Stability analysis of an epidemic model with diffusion and stochastic perturbation

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ABSTRACT

In this paper, we investigate the stability of an epidemic model with diffusion and stochastic perturbation. We first show both the local and global stability of the endemic equilibrium of the deterministic epidemic model by analyzing corresponding characteristic equation and Lyapunov function. Second, for the corresponding reaction–diffusion epidemic model, we present the conditions of the globally asymptotical stability of the endemic equilibrium. And we carry out the analytical study for the stochastic model in details and find out the conditions for asymptotic stability of the endemic equilibrium in the mean sense. Furthermore, we perform a series of numerical simulations to illustrate our mathematical findings.

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1. Introduction

It is well known that epidemiology is the study of the spread of diseases with the objective to trace factors that are responsible for or contribute to their occurrence [1,2]. Mathematical models in epidemiology have been evolving in complexity and realism during the past 80 yr [3]. Most models for the transmission of infectious diseases descend from the classical susceptible-infective-removed (SIR) model of Kermack and McKendrick [4]. The dynamic behaviors of the SIR epidemic model and a lot of its extensions are well investigated by many scholars [5–13].

The spatial component of ecological interactions has been identified as an important factor in how ecological communities are functioning and shaped yet, understanding the role of space is challenging both theoretically and empirically [14,15]. There has been a growing awareness of the importance of including a spatial aspect when constructing realistic models of biological systems, with a consequent development of both approximate and mathematical rigorous methods of analysis [16]. From a biological perspective, individual organisms are distributed in space and typically interact with the physical environment and other organisms in their spatial neighborhood [17]. The diffusion of individuals may be connected with other things, such as searching for food, escaping high infection risks and so on [18]. In the first case, individuals tend to diffuse in the direction of lower density of population, where there are richer resources. In the second case, to avoid higher infections, individuals may move along the gradient of infectious individuals [19]. Keeping these in view, epidemic models are considered where the diffusion of individuals is influenced by intraspecific competition pressures and is affected by different classes [19,20].

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On the other hand, environmental fluctuations are important components in an ecosystem. Most natural phenomena do not follow strictly deterministic laws but rather oscillate randomly about some average value so that the deterministic equilibrium is no longer an absolutely fixed state. And there are benefits to be gained in using stochastic models because real environments are full of random perturbations (i.e., the effects of noise), which are uncertain and stochastic. In ecology, deterministic models do not usually incorporate environmental fluctuations based on the idea that for large populations, the effect of random environmental fluctuations are small enough to be ignored [21]. The basic mechanism and factors of population growth like resources and vital rates—birth, death, immigration and emigration, change non-deterministically due to continuous fluctuations in the environment (e.g. variation in intensity of sunlight, temperature, water level, etc.) [22,23]. Then the parameters which involved in population models characterizing real environments all are not absolute constants and exhibit random fluctuation to greater or lesser degree. Using a stochastic model one can build up a distribution of the predicted outcomes, for example, the number of infected cells at time t, whilst a deterministic model just gives a single predicted value. It can be seen that stochastic models produce more useful output than deterministic models [24]. In addition, from the points of epidemiological view, it is important to discover the properties of the stochastic system, such as global stability and stochastic permanence, and whether the presence of such noise affects some known results.

Recent advances in stochastic differential equations enable a lot of authors to introduce noise into the model of physical phenomena to study stochastic epidemic models, whether it is a random noise in the system of differential equations or environmental fluctuations in parameters [24–31]. Of them, in [25,26], Mao et al. shown that the noise cannot only have a destabilizing effect but can also have a stabilizing effect in the control theory, and obtained the conclusion that even a sufficiently small noise can suppress explosions in population dynamics. Dalal et al. [24] considered the stochasticity in the HIV model by the situation of parameter perturbation. In [29], Yu et al. proved the endemic equilibrium of the two-group SIR model by the random perturbation. Meng [30] presented the stability conditions of the disease-free equilibrium of the SIR model without stochastic perturbation and with stochastic perturbation. These results reveal the significant effect of the environmental noise on some epidemic models, because the stochastic models can provide some additional degree of realism compared to their deterministic counterparts [32].

However, to the best of our knowledge, the dynamics of the epidemic model with nonlinear incidence rate and diffusion or stochastic perturbation seems rare.

Based on the discussions above, we will focus on dynamical properties of the epidemic model with nonlinear incidence rate and diffusion, stochastic perturbation. The organization of this paper is as follows. In the next section, we introduce the deterministic epidemic model with nonlinear incidence rate and give a general survey of the stability analysis. Section 3 investigates the deterministic epidemic model with diffusion. In Section 4, the stochastic version of the epidemic model is presented and we study our main results on its stability. Finally, we give a concluding remark section.

2. The deterministic epidemic model

Considering a mathematical model of the spread of infectious diseases, we study a deterministic epidemic model with nonmonotonic incidence rate deriving from Xiao and Ruan [33], in which the population is divided into susceptible, infectious and recovered individuals with sizes $S(t)$, $I(t)$ and $R(t)$ at time $t$, respectively, the model takes the following form

$$
\begin{cases}
\frac{dS}{dt} = b - dS - \frac{kSI}{1+Ip} + \gamma R, \\
\frac{dI}{dt} = \frac{kSI}{1+Ip} - (d + \mu)I, \\
\frac{dR}{dt} = \mu I - (d + \gamma)R,
\end{cases}
$$

(1)

where $b$ is the recruitment rate of the population, $d$ the death rate of the population, $k$ the proportionality constant, the parameter $x$ a nonnegative constant, $\gamma$ the rate of removed individuals who lose immunity and return to susceptible class and $\mu$ the recovery rate of infective individuals. $\frac{k}{1+Ip}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals.

