Existence and multiplicity of solutions for asymptotically linear nonperiodic Hamiltonian elliptic system

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1. Introduction and main results

The goal of this paper is to establish the existence and multiplicity of solutions to the nonperiodic Hamiltonian elliptic system

\[
\begin{align*}
-\Delta u + V(x)u &= H_u(x, u, v) \\
-\Delta v + V(x)v &= H_v(x, u, v) \\
u(x), v(x) &\rightarrow 0 \text{ as } |x| \rightarrow \infty,
\end{align*}
\]

where \(z = (u, v) : \mathbb{R}^N \rightarrow \mathbb{R} \times \mathbb{R}, N \geq 3\), \(V \in C(\mathbb{R}^N, \mathbb{R})\) and \(H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})\).

For the case of a bounded domain these systems were studied by a number of authors. For instance, see [1–6] and the references therein. Problems similar to (ES) in the whole space \(\mathbb{R}^N\) was considered recently in some works. For instance, see [7–18] and the references therein. Most of them focused on the case \(V \equiv 1\). The main difficulty of this problem is the lack of compactness for Sobolev’s embedding theorem. A usual way to overcome this difficulty is working on the radially symmetric function space which possesses compact embedding. In [9], De Figueiredo and Yang considered the system

\[
\begin{align*}
-\Delta u + u &= g(x, u) \\
-\Delta v + v &= f(x, u)
\end{align*}
\]

They proved the system (1.1) has a strong radial solution pair under the assumptions that \(f(x, t)\) and \(g(x, t)\) are superlinear in \(t\) and radially symmetric with respect to \(x\), \(|f(x, t)| \leq c(1 + |t|^p)\) and \(|g(x, t)| \leq c(1 + |t|^q)\) with \(1 \leq p, q < \frac{N+2}{N-2}, N > 2\).
This result was later generalized by Sirakov [15] to the system
\begin{align}
-\Delta u + b(x)u &= g(x, u) \\
-\Delta v + b(x)v &= f(x, v)
\end{align}
\tag{1.2}
with \(f, g\) satisfying the above growth condition and \(p, q\) satisfying \(\frac{1}{p-1} + \frac{1}{q-1} > \frac{N-2}{N}\). Bartsch and De Figueiredo [11] proved that the system admits infinitely many radial as well as non-radial solutions. In [13], by using the dual variational method, Alves, Carrião and Miyagaki proved the existence of positive solutions for
\begin{align}
-\Delta u + u &= W_1(x)|u|^{p-1}u \\
-\Delta v + v &= W_2(x)|u|^{q-1}u
\end{align}
\tag{1.3}
with asymptotically periodic nonlinearities and \(p, q\) satisfying \(\frac{1}{p-1} + \frac{1}{q-1} > \frac{N-2}{N}\). The same method was also used in [14,16]. Recently, Bartsch and Ding [19] developed a generalized linking theorem for the strongly indefinite functional (also see [20,21]), which provided another way to deal with such problems. By applying the theorems of [19], Zhao et al. [7] considered the periodic asymptotically linear elliptic system
\begin{align}
-\Delta \varphi + V(x)\varphi &= G_\varphi(x, \varphi, \psi) \\
-\Delta \psi + V(x)\psi &= G_\psi(x, \varphi, \psi)
\end{align}
\tag{1.4}
where the potential \(V\) is periodic and 0 lies in a gap of \(\sigma(\Delta + V)\), \(G(x, \varphi, \psi)\) is periodic in \(x\) and asymptotically quadratic in \((\varphi, \psi)\), they obtained the infinitely many geometrically distinct solutions. In [12], Zhao et al. considered the following system
\begin{align}
-\Delta u + V(x)u &= g(x, u) \\
-\Delta v + V(x)v &= f(x, v)
\end{align}
\tag{1.5}
where the potential \(V\) is periodic and has a positive bound from below, \(f(x, t)\) and \(g(x, t)\) are periodic in \(x\), asymptotically linear in \(t\) as \(|t| \to \infty\). Existence of a positive ground state solution as well as infinitely many geometrically distinct solutions for odd \(f\) and \(g\) are obtained. For the nonperiodic case, we refer readers to [8,17,18]. Moreover, Zhao and Ding [22] also considered following periodic or non-periodic diffusion system
\begin{align}
-\Delta u + b(x) \cdot \nabla u + V(x)u &= H_u(x, u, v) \\
-\Delta v + b(x) \cdot \nabla v + V(x)v &= H_v(x, u, v)
\end{align}
\tag{1.6}
where \(b = (b_1, \ldots, b_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N), V \in C(\mathbb{R}^N, \mathbb{R})\) and \(H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})\). For the potential \(V\), they assumed

\((V^*)\) \(V \in C(\mathbb{R}^N, \mathbb{R}), V(x) \geq V_0 > 0\), and there exists some \(M > 0\) such that the set \(\Omega_M := \{x \in \mathbb{R}^N | V(x) \leq M\}\) is nonempty and has finite Lebesgue measure.

