General impulsive control of chaotic systems based on a TS fuzzy model

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Abstract

In this paper, a time-delayed impulsive controller is constructed and the uniformly asymptotic stability of chaotic systems based on a Takagi–Sugeno (TS) fuzzy model is investigated and some new and useful criteria are derived. In addition, the uniformly asymptotic synchronization of chaotic systems based on the TS fuzzy model is discussed using similar techniques. Finally, some numerical simulations are presented to illustrate the effectiveness and feasibility of the derived results.

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1. Introduction

The term chaos, associated with an interval map, was first formally introduced in mathematics by Li and Yorke in 1975, whereby they established a simple criterion for chaos in one-dimensional difference equations, i.e., the well-known “period three implies chaos.” Over the past three decades, chaos has matured as a science (although it is still evolving) and has provided insights into previously intractable and inherently nonlinear natural phenomena [1]. There is, however, still no unified, universally accepted and rigorous mathematical definition of chaos in the scientific literature to provide a fundamental basis for studying such exotic phenomena. Roughly speaking, chaotic behavior is the seemingly random behavior of a deterministic system that is characterized by sensitive dependence on initial conditions.

Although chaos is a very attractive subject for study, it was once believed to be neither predictable nor controllable owing to its intrinsic topological complexity. However, in 1990, the pioneering work of Ott, Grebogi and Yorke denied completely the above viewpoint. On the one hand, recent research has shown that chaos can actually be useful under certain circumstances, such as enhanced mixing of chemical reactants. On the other hand, chaos should be weakened or completely suppressed when it is harmful. Therefore, to take full advantage of chaos, it is necessary to study the phenomenon of chaos and chaos control.

Recently, impulsive control has been widely recognized as a powerful control approach for stabilizing and synchronizing nonlinear systems [2–9]. These impulsively controlled systems are described by impulsive differential equations.
Impulsive control represents an efficient method for dealing with plants that cannot endure continuous control inputs and in some applications it is impossible to provide continuous control inputs. For example, a government cannot change the savings rates of its central bank every day. In view of its advantages, impulsive control has been successfully applied to various chaotic systems [3,10–14]. Chen et al. discussed the exponential and asymptotic stability of impulsive systems using impulsive control theory [10]. The authors then designed an impulsive control law for Lü’s system by applying those conditions. Chen et al. [11] and Zhang et al. [14] proposed impulsive control and synchronization of a unified chaotic system. Liu investigated impulsive stabilization of chaotic systems using Lyapunov stability theory [3]. Sun and Zhang [12] and Yang et al. [13] investigated asymptotic stabilization of the origin of Chua’s oscillator and Lorenz system, respectively, using impulsive control.

Delayed feedback control (DFC), as an important method for chaos control, was introduced by Lithuanian physicist K. Pyragas in 1992 to stabilize an unstable periodic orbit of a nonlinear control system [1]. A main feature of the DFC method is that the internal dynamics of the system considered do not need to be modeled, and prior information on the desired unstable periodic orbit is not required, apart from its period. Based on this method, the control input is fed by the difference between the current state and the delayed state. Some studies on DFC applications have been published [15–23]. However, the combination of impulsive control and DFC to yield time-delayed impulsive control has not been considered in such studies.

In the field of nonlinear control systems, Takagi–Sugeno (TS) fuzzy control as a non-traditional control technique has attracted much attention for modeling of nonlinear systems using a set of if-then rules. The TS fuzzy dynamic model was first proposed by Takagi and Sugeno [1]. The main idea of the model is to express the local dynamics of each fuzzy rule using a linear system model, and to express the overall system via fuzzy “blending” of the local linear system models. As a result, this model utilizes the well-established linear system theory to analyze and synthesize a highly nonlinear dynamic system. To date, many results concerning chaotic systems based on TS fuzzy models have been presented [24–30]. Hu et al. investigated the fuzzy impulsive control and synchronization of chaotic systems based on a TS model [24]. Wang et al. investigated asymptotic synchronization of chaotic systems based on a TS model [27]. Different criteria have been obtained concerning the impulsive stability of chaotic systems based on TS models [25,28–30]. Nevertheless, the uniformly asymptotic stability and synchronization of chaotic systems based on a TS fuzzy model have not been considered in previous studies and time-delayed impulsive controllers have not been investigated.

Motivated by the above discussion, this paper addresses the chaos control problem using a TS fuzzy model via impulsive control. The main contributions of the paper are as follows. First, based on a TS fuzzy model, a time-delayed impulsive controller is proposed by applying DFC and impulsive control theory, which guarantee the uniformly asymptotic stability of the origin for the chaotic system considered. The uniformly asymptotic synchronization of this chaotic system is also investigated. It is worth noting that the impulsive functions can be nonlinear in this study, which extends previous results for which linear impulsive control functions are required [10–14,27–29]. In addition, in many previous results the impulsive control functions are independent of time delays [2,3,5,6,10–14,24,25,27–31]. However, in our approach, because a DFC technique is applied, delayed impulsive functions are proposed to control the chaotic system. Moreover, our results are independent of the feedback time delay, which extends previous results [22]. Most importantly, compared with previous studies, the upper bound of the impulsive interval derived from our results is greater. Thus, our control scheme is more effective and economic than previous results, which was one of the main motivations for investigating synchronization using time-delayed impulsive control.