The limit set of model (1) is on the plane $S + I + R = \frac{b}{d}$, and it can be written as follows

$$
\begin{cases}
\frac{dI}{dt} = \frac{kI}{1+Ip}(\frac{b}{d} - I - R) - (d + \mu)I, \\
\frac{dR}{dt} = \mu I - (d + \gamma)R.
\end{cases}
$$

(2)

To be concise in notations, re-scale the model (2) by letting $x = \frac{k}{b+R}$, $y = \frac{I}{b+R}$, $\tau = (d + \gamma)t$. For simplicity, we still use variables $I$, $R$, $t$ instead of $x$, $y$, $\tau$. Then we obtain the following model

$$
\begin{cases}
\frac{dl}{dt} = \frac{1}{1+Ip} (A - I - R) - ml, \\
\frac{dR}{dt} = qI - R, \\
I(0) > 0, \quad R(0) > 0,
\end{cases}
$$

(3)

where $A = \frac{bk}{b+R}$, $m = \frac{d+\mu}{b+R}$, $p = \frac{q(x+1)}{x}$, $q = \frac{\mu}{b+R}$. For more details, refer to reference [33].
Substituting the expressions of $W$ in (11), we have a sufficient condition given for the local stability of $E^*$, and it is given by

$$I' = \frac{\sqrt{(1+q)^2 + 4mp(A-m) - (1+q)}}{2mp}, \quad R' = \frac{q(\sqrt{(1+q)^2 + 4mp(A-m) - (1+q)})}{2mp} = qI'.$$

**Theorem 2.1.** All the solutions of model (3) which are initiated in $B^2_+$ are uniformly bounded.

**Proof.** Summing up the two equations in model (3) and denoting $W(t) = I(t) + R(t)$, we have

$$\frac{dW}{dt} = \frac{I}{1 + pl^2}(A-I-R) - (m-q)I - R.$$

For each $\epsilon > 0$, the following inequality holds

$$\frac{dW}{dt} + \epsilon W = \frac{I}{1 + pl^2}(A-I-R) - (m-q-\epsilon)I - (1-\epsilon)R \leq \frac{1}{2\sqrt{p}}(A-W) - (m-q-\epsilon)I - (1-\epsilon)R.$$

Take $\epsilon = \min\{1, m-q, \frac{1}{\sqrt{p}}\}$, then the right-hand side of the above inequality is bounded. That is,

$$\frac{dW}{dt} + \epsilon W \leq \frac{1}{2\sqrt{p}}(A-W).$$

From the above inequation, we have $\frac{dW}{dt} \leq \frac{A}{2\sqrt{p}}(1 + e\sqrt{p} - \epsilon)W$. There exists a $T$, such that for $t > T$,

$$\limsup_{t \to \infty} W(t) = \limsup_{t \to \infty} (I(t) + R(t)) \leq \frac{A}{1 + 2e\sqrt{p}}.$$

We conclude the proof. \(\square\)

The Jacobian matrix corresponding to $E^* = (I^*, R^*)$ of model (3) is that

$$J = J(E^*) = \left(\begin{array}{c} pl^2 - \frac{p(1+q)}{(1+pl^2)^2} \frac{A-I^*}{q} - 1 \end{array} \right).$$

The characteristic equation is given by

$$\lambda^2 - \text{tr}(J)\lambda + \text{det}(J) = 0,$$

where

$$\text{tr}(J) = \frac{pl^2 - 2p(R^* - A)l^2 - I^*}{(1 + pl^2)^2} - 1 < 0,$$

$$\text{det}(J) = \frac{(1+q)I^* - 2ApI^2 - p(1+q)l^2}{(1 + pl^2)^2} > 0.$$

Then, we have a sufficient condition given for the local stability of $E^* = (I^*, R^*)$ of model (3) by Roth–Hurwitz criterion.

**Theorem 2.2.** The endemic equilibrium $E^* = (I^*, R^*)$ of model (3) is locally asymptotically stable.

For more details about Theorem 2.2, refer to Section 3 in [33].

**Theorem 2.3.** The endemic equilibrium $E^* = (I^*, R^*)$ of model (3) is globally asymptotically stable in the interior of the first octant.

**Proof.** Define a Lyapunov function

$$V_1(I, R) = \int_I^1 \frac{\zeta - I}{\Psi(I)} d\zeta + \frac{1}{q} \int_R^R (\xi - R) d\xi,$$

where $\Psi(I) = \frac{1}{1 + pl^2}$. We note that $V_1(I, R)$ is non-negative and $V_1(I, R) = 0$ if and only if $(I_t, R_t) = (I^*, R^*)$. Furthermore, the time derivative of $V_1$ along the solutions of model (3) is

$$\frac{dV_1}{dt} = \frac{1 - I^*}{\Psi(I)} \frac{dl}{dt}(R - R^*) + \frac{R - R^*}{q} (ql - R).$$

Substituting the expressions of $\frac{dl}{dt}$ and $\frac{dR}{dt}$ from model (3), we obtain

$$\frac{dV_1}{dt} = \frac{1 - I^*}{\Psi(I)} \left( \frac{I}{1 + pl^2}(A-I-R) - ml \right) + \frac{R - R^*}{q} (ql - R).$$

In this paper, we mainly focus on model (3) in the case of $A > m$, which has two nonnegative real equilibria, namely,

(i) a disease free equilibrium $E_0 = (0, 0)$, which is a hyperbolic saddle and corresponds to extinction of the epidemic.