Existence and multiplicity of solutions via critical point theory. In this paper, we consider the more general case and weaken the condition of \((V^*)\). We assume

\((V_0)\) \(V \in C(\mathbb{R}^N, \mathbb{R}), \) and there exists some \(M > 0\) such that the set \(\Omega_M := \{x \in \mathbb{R}^N | V(x) \leq M\}\) is nonempty and has finite Lebesgue measure.

To our knowledge, for sign-changing potential and nonperiodic asymptotically linear case, there are no previous results. Inspired by recent works of Ding and L. Jeanjean [23], we are going to consider this case. By using the critical point theory of strongly indefinite functionals, the existence and multiplicity of solutions are obtained. However, the main difficulty of this problem is the lack of compactness of Sobolev’s embedding theorem, since our domain is the whole space. In order to overcome this difficulty, we assume that \(|H_2(x, z)|/|z|\) does not interfere with the essential spectrum of \(S\) (\(S\) will be defined later) for large \(|x|\) (see condition \((H_d)\)). We would mention here a similar work for Schrödinger equations in Liu et al. [24].

**Remark 1.1.** The assumption \((V_0)\) implies that the potential \(V\) is not periodic and changes sign. Here we say that \(V\) changes sign if \(V(x_1) < 0 < V(x_2)\) for some \(x_1, x_2 \in \mathbb{R}^N\).

It is well known that, under \((V_0)\), the Schrödinger operator \(S := -\Delta + V\) is selfadjoint and semibounded on \(L^2(\mathbb{R}^N)\). Moreover, 0 may be a spectrum of \(S\). We denote by \(\sigma(S), \sigma_e(S), \sigma_p(S)\) the spectrum of \(S\), the essential spectrum of \(S\) and the pure point spectrum of \(S\), respectively.

For the nonlinearity, we assume

\((H_0)\) \(H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0, \infty))\);

\((H_1)\) \(H(x, z) = o(|z|^2)\) as \(|z| \to 0\);

\((H_2)\) \(|H_2(x, z) - H_\infty(x)|/|z| \to 0\) as \(|z| \to \infty\), where \(H_\infty(x)\) is a bounded, continuous real-valued function;

\((H_3)\) \(F_0 := \inf_{x, z \in \mathbb{R}^N} H_\infty(x) > a := \inf\{0, \infty\} \cap \sigma(S)\);

\((H_4)\) \(H := \lim_{|x| \to \infty} \sup_{z \neq 0} \inf_{x \in \mathbb{R}^N} \frac{|H_2(x, z)|}{|z|} < M_0\), where \(M_0 := \sup\{M > 0 | \Omega_M < \infty\}, |\Omega_M|\) denotes the Lebesgue measure of the set \(\Omega_M\).
Clearly, we have $\sigma(S) \not\in \sigma(S - H_{\infty})$.

Lemma 2.1. Even in $z$, then $(ES)$ has at least $k$ pairs of solutions.

Proof. Let $M$ which is a contradiction. Now, the desired result follows from the arbitrariness of $\sigma_e(S \cap (-M, M))$. Then there exists $\{u_n\} \subset D(S)$ with $|u_n|_2 = 1$ such that $u_n \to 0$ and $(S - \mu)u_n \to 0$ in $L^2(\mathbb{R}^N)$. By $(V_0)$, similar to Bartsch et al. [25], one can check that the multiplication operator $u \mapsto W^0u$ is compact, then $W^0u_n \to 0$ in $L^2(\mathbb{R}^N)$ since $u_n \to 0$. We get

$$o(1) = ((S - \mu)u_n, u_n)_2 = ((-\Delta + W^+ + M)u_n, u_n)_2 - \mu(u_n, u_n)_2 + o(1) \ge M|u_n|^2 - \mu \int_{\mathbb{R}^N} u_n^2dx + o(1) \ge M - \mu + o(1),$$

which is a contradiction. Now, the desired result follows from the arbitrariness of $M$.

Below by $\| \cdot \|_q$ we denote the usual $L^q$ norm, $(\cdot, \cdot)_2$ denote the usual $L^2$ inner product, $c$ or $c_i$ stand for different positive constants. Let $X$ and $Y$ be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. We always choose the equivalent norm $\| (x, y) \|_{X \times Y} = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}$ on the product space $X \times Y$. In particular, if $X$ and $Y$ are two Hilbert spaces with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, we choose the inner product $((x, y), (w, z)) = (x, w)_X + (y, z)_Y$ on the product space $X \times Y$.

