius in the $x$ direction is $4.23$ mm at the CCD plane and $2.28$ mm at the plane of the particle, with a magnification of $1.85$. Meanwhile, in the $y$ direction, the beam radius maintains $3$ mm from the particle to the CCD, and it implies that no magnification occurs. The major axes in the $x$ direction of the elliptical particle image reconstructed with the asymmetric wavelet method, modified Fresnel approximation method, and FRFT are $264, 260$, and $256$ mm, with magnifications of $1.88, 1.85$, and $1.83$, respectively, all corresponding to approximately $140 \mu m$. The minor axes for the asymmetric wavelet and FRFT reconstruction are $140 \mu m$, and equal the original size of the particle. The results agree well with the value predicted by the theory. In summary, for the reconstructed image of a spherical particle hologram in the presence of astigmatism, both the image size and the pixel pitch are different in the two directions for the modified Fresnel approximation method and FRFT, while the asymmetric wavelet method maintains a constant pixel size and image resolution. Although the particle images reconstructed with the three methods might be different in pixel, they turn into the same ellipse by rescaling the pixel size, from which the real particle size can be evaluated by taking the magnifications in the two directions into account.

The asymmetric wavelet method was also validated using experimental holograms. The holograms were obtained with the typical experimental setup of digital inline holography [20]. A helium–neon circular Gaussian beam, with a wavelength of $632.8 \text{ nm}$, passed through a cylinder lens, and was transformed into an elliptical Gaussian beam. The focal length cylinder lens is $100 \text{ mm}$ along the $x$ direction and infinite along the $y$ direction. Then the elliptical Gaussian beam illuminated the coal particle field.
The hologram was recorded by the CCD, with a pixel number of 1024 × 1024 and a square pixel size of 7.4 μm. Figure 4(a) shows the experimental hologram of a nonspherical coal particle. Figure 4(b) shows the coal particle reconstructed with the asymmetric wavelet method, and the reconstructed image field with the modified Fresnel approximation method and FRFT are rescaled to have the same size and resolution as the recorded hologram, as shown in Figs. 4(c) and 4(d). The reconstructed coal particle image by the wavelet method is compared with the rescaled images reconstructed by FRFT and the modified Fresnel approximation method. Results show that the two methods reconstruct the same nonspherical particle shape, and therefore the proposed wavelet method can reconstruct the particle hologram illuminated by an elliptical Gaussian beam.

4. Conclusion
In conclusion, an asymmetric wavelet method was proposed to reconstruct the particle from the hologram illuminated by an elliptical, astigmatic Gaussian beam. Two scale factors are introduced to compensate for the astigmatism caused by the elliptical Gaussian beam. Simulated holograms of spherical particles and experimental holograms of nonspherical opaque coal particles were reconstructed by the proposed wavelet method, and the reconstructed image is compared with those from the modified Fresnel approximation method and FRFT for validation. This approach can enable the reconstructed image to have the same size and resolution as the recorded hologram, and thus the transverse position of the reconstructed particle does not change along the depth position and the reconstructed 3D field can be easily extended for automatic particle locating and sizing.

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