(ii) an endemic equilibrium $E^* = (I^*, R^*)$, which corresponds to the coexistence of $I$ and $R$, and is given by

$$I' = \sqrt{(1+q)^2 + 4mp(A-m) - (1+q)}, \quad R' = \frac{q(\sqrt{(1+q)^2 + 4mp(A-m) - (1+q)})}{2mp} = qI'.$$
Using the fact that $I_1 + pI_2 (A - I^* - R^*) - ml^* = 0, qI^* - R^* = 0$, Eq. (5) can be re-written as

$$\frac{dI}{dt} = -(1 + mp(I + I^*))(I - I^*)^2 - \frac{1}{q}(R - R^*)^2.$$  

(6)

It is clear that $\frac{dV}{dt} < 0$, the equilibrium $E^*$ is globally asymptotically stable. \[\Box\]

When we choose the value of parameters $A = 4.15, m = 1.2, p = 0.2, q = 0.8$ for model (3) with initial value $(I_0, R_0) = (0.2, 1.8)$, the endemic equilibrium $(I^*, R^*) = (1.383630944, 1.106904755)$ exists. By Theorem 2.3, the equilibrium $E^* = (I^*, R^*)$ is globally asymptotically stable. The trajectory of the infectious individuals and the removed individuals population of model (3) is shown in Fig. 1. In addition, Fig. 1 shows the direction field for model (3). From the phase graph, we have that the endemic equilibrium would be stable and steady-state population densities would occur.

### 3. The epidemic model with diffusion

In this section, we assume that infectious individuals $I$ and recovered individuals $R$ move spatially randomly, the corresponding spatial model to (3) is as follows

$$\begin{align*}
\frac{\partial I}{\partial t} &= I_1(A - I - R) - ml + d_1 \nabla^2 I, \\
\frac{\partial R}{\partial t} &= qI + d_2 \nabla^2 R,
\end{align*}$$

(7)

where the nonnegative constants $d_1$ and $d_2$ are the diffusion coefficients of $I$ and $R$, respectively. $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian operator in two-dimensional space.

Model (7) is to be analyzed under the following non-zero initial condition and zero-flux boundary condition

$$\begin{align*}
I(r, 0) &= I_0(r) \geq 0, & R(r, 0) &= R_0(r) \geq 0, & r = (x, y) \in \Omega = [0, L] \times [0, L], \\
\frac{\partial I}{\partial n} - \frac{\partial R}{\partial n} &= 0, & (x, y) \in \partial \Omega.
\end{align*}$$

(8)

(9)

In the above, $n$ is the outward unit normal vector of the boundary $\partial \Omega$. Zero-flux boundary conditions imply that the model domain boundary is simply reflective, and that the domain is isolated or insulated from the external environment [34]. And zero-flux conditions imply that there is no fluxes of populations through the boundary, that is, no external input is imposed from outside [15,35].

Next, we investigate some relevant properties of the reaction–diffusion model (7).
Theorem 3.1. For any solution \((l(t), r(t))\) of model (7),
\[
\limsup_{t \to \infty} \max_{\bar{u}} l(t, \bar{u}) \leq A \left( \frac{1}{2m^2} \right), \quad \limsup_{t \to \infty} \max_{\bar{u}} r(t, \bar{u}) \leq Aq \left( \frac{1}{2m^2} \right).
\]
Thus, for any \(\varepsilon > 0\), the rectangle \([0, A \left( \frac{1}{2m^2} \right) + \varepsilon] \times [0, Aq \left( \frac{1}{2m^2} \right) + \varepsilon]\) is a global attractor of model (7) in the first octant.

Proof. Since \(I\) satisfies
\[
\begin{align*}
\frac{dl}{dt} - d_1 \nabla^2 l & \leq A \left( \frac{1}{2m^2} \right) - ml, \quad r \in \Omega, \quad t > 0, \\
\frac{dl}{dn} & = 0, \quad r \in \partial \Omega, \quad t > 0, \\
l(r, 0) & = l_0(r), \quad r \in \Omega.
\end{align*}
\] (10)
Let \(\chi(t)\) be a solution to the ordinary differential equation
\[
\dot{\chi} = \frac{A}{2m^2} - m \chi \quad (t \geq 0), \quad \chi(0) = \max_{\bar{u}} l(0, \bar{u}) > 0,
\]
then \(\lim_{t \to \infty} \chi(t) = \frac{A}{2m^2}\). From the comparison principle, one can get \(l(t, \bar{u}) \leq \chi(t)\), hence,
\[
\limsup_{t \to \infty} \max_{\bar{u}} l(t, \bar{u}) \leq A \left( \frac{1}{2m^2} \right).
\]
As a result, for any \(\varepsilon > 0\), there exists a \(T > 0\), such that \(l(t, \bar{u}) \leq A \left( \frac{1}{2m^2} \right) + \varepsilon\) for all \(r \in \Omega\) and \(t \geq T\).

Similarly, \(R\) satisfies
\[
\begin{align*}
\frac{dR}{dt} - d_2 \nabla^2 R & \leq q \left( A \left( \frac{1}{2m^2} \right) + \varepsilon \right) - R, \quad r \in \Omega, \quad t > 0, \\
\frac{dR}{dn} & = 0, \quad r \in \partial \Omega, \quad t > 0, \\
R(r, 0) & = R_0(r), \quad r \in \Omega.
\end{align*}
\] (11)
In the same way, from the arbitrariness of \(\varepsilon > 0\), we can get that
\[
\limsup_{t \to \infty} \max_{\bar{u}} R(t, \bar{u}) \leq Aq \left( \frac{1}{2m^2} \right).
\] (12)
This completes the proof. □

Next, we will analyze the local and global stability of the endemic equilibrium \(E^* = (I^*, R^*)\) of model (7). Let \(0 < \mu_1 < \mu_2 < \cdots\) be the eigenvalues of the operator \(-\nabla^2\) on \(\Omega\) with the homogeneous zero-flux boundary condition. Set
\[
Y = \left\{ (l, R) \in [C^1(\Omega)]^2 \mid \frac{dl}{dn} = \frac{dR}{dn} = 0 \text{ on } \partial \Omega \right\},
\]
and consider the decomposition \(Y = \bigoplus_{i=0}^{\infty} Y_i\), where \(Y_i\) is the eigenspace corresponding to \(\mu_i\).