If $(V_0)$ holds, $0$ is at most an eigenvalue of finite multiplicity of $S$, and there is an orthogonal decomposition

$$L^2 = L^- \oplus L^+ \oplus L^0, \quad u = u^- + u^0 + u^0$$

such that $S$ is negative definite (resp. positive definite) in $L^- \ (resp. L^+)$ and $L^0 = \ker S$. Let $|S|$ denote the absolute of $S$ and $|S|^{\frac{1}{2}}$ be the square root of $|S|$. Let $H = D(|S|^{\frac{1}{2}})$ be the Hilbert space with the inner product

$$(u, v)_H = \left| |S|^{\frac{1}{2}}u, |S|^{\frac{1}{2}}v \right|_2 + (u^0, v^0)_2$$

and norm $\| u \|_H = (u, u)_H^{\frac{1}{2}}$. There is an induced decomposition

$$H = H^- \oplus H^0 \oplus H^+, \quad \text{where } H^\pm = H \cap L^\pm \text{ and } H^0 = L^0$$

which is orthogonal with respect to the inner products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)_H$. Let $E = H \times H$ with the inner product

$$(u, v, (\varphi, \psi)) = (u, \varphi)_H + (v, \psi)_H$$

and the corresponding norm

$$\| (u, v) \| = (\| u \|_H^2 + \| v \|_H^2)^{\frac{1}{2}}.$$

Setting

$$E^+ = H^+ \times H^-, \quad E^- = H^- \times H^+, \quad E^0 = H^0 \times H^0,$$

Then for any $z = (u, v) \in E$, we have $z = z^- + z^0 + z^+$, where $z^- = (u^-, v^-), z^0 = (u^0, v^0), z^+ = (u^+, v^+)$. Clearly, $E^-, E^0, E^+$ are orthogonal with respect to the products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)_H$. Hence $E = E^- \oplus E^0 \oplus E^+$. $\square$

Lemma 2.2. If $(V_0)$ holds, then $E \hookrightarrow L^p_0(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for $p \in [2, 2^*)$ and $E \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for $p \in [2, 2^*)$, where $2^*$ is the Sobolev critical exponent.

On $E$ we define the following functional

$$\Phi(z) = \frac{1}{2}(\| z^+ \|^2 - \| z^- \|^2) - \Psi(z)$$

(2.1)
where \( \Psi(z) = \int_{R^N} H(x, z)dx \). Our hypotheses imply that \( \Phi \in C^1(E, R) \) (see [26]), and a standard argument shows that critical points of \( \Phi \) are solutions of (ES). The functional \( \Phi \) is strongly indefinite; such types of functionals have appeared extensively in the study of differential equations via critical point theory.

Now we discuss the linking structure of \( \Phi \).

**Lemma 2.3.** Suppose \((H_0)\)–\((H_2)\) are satisfied. Then there is a \( \rho > 0 \) such that \( \kappa := \inf \Phi(\partial B_\rho \cap E^+) > 0 \).

**Proof.** Observe that, given \( \varepsilon > 0 \), there is \( C_\varepsilon > 0 \) such that

\[
|H(x, z)| \leq \varepsilon |z| + C_\varepsilon |z|^{p-1}
\]

(2.2)

and

\[
|H(x, z)| \leq \varepsilon |z|^2 + C_\varepsilon |z|^p
\]

(2.3)

for all \((x, z)\), where \( p > 2 \). For \( z \in E^+ \), by Lemma 2.2 and (2.3) we have

\[
\Phi(z) = \frac{1}{2} \|z\|^2 - \int_{R^N} H(x, z)dx \\
\geq \frac{1}{2} \|z\|^2 - C(\varepsilon \|z\|^2 + C_\varepsilon \|z\|^p).
\]

Choosing an appropriate \( \varepsilon \) we see that the desired conclusion holds for some \( \rho > 0 \).

In the following, we arrange all the eigenvalues (counted with multiplicity) of \( S \) in \((0, F_0)\) by \( 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k < F_0 \) and let \( e_j \) denote the corresponding eigenfunctions: \( Se_j = \mu_j e_j \) for \( j = 1, 2, \ldots, k \). Similarly, we choose the eigenvalues of \( S \) in \((-F_0, 0)\) such that \(-\mu_j \leq \lambda_j \leq \cdots \leq \lambda_1 \leq -\mu_1 < 0 \), and let \( \beta_j \) denote the corresponding eigenfunctions: \( S\beta_j = \lambda_j \beta_j \) for \( j = 1, 2, \ldots, n \). Setting \( Y_0 := \text{span}\{e_1, e_2, \ldots, e_k\}, Y_1 := \text{span}\{\beta_1, \beta_2, \ldots, \beta_n\} \). If there were no such \( \{\lambda_j\} \), then \( Y_1 \) is zero space. Note that

\[
\mu_1 |u_j|^2 \leq \|u_j\|^2 \leq \mu_k |u_j|^2
\]

for all \( u_j \in Y_0 \),

\[
|v_j|^2 \leq \|v_j\|^2 \leq |\mu_k v_j|^2
\]

for all \( v_j \in Y_1 \).

Set \( W_0 = Y_0 \times Y_1 \) is an finite dimensional subspace of \( E^+ \) and

\[
\mu_1 |w_j|^2 \leq \|w_j\|^2 \leq \mu_k |w_j|^2
\]

for all \( w_j \in W_0 \).

(2.4)

For any subspace \( W \) of \( W_0 \) set \( E_W = E^- \oplus E^0 \oplus W \).