Theorem 3.2. The endemic equilibrium \(E^* = (I^*, R^*)\) of model (7) is uniformly asymptotically stable.

Proof. The linearization of model (7) at endemic equilibrium \(E^*\) is
\[
\frac{\partial}{\partial \bar{u}} \left( \begin{array}{c} I \\ R \end{array} \right) = \Lambda \left( \begin{array}{c} I \\ R \end{array} \right) + \left( \begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array} \right),
\]
where \(L_j(z_1, z_2) = O(z_1^j + z_2^j)(j = 1, 2)\). We note \(L_{11} < 0\).

For each \(i (i = 0, 1, 2, \ldots)\), \(Y_i\) is invariant under the operator \(\Gamma\), and \(\lambda\) an eigenvalue of \(\Gamma\) on \(Y_i\), if and only if \(\lambda\) is an eigenvalue of the matrix
\[
H_i = \left( \begin{array}{cc} J_{11} - d_1 \mu_i & J_{12} \\ J_{21} & J_{22} - d_2 \mu_i \end{array} \right).
\] (13)
Note that
\[
\text{tr}(H_i) = -(d_1 + d_2)\mu_i + \text{tr}(J) < 0, \\
\det(H_i) = d_1d_2\mu_i^2 - (d_1J_{22} + d_2J_{11})\mu_i + \det(J) > 0.
\] (14)
Because \( \text{tr} (H_i) < 0 \) and \( \det (H_i) > 0 \), the two eigenvalues \( \lambda_i^+ \) and \( \lambda_i^- \) have negative real parts. For any \( i \geq 0 \), the following hold

(1) For \( i = 0 \), if \((\text{tr}(J))^2 - 4\det(J) \leq 0\),
\[
\text{Re}(\lambda_i^+) = \frac{1}{2} \text{tr}(J) < 0,
\]
and if \((\text{tr}(J))^2 - 4\det(J) > 0\),
\[
\text{Re}(\lambda_i^+) = \frac{\text{tr}(J) + \sqrt{(\text{tr}(J))^2 - 4\det(J)}}{2} < 0,
\]
\[
\text{Re}(\lambda_i^-) = \frac{\text{tr}(J) - \sqrt{(\text{tr}(J))^2 - 4\det(J)}}{2} < 0.
\]

(II) For \( i \geq 1 \), if \((\text{tr}(H_i))^2 - 4\det(H_i) \leq 0\), then
\[
\text{Re}(\lambda_i^+) = \frac{1}{2} \text{tr}(H_i) \leq \frac{1}{2} \text{tr}(J) < 0,
\]
and if \((\text{tr}(H_i))^2 - 4\det(H_i) > 0\), since \( \text{tr}(H_i) < 0 \) and \( \det(H_i) > 0 \),
\[
\text{Re}(\lambda_i^+) = \frac{\text{tr}(H_i) + \sqrt{(\text{tr}(H_i))^2 - 4\det(H_i)}}{2} \leq \frac{1}{2} \text{tr}(H_i) \leq \frac{1}{2} \text{tr}(J) < 0,
\]
\[
\text{Re}(\lambda_i^+) = \frac{\text{tr}(H_i) + \sqrt{(\text{tr}(H_i))^2 - 4\det(H_i)}}{2} = \frac{2\det(H_i)}{\text{tr}(H_i) - \sqrt{(\text{tr}(H_i))^2 - 4\det(H_i)}} \leq \frac{\det(H_i)}{c} < 0,
\]
for some positive \( c \) which is independent of \( i \).

This shows that there exists a positive constant \( C \), which is independent of \( i \), such that \( \text{Re}(\lambda_i^+) < C \) for all \( i \). Hence, the spectrum of \( C \) which consists of eigenvalues lies on \( (\text{Re}(\lambda) < C) \). Referring to [36], we have that the endemic equilibrium \( E^* \) of model (7) is uniformly asymptotically stable. The proof is complete. \( \square \)

In the following, we consider the global stability behavior of the endemic equilibrium \( E^* = (I^*, R^*) \) of model (7). The statement of global stability of the endemic equilibrium means that, however quickly or slowly the two individuals diffuse, the disease will spatially homogeneously exists over time.

**Theorem 3.3.** Suppose that \( (9A^2p + 6A\sqrt{b}(1+q))^{-1} > I > 2(mp(A-m))^2 \) holds, then the endemic equilibrium \( E^* = (I^*, R^*) \) of model (7) is globally asymptotically stable.

**Proof.** We select a Lyapunov function for model (7)
\[
V_2(t) = \int_{\Omega} V_1(I, R) \, dA,
\]
where \( V_1(I, R) \) is the same as that defined in (4). So, differentiating \( V_2(t) \) with respect to time \( t \) along the solutions of model (7), we obtain
\[
\frac{dV_2(t)}{dt} = \int_{\Omega} \frac{dV_1}{dt} \, dA + \int_{\Omega} \left( \frac{\partial V_1}{\partial I} \frac{\partial I}{\partial t} + \frac{\partial V_1}{\partial R} \frac{\partial R}{\partial t} \right) \, dA.
\]
Using Green's first identity and considering zero-flux boundary condition (9), we can show that
\[
\frac{dV_2(t)}{dt} = \int_{\Omega} \frac{dV_1}{dt} \, dA - \int_{\Omega} \left( \frac{\partial^2 V_1}{\partial I^2} \left( \frac{\partial I}{\partial x} \right)^2 + \left( \frac{\partial I}{\partial y} \right)^2 \right. \, dA - d_2 \int_{\Omega} \frac{\partial^2 V_1}{\partial R^2} \left( \frac{\partial R}{\partial x} \right)^2 + \left( \frac{\partial R}{\partial y} \right)^2 \, dA
\]
\[
= \int_{\Omega} \frac{dV_1}{dt} \, dA - d_1 \int_{\Omega} \left( 2pI^3 - pI^2 R + I^r \right) \left( \frac{\partial I}{\partial x} \right)^2 + \left( \frac{\partial I}{\partial y} \right)^2 \, dA - d_2 \int_{\Omega} \frac{1}{q} \left( \frac{\partial R}{\partial x} \right)^2 + \left( \frac{\partial R}{\partial y} \right)^2 \, dA.
\]
Define \( \Phi(I) = 2pI^3 - pI^2 R + I^r \), then we can obtain \( \Phi(0) = I^r > 0 \) and \( \Phi'(I) = 2pI(3I - I^r) \). If \( \Phi'(I) > 0 \), we have \( I > \xi \). Based on **Theorem 3.1**, we know \( 0 < I < \frac{A}{mp} \) and note that
\[
I^r = \frac{\sqrt{(1+q)^2 + 4mp(A-m)} - (1+q)}{2mp},
\]
hence, when $9A^2p + 6A\sqrt{p}(1 + q) > I^2 > 4mp(A - m)$, we have
\[
\frac{dV_2(t)}{dt} \leq \int \int \frac{dV_1}{dt} dA \leq 0.
\]
Thus, the endemic equilibrium $E^* = (l^*, R^*)$ of model (7) is globally asymptotically stable. The whole proof of Theorem 3.3 is complete. □

4. The epidemic model with stochastic perturbation

The above discussion rests on the assumption that the environmental parameters involved with the model system are all constants irrespective of time and environmental fluctuations. In reality all such parameters exhibit random variations to a greater or lesser extent. In this section, we consider the corresponding stochastic model of the deterministic model (3) by introducing white noise. Motivated by [21], we assume that the white noise type is directly proportional to the distances $h_i$, $h_j$ of any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition [37]. To show that model (18) has a positive global solution, let us firstly prove that the model has a positive local solution by making the change of variables [38].

Proof. Set $u(t) = \ln I(t)$, $v(t) = \ln R(t)$ and we consider the equations
\[
\begin{align*}
  du & = \left[\frac{4e^{uv(t)} - m - \sigma_1^2}{2} \left(1 - e^{uv(t)}\right)\right] dt + \sigma_1 \left(1 - e^{uv(t)}\right) dB_1(t), \\
  dv & = \left[\frac{\sigma_2^2}{2} \left(1 - e^{uv(t)}\right)\right] dt + \sigma_2 \left(1 - e^{uv(t)}\right) dB_2(t), \\
  u(0) & = \ln I_0, \quad v(0) = \ln R_0,
\end{align*}
\]
on $t > 0$. Then we know that the coefficients of (21) satisfy the local Lipschitz condition, and there is a unique local solution $(u(t), v(t))$ on $[0, \tau_e)$. By Itô’s formula, we can see that $I(t) = e^{u(t)}, R(t) = e^{v(t)}$ is the unique positive local solution to (18) with initial value $I_0 > 0, R_0 > 0$. □

Next, we will show this solution is global, i.e., $\tau_e = \infty$.

Theorem 4.1. There is a unique local solution $(l(t), R(t))$ for $t \in [0, \tau_e)$ to model (18) almost surely for the initial value $I_0 > 0, R_0 > 0$, where $\tau_e$ is the explosion time.

Proof. For model (18) and any given initial value $(I_0, R_0) \in \mathbb{R}_+^2$, there is a unique solution $(l(t), R(t))$ on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^2$ with probability one.

Proof. Referring to the work of Luo and Mao [39], let $n_0 > 0$ be sufficiently large for $l_0$ and $R_0$ lying within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, define the stopping times
\[
\tau_n = \inf \left\{ t \in [0, \tau_e) : \; l(t) \neq \frac{1}{n}, n \quad \text{or} \quad R(t) \neq \frac{1}{n}, n \right\},
\]
where we set $\inf \emptyset = \infty$ ($\emptyset$ represents the empty set) in this paper. $\tau_n$ is increasing as $n \to +\infty$. □
Let
\[ \tau_\infty = \lim_{n \to \infty} \tau_n, \]
then \( \tau_\infty \leq \tau_\infty \) a.s. In the following, we need to show \( \tau_\infty = \infty \) a.s. If this statement is violated, there exist constants \( T > 0 \) and \( \epsilon \in \langle 0, 1 \rangle \) such that
\[ \mathbb{P}\{\tau_\infty \leq T\} > \epsilon. \]
Then, there is an integer \( n_1 \geq n_0 \) such that
\[ \mathbb{P}\{\tau_n \leq T\} \geq \epsilon, \quad n \geq n_1. \tag{22} \]
Define a \( C^2 \)-function \( V_3 : \mathbb{R}^+_0 \to \mathbb{R}_+ \) by
\[ V_3(I, R) = I + R - (\ln I + \ln R) - 2, \tag{23} \]
which is a non-negativity function. If \( I(t), R(t) \in \mathbb{R}_+^2 \), by the Itô’s formula, we compute
\[
dV_3(I, R) = \left[ \frac{\partial V_3(Z(t))}{\partial t} + \frac{\partial V_3(Z(t))}{\partial Z(t)} F(Z(t)) + \frac{1}{2} \text{tr}\left( G^T(Z(t)) \frac{\partial^2 V_3(Z(t))}{\partial Z(t)^2} G(Z(t)) \right) \right] dt + \frac{\partial V_3(Z(t))}{\partial Z(t)} G(Z(t)) dB(t)
\]
\[ = \left[ \frac{(I-1)(A-I-R)}{1 + pl^2} + (q-m)I - R - \frac{ql}{R} + m + 1 + \frac{\sigma_1^2(I-I')^2}{2l^2} + \frac{\sigma_2^2(R-R')^2}{2R^2} \right] dt + \sigma_1 \left( \frac{1}{I} \right) (I-I') dB_1(t) + \sigma_2 \left( \frac{1}{R} \right) (R-R') dB_2(t) \]
\[ \leq \left[ \frac{(A+1)I - A}{1 + pl^2} + (q-m)I + m + 1 + \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{\sigma_1^2 I^2}{2l^2} + \frac{\sigma_2^2 R^2}{2R^2} \right] dt + \sigma_1 \left( \frac{1}{I} \right) (I-I') dB_1(t) + \sigma_2 \left( \frac{1}{R} \right) (R-R') dB_2(t), \tag{24} \]
where \( M \) is a positive constant. Integrating both sides of the above inequality from 0 to \( \tau_n \cap T \) and then taking the expectations leads to
\[ \mathbb{E} V_3(I(\tau_n \cap T), R(\tau_n \cap T)) \leq V_3(I_0, R_0) + MT. \tag{25} \]
Set \( \Omega_n = \{ \tau_n \leq T \} \) and by inequality (22), we get \( \mathbb{P}(\Omega_n) \geq \epsilon \). For every \( \sigma \in \Omega_n, I(\tau_n, \sigma) \) and \( R(\tau_n, \sigma) \) equal either \( n \) or \( \frac{1}{n} \), hence
\[ V_3(I(\tau_n, \sigma), R(\tau_n, \sigma)) \geq \min \left\{ \left( \sqrt{n} - \ln \sqrt{n} - 1 \right), \left( \sqrt{n} - \ln \sqrt{n} - 1 \right) \right\}. \]
Then we obtain,
\[ V_3(I_0, R_0) + MT \geq \mathbb{E} \left[ 1_{\Omega_n}(\sigma) V_3(I(\tau_n), R(\tau_n)) \right] \geq \epsilon \min \left\{ \left( \sqrt{n} - \ln \sqrt{n} - 1 \right), \left( \sqrt{n} - \ln \sqrt{n} - 1 \right) \right\}, \]
where \( 1_{\Omega_n} \) is the indicator function of \( \Omega_n \). Letting \( n \to +\infty \) leads to the contradiction \( \infty > V_3(I_0, R_0) + MT > \infty \). This completes the proof. \( \square \)