**Lemma 2.4.** Let \((H_0)\) and \((H_2)\)–\((H_3)\) be satisfied and \( \rho > 0 \) be given by Lemma 2.3. Then for any subspace \( W \) of \( W_0 \), \( \sup \Phi(E_W) \) \( < \infty \), and there is \( R_{\rho} > 0 \) such that \( \sup \Phi(E_W \setminus B_{R_{\rho}}) \) \( < \inf \Phi(B_{\rho} \cap E^+) \).

**Proof.** It is sufficient to prove that \( \Phi(z) \to -\infty \) in \( E_W \) as \( \|z\| \to \infty \). If not, then there are \( M > 0 \) and \( \{z_j\} \subseteq E_W \) with \( \|z_j\| \to \infty \) such that \( \Phi(z_j) \geq -M \) for all \( j \). Denote \( y_j := z_j/\|z_j\| \), passing to a subsequence if necessary, \( y_j \to y, y^-_j \to y^-, y^+_j \to y^0 \) and \( y^+_j \to y^+ \). Now, by (2.1) we have

\[
\frac{1}{2} (\|y^+_j\|^2 - \|y^-_j\|^2) \geq \frac{1}{2} (\|y_j^+\|^2 - \|y_j^-\|^2) - \frac{\Psi(x, z_j)}{\|z_j\|^2} - \frac{\Phi(z_j)}{\|z_j\|^2} \geq \frac{-M}{\|z_j\|^2},
\]

(2.5)

here we use the fact that \( \Psi(z) \geq 0 \), from where it follows that

\[
\frac{1}{2} \|y^-_j\|^2 \geq \frac{1}{2} \|y^+_j\|^2 + \frac{M}{\|z_j\|^2}.
\]

(2.6)

We claim that \( y^+ \neq 0 \). Indeed, if not, (2.6) yields that \( \|y^-\| \to 0 \), and thus \( y_j \to y^0 \), \( \int_{R^N} \frac{H(x, z)}{|z|^2} dx \to 0 \). Let \( R(x, z) = H(x, z) - \frac{1}{2} H_{\infty}(x)|z|^2 \), then by (H2), we have \( |R(x, z)| \leq c|z|^2 \) for some \( c \), \( \frac{R(x, z)}{|z|^2} \to 0 \) as \( |z| \to \infty \) uniformly in \( x \). Hence

\[
o(1) = \int_{R^N} \frac{H(x, z_j)}{|z_j|^2} dx = \frac{1}{2} \int_{R^N} H_{\infty}(x)|y_j|^2 dx + \int_{R^N} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx \\
\geq \frac{1}{2} \int_{R^N} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx - \int_{y(x) \neq 0} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx \\
= \frac{1}{2} \int_{R^N} |y_j|^2 dx - o(1),
\]
which implies that $|y_j|^2 \to 0$ and hence $\|y_j\| \to 0$. This contradicts $\|y_j\| = 1$. By (H3), (2.4), since
\[
\|y^+\|^2 - \|y^-\|^2 - \int_{\mathbb{R}^N} H_\infty(x)|y|^2\,dx \leq \|y^+\|^2 - \|y^-\|^2 - F_0|y|^2 \leq -(F_0 - \mu_k)|y^+|^2 < 0,
\]
then there exists a bounded $\Omega \subset \mathbb{R}^N$ such that
\[
\|y^+\|^2 - \|y^-\|^2 - \int_{\Omega} H_\infty(x)|y|^2\,dx < 0. \tag{2.7}
\]
By Lebesgue’s dominated convergence theorem we have
\[
\lim_{j \to \infty} \int_{\Omega} \frac{|R(x, z_j)|}{\|z_j\|^2} \,dx = \lim_{j \to \infty} \int_{\Omega} \frac{|R(x, z_j)|}{|z_j|^2} |y_j|^2\,dx = 0. \tag{2.8}
\]

Thus (2.5), (2.7) and (2.8) imply that
\[
0 \leq \lim_{j \to \infty} \left[ \frac{1}{2} (\|y_j^+\|^2 - \|y_j^-\|^2) - \int_{\Omega} \frac{H(x, z_j)}{\|z_j\|^2} \,dx \right]
= \lim_{j \to \infty} \left[ \frac{1}{2} (\|y_j^+\|^2 - \|y_j^-\|^2) - \frac{1}{2} \int_{\Omega} H_\infty(x)|y_j|^2\,dx - \int_{\Omega} \frac{R(x, z_j)}{\|z_j\|^2} \,dx \right]
\leq \frac{1}{2} \left[ \|y^+\|^2 - \|y^-\|^2 - \int_{\Omega} H_\infty(x)|y|^2\,dx \right] < 0.
\]

Now the desired conclusion follows from this contradiction. \qed

As a consequence, we have

**Lemma 2.5.** Let (H0) and (H2)–(H4) be satisfied and $\kappa > 0$ be given by Lemma 2.3. Then letting $e \in W_0$ with $\|e\| = 1$, there is $R_1 > \rho$ such that $\Phi|_{Q} \leq \kappa$, where $Q := \{z = z^+ + z^- + s: z^+ + z^- \in E^- \oplus E^0, s \geq 0, \|z\| \leq R_1\}$.

3. The (C)$_\kappa$-condition

In this section, we discuss the properties of the (C)$_\kappa$-sequences. Recall that a sequence $\{z_j\} \subset E$ is said to be a (C)$_\kappa$-sequence if $\Phi(z_j) \to c$ and $(1 + \|z_j\|)\Phi'(z_j) \to 0$. $\Phi$ is said to satisfy the (C)$_\kappa$-condition if any (C)$_\kappa$-sequence has a convergent subsequence.

**Lemma 3.1.** Suppose that (H0)–(H4) and (H5) or (H5') are satisfied. Then any (C)$_\kappa$-sequence of $\Phi$ is bounded.

**Proof.** Let $\{z_j\}$ be such that $\Phi(z_j) \to c$ and $(1 + \|z_j\|)\Phi'(z_j) \to 0$. Suppose to the contrary that $\{z_j\}$ is unbounded. Setting $y_j := z_j/\|z_j\|$, then $\|y_j\| = 1$. Without loss of generality, we can assume that $y_j \to y$ in $E$. There are only two cases need to be considered: $y = 0$ or $y \neq 0$.

If $y \equiv 0$, then $y_j \to 0$ in $E$ and $y_j \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $p \in [2, 2^*)$. By (H4), we choosing some number $K$ such that $\tilde{K} < K < M_0$. Let $P_1$ be the projection associated with $[-K, K]$ and $P_2 = \text{id} - P_1$. It is obvious that $P_1y_j \to 0$ in $E$ since $P_1$ has finite-dimensional range, hence $\|y_j\|^2 = \|P_2y_j\|^2 + o(1)$. Moreover,
\[
K|z_j|^2 \leq \|z_j\|^2 \quad \text{for all } z_j \in P_2E. \tag{3.1}
\]

Note that
\[
o(1) = \frac{\Phi'(z_j)(P_2z_j^+ - P_2z_j^-)}{\|z_j\|^2}
\geq \|P_2y_j\|^2 - \int_{\mathbb{R}^N} \frac{H_\infty(x, z_j)(P_2y_j^+ - P_2y_j^-)}{|z_j|} |y_j|\,dx. \tag{3.2}
\]

By (H4) again, for fixed $\epsilon > 0$ there exists some $R_\epsilon > 0$ such that $|H_\infty(x, z)\| \leq (K - \epsilon)|z|$ for $|x| \geq R_\epsilon$ uniformly in $x$. Set $B_{R_\epsilon}(0) = \{x \in \mathbb{R}^N: |x| \leq R_\epsilon\}$ and $B_{R_\epsilon}^c(0) = \mathbb{R}^N \setminus B_{R_\epsilon}(0)$. Using (3.1) and (3.2) we get
Lemma 3.2. Under the assumption of Lemma 3.1, Φ(·) and Φ′(·) is BL-split.

\[
\|P_2 y_l\|^2 = \int_{\mathbb{R}^N} H_2(x, z_j) (P_2 y_l^+ - P_2 y_l^-) |y_l| dx
\]
\[
= \int_{B_{\rho_0}(0)} H_2(x, z_j) (P_2 y_l^+ - P_2 y_l^-) |y_l| dx + \int_{B_{\rho_0}(0)} H_2(x, z_j) (P_2 y_l^+ - P_2 y_l^-) |y_l| dx
\]
\[
\leq C \int_{B_{\rho_0}(0)} |P_2 y_l^+ - P_2 y_l^-| |y_l| dx + (K - \epsilon) \int_{B_{\rho_0}(0)} |P_2 y_l^+ - P_2 y_l^-| |y_l| dx
\]
\[
\leq (K - \epsilon) |P_2 y_l|^2 + o(1)
\]
\[
\leq \frac{K - \epsilon}{K} \|P_2 y_l\|^2 + o(1).
\] (3.3)

This is a contradiction. Therefore, the case \( y \equiv 0 \) cannot occur.

If \( y \neq 0 \). For each \( \eta = (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \), now from (H2) there holds

\[
\Phi'(z_j) \eta = (z_j^+ - z_j^-, \eta) - (H_\infty(x)z_j, \eta)_2 - \int_{\mathbb{R}^N} R_\epsilon(x, z_j) \eta dx
\]
\[
= \| z_j \| \left[ (y_j^+ - y_j^-, \eta) - (H_\infty(x)y_j, \eta)_2 - \int_{\mathbb{R}^N} R_\epsilon(x, z_j) \frac{|y_j|}{|z_j|} dx \right].
\]

Hence we have

\[
(y_j^+ - y_j^-, \eta) - (H_\infty(x)y_j, \eta)_2 - \int_{\mathbb{R}^N} R_\epsilon(x, z_j) \frac{|y_j|}{|z_j|} dx = o(1).
\] (3.4)