**Theorem 4.3.** The solutions of model (18) are stochastically ultimately bounded for any initial value \( (I_0, R_0) \in \mathbb{R}_+^2 \).

**Proof.** The solution will remain in \( \mathbb{R}_+^2 \) for all \( t \geq 0 \) with probability one from Theorem 4.2. Set the function \( V_4 = e^{\theta t} \) for \( (I, R) \in \mathbb{R}_+^2 \) and \( \theta > 0 \). By the Itô’s formula we obtain
\[
dV_4 = e^{\theta t} \left[ 1 + \theta \left( \frac{A-I-R}{1 + pl^2} - m \right) + \frac{\sigma_1^2 \theta (\theta - 1)}{2} \left( 1 - \frac{l}{T} \right)^2 \right] dt + \sigma_1 \theta e^{\theta t} \left( 1 - \frac{l}{T} \right) dB_1 \leq C_1 e^{\theta t} dt + \sigma_1 \theta e^{\theta t} \left( 1 - \frac{l}{T} \right) dB_1, \]
then \( e^{\theta t} E_0 \leq C_1 e^{\theta t}. \) So we get
\[ \lim_{t \to \infty} \sup_t E_0 \leq C_1 < +\infty. \]
Define \( V_5 = e^{\theta t} R \), then
\[
dV_5 = e^{\theta t} \left[ 1 - \theta \left( \frac{1-ql}{R} \right) + \frac{\sigma_2^2 \theta (\theta - 1)}{2} \left( 1 - \frac{R}{R} \right)^2 \right] dt + \sigma_2 \theta e^{\theta t} \left( 1 - \frac{R}{R} \right) dB_2 \leq C_2 e^{\theta t} dt + \sigma_2 \theta e^{\theta t} \left( 1 - \frac{R}{R} \right) dB_2, \]
we also have
\[ \limsup_{t \to \infty} ER^0 \leq C_2 < +\infty. \]

Therefore, we obtain
\[ \limsup_{t \to \infty} E(I(t)^2 + R(t))^2 \leq 2^2(C_1 + C_2) < +\infty. \]

We could have the required assertion by taking the Chebyshev inequality [37].

Following the definition of stochastic permanence of [37], the property of permanence means the long time survival in a population dynamics, then we give the following theorem.

**Theorem 4.4.** For any initial value \((I_0, R_0) \in \mathbb{R}^2_+\), the solution \((I(t), R(t))\) satisfies that

\[
\limsup_{t \to \infty} E \left[ \frac{1}{I^2 + R^2} \right] \leq \frac{2^2 C}{k},
\]

where \(C\) is a positive constant and \(\rho, k\) are arbitrary positive constants satisfying

\[
\rho \min \left\{ \frac{A \sqrt{p} - 2}{2p}, q \right\} > \frac{\rho (\rho + 1)}{2} \left( \max \{\sigma_1, \sigma_2\} \right)^2 + k. \tag{26}
\]

**Proof.** Referring to the method of [40]. Set a function \(V_6(I, R) = \frac{1}{1 + p I^2} \) for \((I(t), R(t)) \in \mathbb{R}^2_+\), by the Ito’s formula, we have

\[
dV_6 = -V_6 \left[ \frac{I(A - I - R)}{1 + p I^2} + (q - m)I - R \right] dt + V_6^3 \left[ \sigma_1^2(I - I')^2 + \sigma_2^2(R - R')^2 \right] dt - V_6^2 \left[ \sigma_1^2(I - I') + \sigma_2^2(R - R') \right] dB_1 + \sigma_2(R - R') dB_2.
\]

Then we choose a positive constant \(\rho\) and use the Ito’s formula, we can get

\[
L(1 + V_6)^\rho = \rho(1 + V_6)^{\rho - 1} \left[ -V_6 \left( \frac{I(A - I - R)}{1 + p I^2} + (q - m)I - R \right) + V_6^3 \left( \sigma_1^2(I - I')^2 + \sigma_2^2(R - R')^2 \right) \right] + \frac{\rho (\rho - 1)}{2} V_6^2 \left( \sigma_1^2(I - I')^2 + \sigma_2^2(R - R')^2 \right).
\]

Let \(k > 0\) sufficiently small such that it satisfies (26), by the Ito’s formula, then

\[
Le^{k(I + V_6)^\rho} = e^{k} \left[ k(1 + V_6)^2 + \rho(1 + V_6) \left( -V_6 \left( \frac{I(A - I - R)}{1 + p I^2} + (q - m)I - R \right) + V_6^3 \left( \sigma_1^2(I - I')^2 + \sigma_2^2(R - R')^2 \right) \right) \right] + \frac{\rho (\rho - 1)}{2} V_6^2 \left( \sigma_1^2(I - I')^2 + \sigma_2^2(R - R')^2 \right).
\]

Motivated by the reference [40] and the following inequalities

\[
V_6^2 \left( \sigma_1^2 I^2 + \sigma_2^2 R^2 \right) \leq \left( \max \{\sigma_1, \sigma_2\} \right)^2 V_6, \quad V_6^2 \left( \sigma_1^2 I^2 + \sigma_2^2 R^2 \right) \leq \left( \max \{\sigma_1, \sigma_2\} \right)^2 V_6.
\]

Therefore, we obtain

\[
Le^{k(I + V_6)^\rho} \leq e^{k} \left[ k + \rho \max \left\{ m, 1 + \frac{1}{2 \sqrt{p}} \right\} + \left( 2k + \rho \max \left\{ m, 1 + \frac{1}{2 \sqrt{p}} \right\} - \rho \min \left\{ \frac{A \sqrt{p} - 2}{2p}, q \right\} \right) \right] V_6^2 - \left( \rho \min \left\{ \frac{A \sqrt{p} - 2}{2p}, q \right\} - \frac{\rho (\rho + 1)}{2} \left( \max \{\sigma_1, \sigma_2\} \right)^2 \right) V_6^2.
\]

There exists a positive constant \(C\) such that \(L e^{k(I + V_6)^\rho} \leq Ce^{k}\). Then

\[
E[e^{k(I + V_6)^\rho}] \leq (1 + V_6(0)) e^{C} \frac{k}{k} e^{k}.
\]

So we can have

\[
\limsup_{t \to \infty} E V_6^2(t) \leq \limsup_{t \to \infty} E (1 + V_6(t))^\rho \leq \frac{C}{k}.
\]
In addition, we know that \((I + R)^p \leq 2^p (I^2 + R^2)^{\frac{p}{2}}\), consequently,

\[
\limsup_{t \to \infty} E \left[ \frac{1}{(I^2 + R^2)^2} \right] \leq 2^p \limsup_{t \to \infty} EV_0^p(t) \leq \frac{2^p C}{k}.
\]

The proof is complete. □

Based on the results of Theorems 4.3 and 4.4, and the Chebyshev inequality, we can get.

**Theorem 4.5.** Assume that \(\frac{1}{2} (\max(\sigma_1, \sigma_2))^2 < \min \left\{ \frac{A(\beta-2)}{2p}, q \right\} \), the solutions of model (18) is stochastically permanently.

Next let us center model (18) on the endemic equilibrium \(E^* = (I^*, R^*)\) by the change of variables \(X = I - I^*, Y = R - R^*\), then the linearized stochastic model around \(E^*\) as following

\[
d\hat{Z}(t) = \hat{F}(\hat{Z}(t))dt + \hat{G}(\hat{Z}(t))dB(t),
\]

where

\[
\hat{Z}(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} J_{11}X + J_{12}Y \\ qX - Y \end{pmatrix}, \quad \hat{G} = \begin{pmatrix} \sigma_1 X & 0 \\ 0 & \sigma_2 Y \end{pmatrix}.
\]

We know that in (27) the endemic equilibrium \(E^*\) corresponds to the trivial solution \((X(t), Y(t)) = (0, 0)\).

**Theorem 4.6** [41]. Suppose there is a function \(V(\hat{Z}, t) \in C^2(\Omega)\) satisfying the inequalities

\[
\begin{align*}
\phi_i \hat{Z}^\alpha &\leq V(\hat{Z}, t) \leq \phi_i |\hat{Z}|^\alpha, \\
LV(\hat{Z}, t) &\leq -\phi_i |\hat{Z}|^\alpha, \quad \phi_i > 0 \quad (i = 1, 2, 3), \quad \omega > 0,
\end{align*}
\]

then the zero solution of model (27) is exponentially \(\omega\)-stable for all time \(t \geq 0\).

Then we can get.

**Theorem 4.7.** Assume that

\[
\Theta = \frac{I}{q(1 + pr^2)}, \quad \sigma_1^2 < \frac{2I(1 + 2p(A - R^*) X^2 - pr^2)}{(1 + pr^2)^2}, \quad \sigma_2^2 < 2
\]

hold, then the zero solution of model (27) is asymptotically mean square stable.

**Proof.** Consider a Lyapunov function

\[
V_3(\hat{Z}(t)) = \frac{1}{2} (X^2 + \Theta Y^2),
\]

where \(\Theta\) is a positive real constant to be chosen later. It can be check that the first inequality of (29) holds true with \(\omega = 2\).