Letting \( j \to \infty \), we have

\[
(y_j^+ - y_j^-, \eta) - (H_\infty(x)y_j, \eta)_2 = 0.
\] (3.5)

this is impossible if (H3) is satisfied. Now we assume (H3) holds, then there exists some \( \alpha > 0 \) such that \( \hat{H}(x, z) \geq \delta_0 \) whenever \( |z| \geq \alpha \). Note that there holds

\[
C \geq \Phi(z_j) - \frac{1}{2} \Phi'(z_j) z_j = \int_{\mathbb{R}^N} \hat{H}(x, z_j) dx
\]
\[
\geq \int_{|z| \geq \alpha} \hat{H}(x, z) dx \geq \int_{|z| \geq \alpha} \delta_0 dx
\]
\[
= \delta_0 \{ \{ x \in \mathbb{R}^N : |z(x)| \geq \alpha \} \}
\]

hence

\[
\{ \{ x \in \mathbb{R}^N : |z(x)| \geq \alpha \} \} \leq C / \delta_0.
\]

By (3.5) and the unique continuation arguments in Heinz [27], we deduce that \( y(x) \neq 0 \) a.e. on \( \mathbb{R}^N \). Hence there exist \( \beta > 0 \) and \( \Omega \subset \mathbb{R}^N \) such that \( |y(x)| \geq 2\beta \) for \( x \in \Omega \) and \( 2C / \delta_0 \leq |\Omega| < \infty \). By Egoroff’s theorem we can find a set \( \Omega' \subset \Omega \) with \( |\Omega'| > C / \delta_0 \) such that \( y_j \to y \) uniformly on \( \Omega' \). So for almost all \( j \), \( |y_j(x)| \geq \beta \) and \( |z_j(x)| \geq \alpha \) in \( \Omega' \). Then

\[
\int_{\mathbb{R}^N} \{ x \in \mathbb{R}^N : |z(x)| \geq \alpha \} \leq C / \delta_0.
\]

This is also a contradiction. Therefore \( \{ z_j \} \) is bounded in \( E \).

Let \( \{ z_j \} \subset E \) be a \( C_0 \)-sequence of \( \Phi \), by Lemma 3.1, it is bounded, up to a subsequence, we may assume \( z_j \to z \) in \( E, z_j \to z \) in \( H^1_{0c}(\mathbb{R}^N) \) for \( p \in [2, 2') \) and \( z_j(x) \to z(x) \) a.e. on \( \mathbb{R}^N \). Obviously \( z \) is a critical point of \( \Phi \).

Recall that a mapping \( f \) from a Banach space \( X \) to another Banach space \( Y \) is called a BL-split, if for every sequence \( \{ x_n \} \) in \( X \) with \( x_n \to x \) it holds that \( f(x_n) - f(x_n - x) \to f(x) \) in \( Y \) (see [28]). Let \( \eta : [0, \infty) \to [0, 1] \) be a smooth function satisfying \( \eta(s) = 1 \) if \( s \leq 1 \), \( \eta(s) = 0 \) if \( s \geq 2 \). Define \( \tilde{z}_j(x) = \eta(2|x|/\epsilon)z(x) \), then \( \tilde{z}_j \to z \) in \( E \). In a similar way to Lemma 4.4 in Ding and Jeanjean [23] (also see [28,22]), we can obtain the following: \( \square \)

Lemma 3.2. Under the assumptions of Lemma 3.1, \( \Phi(\cdot) \) and \( \Phi'(\cdot) \) is BL-split.
Lemma 3.3. Under the assumptions of Lemma 3.1, $\Phi$ satisfies the $(C)_c$-condition.

Proof. Let $\{z_j\}$ be a $(C)_c$-sequence of $\Phi$, suppose that $z_j \rightharpoonup z$ in $E$. Let $K, P_1, P_2$ be as in the proof of Lemma 3.1. Set $y_j = z_j - \tilde{z}_j = P_1y_j + P_2y_j$, then $y_j = z_j - z + z - \tilde{z}_j \rightharpoonup 0$. Hence $P_1y_j \rightharpoonup 0$, since $P_1$ has finite-dimensional range. By Lemma 3.2, we have $\Phi(y_j) \rightharpoonup 0$. Similar to (3.2),

$$\alpha(1) = \Phi'(y_j)(P_2y_j^+ - P_2y_j^-) = \|P_2y_j\|^2 - \int_{\mathbb{R}^N} H_2(x, y_j)(P_2y_j^+ - P_2y_j^-)dx.$$

Thus, similar to (3.3), we have

$$\|P_2y_j\|^2 = \int_{\mathbb{R}^N} \frac{H_2(x, y_j)}{|y_j|} (P_2y_j^+ - P_2y_j^-)|y_j|dx$$

$$= \int_{B_{\epsilon_0}(0)} \frac{H_2(x, y_j)}{|y_j|} (P_2y_j^+ - P_2y_j^-)|y_j|dx + \int_{B_{\epsilon_0}(0)} \frac{H_2(x, y_j)}{|y_j|} (P_2y_j^+ - P_2y_j^-)|y_j|dx$$

$$\leq C \int_{B_{\epsilon_0}(0)} |P_2y_j^+ - P_2y_j^-| |y_j|dx + (K - \epsilon) \int_{B_{\epsilon_0}(0)} |P_2y_j^+ - P_2y_j^-| |y_j|dx$$

$$\leq (K - \epsilon)\|P_2y_j\|^2 + o(1)$$

$$\leq K - \epsilon$$

which implies that $\|P_2y_j\| \rightharpoonup 0$ and so $\|y_j\| \rightharpoonup 0$. Then $z_j \rightharpoonup z$ since $z_j - z = y_j + \tilde{z}_j - z$. This ends the proof. \hfill $\Box$

4. Proof of Theorem 1.2

In this section we give the proof for our Theorem 1.2. Let $E$ be a Banach space with direct sum $E = \mathbb{R} \oplus Y$ and corresponding projections $P_X, P_Y$ onto $X, Y$. Let $\delta \subset (X)^*$ be a dense subset, for each $s \in \delta$ there is a semi-norm on $E$ defined by

$$p_s : E \to \mathbb{R}, \; p_s(u) = |s(x)| + \|y\|$$

for $u = x + y \in E$.

We denote by $T_s$ the topology induced by semi-norm family $\{p_s\}$, $w_s$ denote the weak*-topology on $E^*$. Now, some notations are needed. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E | \Phi(u) \geq a\}$, $\Phi_b = \{u \in E | \Phi(u) \leq b\}$, and $\Phi_c = \Phi_a \cap \Phi_b$. Recall that $\Phi$ is said to be weakly sequentially lower semicontinuous if for any $u_j \rightharpoonup u$ in $E$ one has $\Phi(u) \leq \liminf_{j \to \infty} \Phi(u_j)$, and $\Phi'$ is said to be weakly sequentially continuous if $\lim_{j \to \infty} \Phi'(u_j)w = \Phi'(u)w$ for each $w \in E$.

Suppose

$(\Phi_0)$ for any $c \in \mathbb{R}$, superlevel $\Phi_c$ is $T_s$-closed, and $\Phi' : (\Phi_c, T_s) \to (E^*, w^*)$ is continuous;

$(\Phi_1)$ for any $c > 0$, there exists $\xi > 0$ such that $\|u\| < \xi \Rightarrow \|P_Yu\|$ for all $u \in \Phi_c$;

$(\Phi_2)$ there exists $\rho > 0$ such that $c = \inf \Phi(S_{\rho} \cap Y) > 0$, where $S_{\rho} := \{u \in E : \|u\| = \rho\}$;

$(\Phi_3)$ there exists a finite-dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that we have for $E_0 := X \oplus Y_0$ and $B_0 := \{u \in E_0 : \|u\| \leq R\}$ that $c := \sup \Phi(E_0) < \infty$ and $\inf \Phi(E_0 \setminus B_0) < \inf \Phi(S_{\rho} \cap Y)$.

Now we state two critical point theorems which will be used later (see [19,21]).

Theorem 4.1. Let $(\Phi_0) - (\Phi_2)$ be satisfied and suppose there are $R > \rho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi(\partial Q) \leq \kappa$ where $Q := \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. If $\Phi$ satisfies the $(C)_c$-condition for all $c \leq \tilde{c}$ then $\Phi$ has a critical point $u$ with $\kappa \leq \Phi(u) \leq \tilde{c}$.

Theorem 4.2. Assume $\Phi$ is even and $(\Phi_0), (\Phi_2) - (\Phi_3)$ be satisfied. Then $\Phi$ has at least $m := \dim Y_0$ pairs of critical points with critical values less or equal to $\tilde{c}$ provided $\Phi$ satisfies the $(C)_c$-condition for all $c \in [\kappa, \tilde{c}]$.

Lemma 4.3. $\Phi$ satisfies $(\Phi_0)$.

Proof. We first show that $\Phi_a$ is $T_s$-closed for every $a \in \mathbb{R}$. Consider a sequence $\{z_j\} \subset \Phi_a$ which $T_s$-converges to $z \in E$, and write $z_j = z_j^- + z_j^0 + z_j^+$. Then $z_j^+ \rightharpoonup z^+$ in norm topology and hence $\{z_j^+\}$ is bounded in norm topology. Observe that there exists $C > 0$ such that

$$\|z_j^-\|^2 = \|z_j^0\|^2 - 2\Phi(z_j) - 2 \int_{\mathbb{R}^N} H(x, z_j)dx \leq C$$

since $H(x, z) \geq 0$. This implies the boundedness of $\{z_j^-\}$ and hence $z_j^- \rightharpoonup z^-$. Therefore we have $z_j \rightharpoonup z$. Since $\Psi$ is weakly sequentially lower semi-continuous, it follows that

$$a \leq \lim_{j \to \infty} \Phi(z_j) \leq \Phi(z),$$

so $z \in \Phi_a$ and hence $\Phi_a$ is $T_s$-closed.
Next we show that \( \Phi' : (\Phi_c, T_\lambda) \to (E^*, w^*) \) is continuous. It is sufficient to show that \( \Psi' \) has the same property, where \( \Psi(z) := \int_{\mathbb{R}^N} H(x, z) dx \). Suppose \( z_j \rightharpoonup z \) in \( E \). Then \( z_j \to z \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \) for \( p \in [2, 2^*) \). It is obvious that
\[
\Phi'(z_j) \eta = \int_{\mathbb{R}^N} H_z(x, z_j) \eta dx \to \int_{\mathbb{R}^N} H_z(x, z) \eta dx = \Phi'(z) \eta,
\]
for all \( \eta = (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \) as \( j \to \infty \). Now using the density of \( C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \) in \( E \) one can check the desired conclusion. \( \square \)