Moreover,

\[
LV_3(\hat{Z}(t)) = X(J_{11}X + J_{12}Y) + \Theta Y(qX - Y) + \frac{1}{2} \left( \sigma_1^2 X^2 + \Theta \sigma_2^2 Y^2 \right) = (J_{11} + \frac{\sigma_1^2}{2}) X^2 + (\Theta q + J_{12}) XY - \Theta (1 - \frac{\sigma_2^2}{2}) Y^2.
\]

If we let \(\Theta = -\frac{\nu_1}{q} = \frac{r}{q(1 + pr^2)} > 0\), then we have

\[
LV_3(\hat{Z}(t)) = \left( J_{11} + \frac{\sigma_1^2}{2} \right) X^2 - \frac{J_{12}}{q} \left( 1 - \frac{\sigma_2^2}{2} \right) Y^2 = -\hat{Z}^T Q \hat{Z} < 0,
\]

where \(Q = \text{diag} \left\{ -J_{11} + \frac{\sigma_1^2}{2}, \frac{\nu_1}{q} \left( 1 - \frac{\sigma_2^2}{2} \right) \right\} \), and its eigenvalues \(\lambda_i (i = 1, 2)\) are positive real constants if and only if the following conditions hold

\[
\sigma_1^2 < 2J_{11}, \quad \sigma_2^2 < 2.
\]

Set \(\lambda_{\min} = \min(\lambda_1, \lambda_2)\), then from Eq. (30) we get the result

\[
LV_3(\hat{Z}(t)) \leq -\lambda_{\min} |\hat{Z}(t)|^2.
\]

Hence, the theorem holds. □

Next, we perform some numerical simulations to illustrate the above analytical findings and the parameters are taken as \(A = 4.15, m = 1.2, p = 0.2, q = 0.8\) for model (18). In Fig. 2, we show time-series plots of the two species population \(I\) and \(R\) of model (18) with initial value \((I_0, R_0) = (0.2, 1.8)\) and different noise intensity \(\sigma_1^2, \sigma_2^2\). From Fig. 2 (a), without noise (i.e., \(\sigma_1^2 = \sigma_2^2 = 0\)), the endemic equilibrium \((I^*, R^*) = (1.383630944, 1.106904755)\) is globally stable of model (3), and it shows
the stable population distribution of the both species. When we choose noise intensity $r_1^2 = r_2^2 = 1.0$ (Fig. 2 (b)) and $r_1^2 = 2.25; r_2^2 = 1.96$ (Fig. 2 (c)), which satisfy conditions of Theorem 4.7, for model (18), one may observe that the endemic equilibrium is stochastic asymptotically stable, which mean that stochastic perturbations do not change the permanence of the deterministic model. In these cases, we get a stable population distribution. Comparing Fig. 2 (b) with (c), we can see that as the intensity of noise increasing, the two species population give rise to drastically ruleless oscillations. In Fig. 2 (d), when we choose $r_1^2 = r_2^2 = 4.0$ which violate conditions of Theorem 4.7, we can see that the population $I$ and $R$ will go to extinction. In other words, the endemic equilibrium $(I^*, R^*)$ is no longer globally stable if conditions of Theorem 4.7 are not satisfied. By comparing Fig. 2 (a) with (b) and (c), we can observe that if the endemic equilibrium of the deterministic model is globally stable, then the stochastic model will keep stochastic asymptotically stable when the noise satisfies conditions of Theorem 4.7. However, for Fig. 2 (d), when the intensity of environmental noise is getting sufficiently large such that the two species of the stochastic model die out.

5. Conclusions and remarks

In this paper, we propose an epidemic model with a nonlinear incidence rate of the form $\frac{S}{1+IP}$, which is increasing when $I$ is small and decreasing when $I$ is large, see Xiao and Ruan [33]. We extend to consider and analyze the epidemic model with diffusion or stochastic perturbation. The value of this study lies in three aspects. First, it presents local and global stability analysis of the endemic equilibrium for the deterministic epidemic model (3). Second, it shows that diffusion affect the stability of model (7). Third, it verifies some relevant properties of the corresponding stochastic model (18) and reveals the effect of environmental noise on the epidemic model.
The epidemic model with diffusion \((7)\) is introduced in a general form so that it has broad applications to a range of interactions populations. In the present manuscript, we adopt the Laplacian operator \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) to approximate the diffusive process, i.e., the diffusion of the species \(I\) and \(R\) is random in \(xy\)-plane. By qualitative analysis, we show that the endemic equilibrium \(E^e = (I^e, R^e)\) of model \((7)\) is globally asymptotically stable if it is globally asymptotically stable in the absence of diffusion.

Further to study the effect of environmental noise on deterministic model \((3)\), we stochastically perturb model \((3)\) with respect to white noise around its endemic equilibrium. To our knowledge, this model in such a stochastic setting appears to be rare. Using Lyapunov analysis method of Ref.\([37,38,40]\), we show there exits a positive and global solution of model \((18)\). By the Itô's formula, we derive that the solution is stochastically bounded and under a assumption, it is stochastically permanent. In addition, the stochastic model is stochastic asymptotically stable in the mean square sense. The conditions of stochastically asymptotic stability depend on the intensity of environmental noise \(\sigma_1^2\) and \(\sigma_2^2\). When \(\sigma_1^2 < \frac{1}{2} \left( \frac{1}{1 + \rho} - 1 \right) \left( \frac{1}{1 + \rho} - \frac{1}{1 + \rho - \rho^2} \right)\) and \(\sigma_2^2 < 2\) do not hold, large white noise will force stochastic model \((18)\) to become extinct while it may be persistent under small white noise. And our complete analysis of the modified epidemic model will give new suggestion to other models.

What’s more, from the paper of Jiang [31], we know that they only paid attention to the stability of solution around \(E^e\) and did not care whether the solution is positive, due to the stochastic model which stochastic perturbations of the white noise directly fluctuate around the endemic equilibrium, does not always have nonnegative solutions. For more details, we refer to the reader the last of Section 3 in [31].

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