**Lemma 4.4.** \( \Phi \) satisfies \( (\Phi_1) \).

**Proof.** We assume by contradiction that for some \( c \), there is a sequence \( \{z_j\} \) with \( \Phi(z_j) \geq c \) and \( \|z_j\| \geq j\|z_j^+\| \) for each \( j \in \mathbb{N} \). Form \( \Phi \) implies
\[
\|z_j^+ - z_j^0\|^2 \geq (j - 1)\|z_j^+\|^2 \geq (j - 1)(2c + \|z_j^+\|^2 + 2\int_{\mathbb{R}^N} H(x, z_j) dx),
\]

or
\[
\|z_j^0\|^2 \geq (j - 1)2c + (j - 2)\|z_j^-\|^2 + 2(j - 1)\int_{\mathbb{R}^N} H(x, z_j) dx.
\]

Since \( c > 0 \) and \( H(x, z) \geq 0 \), it follows that \( \|z_j^0\| \to \infty \), hence \( \|z_j\| \to \infty \). Set \( y_j = \frac{z_j}{\|z_j\|} \), we have \( \|y_j^+\|^2 \leq \frac{1}{j-1} \to 0 \). By
\[
1 \geq \|y_j^0\|^2 \geq \frac{(j - 1)2c + (j - 2)\|y_j^-\|^2 + 2(j - 1)\int_{\mathbb{R}^N} H(x, z_j) dx}{\|z_j\|^2},
\]

we also have \( \|y_j^0\|^2 \leq \frac{1}{j-1} \to 0 \). Therefore, \( y_j \to y = y^0 \) in \( E \) and \( \|y_j\|^2 = 1 \). Recall that \( R(x, z) = H(x, z) - \frac{1}{2}H_\infty(x)|z|^2 \), then
\[
\frac{R(x, z)}{|z|^2} \to 0 \quad \text{as} \quad |z| \to \infty \quad \text{uniformly in} \ x.\n\]

Therefore, since \( |z_j(x)| \to \infty \) for \( y(x) \neq 0 \), have
\[
\int_{\mathbb{R}^N} \frac{R(x, z)}{|z_j|^2} dx \leq \int_{\mathbb{R}^N} \frac{R(x, z_j)y_j - y_j^2}{|z_j|^2} dx + \int_{\mathbb{R}^N} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx
\]
\[
\leq \int_{y(x) \neq 0} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx + c|y_j - y_j|^2 \to 0.
\]

This implies
\[
\frac{1}{2(j - 1)} \geq \int_{\mathbb{R}^N} \frac{H(x, z)}{|z_j|^2} dx = \frac{1}{2} \int_{\mathbb{R}^N} H_{\infty}(x)|y_j|^2 dx + \int_{\mathbb{R}^N} \frac{R(x, z_j)|y_j|^2}{|z_j|^2} dx
\]
\[
\geq \frac{1}{2}H_0|y_j|^2 + o(1),
\]

consequently, \( y^0 = 0 \), a contradiction. \( \square \)

**Proof of Theorem 1.2** (Existence of a Least Energy Solution). With \( X = E^- \oplus E^0 \) and \( Y = E^+ \) the condition \( (\Phi_0) \) holds by Lemma 4.3 and \( (\Phi_3) \) holds by Lemma 4.4. Lemma 2.3 implies \( (\Phi_2) \). Lemma 2.5 shows that \( \Phi \) possesses the linking structure of Theorem 4.1 and Lemma 3.3 implies \( \Phi \) satisfies the \( (C)_c \)-condition. Therefore, \( \Phi \) has at least one critical point \( z \) with \( \Phi(z) \geq \kappa > 0 \). Moreover, let \( K = \{ z \in E : \Phi'(z) = 0 \} \) be the set of nontrivial critical points of \( \Phi \). Claim that \( c = \inf \{ \Phi(z) : z \in K \} \) is achieved. Letting \( \{z_j\} \subset K \) be a minimizing sequence for \( c \). Clearly, \( \{z_j\} \) is a \( (C)_c \)-sequence of \( \Phi \), hence \( \{z_j\} \) is bounded by Lemma 3.1, and has a convergent subsequence by Lemma 3.3. Therefore, along a subsequence, \( z_j \to z \) and
\[
c = \lim_{j \to \infty} \Phi(z_j) = \Phi(z)
\]
from where it follows that \( c \) is achieved. \( \square \)

**Multiplicity.** \( \Phi \) is even provided \( H(x, z) \) is even in \( z \). Lemma 2.4 says that \( \Phi \) satisfies \( (\Phi_3) \) with \( \dim Y_0 = k \). Therefore, \( \Phi \) has at least \( k \) pairs of nontrivial critical points by Theorem 4.2.

**References**