The remainder of the paper is organized as follows. Section 2 introduces the TS fuzzy model and fuzzy control of chaotic systems. In Section 3, a time-delayed impulsive controller is described and some new and useful criteria leading to uniformly asymptotic stability of chaotic systems based on a TS fuzzy model are derived. The uniformly asymptotic synchronization of chaotic systems based on the TS fuzzy model is described in Section 4. In Section 5, the applicability and feasibility of the methods developed are demonstrated using numerical simulations.

2. TS fuzzy model and fuzzy control of a chaotic system

A TS fuzzy model is described by a set of fuzzy if-then rules that characterize local relations of a nonlinear system in the state space. The main feature of a TS model is that it expresses the local dynamics of each fuzzy rule by a linear state-space system model. The overall fuzzy system is modeled by fuzzy “blending” of these local linear system models through some suitable membership functions.
Consider the following chaotic system

\[
\dot{x}(t) = f(t, x(t)),
\]

where \(x \in \mathbb{R}^n\) is the state variable and \(f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]\) is a nonlinear vector-valued function satisfying \(f(t, 0) \equiv 0\) for all \(t \in \mathbb{R}^+\).

For system (1), using the technique introduced by Li [1], we can construct a TS fuzzy model as follows:

**Rule i:** If \(z_i(t) = M_i^1, z_2(t) = M_i^2, \ldots, z_p(t) = M_i^p\), then

\[
\dot{x}(t) = A_i x(t), \quad i = 1, 2, \ldots, r,
\]

where the premise variables \(z_1(t), \ldots, z_p(t)\) are proper state variables, each \(M_i^j\) denotes the fuzzy set, each \(A_i\) denotes an \(n \times n\) constant matrix and \(r\) is the number of fuzzy rules.

Using the singleton fuzzifier, product fuzzy inference and a weighted average defuzzifier [1], the final output of TS fuzzy system (2) for system (1) is inferred as follows:

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t),
\]

where \(z(t) = (z_1(t), \ldots, z_p(t))\) and

\[
h_i(z(t)) = \frac{w_i(z(t))}{\sum_{j=1}^{r} w_j(z(t))}, \quad w_i(z(t)) = \prod_{j=1}^{p} M_i^j(z_j(t)),
\]

in which \(M_i^j(z_j(t))\) is the grade of membership of \(z_j(t)\) in \(M_i^j\), \(w_i(z(t)) \geq 0\) \((i = 1, 2, \ldots, r)\) and \(\sum_{i=1}^{r} w_i(z(t)) > 0\). It is clear that

\[
\sum_{i=1}^{r} h_i(z(t)) = 1, \quad h_i(z(t)) \geq 0, \quad i = 1, 2, \ldots, r,
\]

for all \(t \in \mathbb{R}^+\), where \(h_i(z(t))\) can be regarded as the normalized weight of the if–then rules.

**Remark 1.** According to TS fuzzy theory [1], the defuzzified output (3) of TS fuzzy model (2) is mathematically identical to chaotic system (1).

To control chaotic system (1), we introduce the following TS fuzzy control model.

**Rule i:** If \(z_i(t) = M_i^1, z_2(t) = M_i^2, \ldots, z_p(t) = M_i^p\), then

\[
\dot{x}(t) = A_i x(t) + u_i(t), \quad i = 1, 2, \ldots, r,
\]

where \(r, A_i, z_i(t)\) and \(M_i^j\) are defined in (2) and \(u_i(t)\) denotes the external control input.

Similar to (2), the defuzzified output of TS fuzzy control system (4) is represented as follows:

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + u_i(t)],
\]

where \(h_i(z(t))\) is defined in (3).

### 3. Fuzzy impulsive control

In this section, we design a time-delayed impulsive controller for chaotic system (1) based on fuzzy control model (4). In the following, we assume that the controller shares the same fuzzy sets with system (4). More precisely, the \(i\)th control rule is formulated as follows...
Control Rule i: If \( z_1(t) \) is \( M^1_i \), \( z_2(t) \) is \( M^2_i \), \ldots, \( z_p(t) \) is \( M^p_i \), then
\[
 u_i(t) = \sum_{k=1}^{\infty} I_k(t)[F_i x(t) + G_i x(t - \tau)] + \sum_{k=1}^{\infty} \delta(t - (t_k - h))[I_{ik}(x(t), x(t - \tau)) - x(t)],
\]
for \( i = 1, 2, \ldots, r \), where \( h > 0 \) is sufficiently small, \( \tau > 0 \) is the feedback time delay, \( F_i \) and \( G_i \) denote \( n \times n \) constant matrices and the time sequence \( \{t_k\} \) satisfies \( 0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} \cdots \) and \( \lim_{k \to \infty} t_k = \infty \). The switching function \( I_k(\cdot) \) is defined as
\[
 I_k(t) = \begin{cases} 
 1, & t_{k-1} \leq t < t_k, \\
 0, & \text{otherwise.}
\end{cases}
\]
\( \delta(\cdot) \) is the Dirac delta function, each \( I_{ik} \in C[\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n] \) satisfies \( I_{ik}(0, 0) = 0 \) and the following condition is also satisfied:

(H) There exist constants \( \alpha_{ik} \geq 0 \) and \( \beta_{ik} \geq 0 \) such that
\[
 \|I_{ik}(u, v)\|^2 \leq \alpha_{ik}\|u\|^2 + \beta_{ik}\|v\|^2,
\]
for any \( u, v \in \mathbb{R}^n \), where \( \alpha_{ik} \) and \( \beta_{ik} \) are not all equal to 0. Here and throughout this paper, \( \|y(t)\| = [y^T(t)y(t)]^{1/2} \) is the Euclidean norm.

From (6), system (5) can be rewritten as
\[
 \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + u_i(t)] = \sum_{i=1}^{r} h_i(z(t))A_i x(t) + \sum_{i=1}^{r} h_i(z(t))\sum_{k=1}^{\infty} I_k(t)[F_i x(t) + G_i x(t - \tau)] \\
 + \sum_{i=1}^{r} h_i(z(t))\sum_{k=1}^{\infty} \delta(t - (t_k - h))[I_{ik}(x(t), x(t - \tau)) - x(t)],
\]
which implies that
\[
 x(t_k) - x(t_k - 2h) = \int_{t_k-2h}^{t_k} \left[ \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + \sum_{i=1}^{r} h_i(z(t))\sum_{k=1}^{\infty} I_k(t)[F_i x(t) + G_i x(t - \tau)] \\
 + \sum_{i=1}^{r} h_i(z(t))\sum_{k=1}^{\infty} \delta(t - (t_k - h))[I_{ik}(x(t), x(t - \tau)) - x(t)] \right] \, dt.
\]
As \( h \to 0^+ \), by applying the properties of the Dirac delta function, we obtain
\[
 x(t_k) - x(t_k^-) = \sum_{i=1}^{r} h_i(z(t_k^-))I_{ik}(x(t_k^-), x(t_k^- - \tau)) - x(t_k^-),
\]
which leads to
\[
 x(t_k) = \sum_{i=1}^{r} h_i(z(t_k^-))I_{ik}(x(t_k^-), x(t_k^- - \tau)).
\]
Thus, system (7) is rewritten as
\[
 \begin{align*}
 \dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))(A_i + F_i)x(t) + \sum_{i=1}^{r} h_i(z(t))G_i x(t - \tau), & t \neq t_k, \\
x(t_k) &= \sum_{i=1}^{r} h_i(z(t_k^-))I_{ik}(x(t_k^-), x(t_k^- - \tau)), & k \in \mathbb{Z}^+.
\end{align*}
\]

It is easy to see that system (9) is an impulsive differential function for which it is necessary to provide an initial function. Hence, we assume that
\[
 x(s) = \phi(s), \quad \forall s \in [-\tau, 0],
\]
where \( \phi(s) \in C([-\tau, 0], \mathbb{R}^n) \).
Remark 2. To stabilize the trivial solution $x = 0$ of chaotic system (1), fuzzy impulsive controller (6) is constructed based on TS fuzzy control model (4) and the final control output (9) is obtained. Evidently, the problem of controlling chaotic system (1) to a stable origin is converted to the stability problem for the zero solution for system (9).

In the following, we study the uniformly asymptotic stability of system (9). For convenience, we first introduce the following mathematical notation.

Let $\delta = \sup_{k \in \mathbb{Z}^+} \{t_k - t_{k-1}\} < \infty$ and let $P$ be an $n \times n$ symmetric and positive definite matrix and let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the smallest and the greatest eigenvalues of $P$, respectively.

Let $\alpha = \sup_{1 \leq i \leq r, \ k \in \mathbb{Z}^+} \{\alpha_{ik}\}$, $\beta = \sup_{1 \leq i \leq r, \ k \in \mathbb{Z}^+} \{\beta_{ik}\}$ and

$$
\Gamma = \frac{(\theta^2 + 1)\lambda_2}{2\lambda_1}, \quad \mathcal{A} = \frac{(\theta^2 + 1)\lambda_2}{2\lambda_1} \beta,
$$

(11)

where $\theta > 0$ is a real number.

Denoting matrices $Q_i = P^{-1}(A_i + F_i)^T P + (A_i + F_i) + (1/(\Gamma + A))I + G_i P^{-1} G_i^T P$, $\gamma_i$ denotes the greatest eigenvalue of $Q_i$ and $\gamma = \max_{1 \leq i \leq r}\{\gamma_i\}$, where $I$ denotes the $n \times n$ identity matrix.

The following theorem is provided to guarantee that the origin of system (9) is uniformly asymptotically stable.

**Theorem 1.** Under assumption (H), if there exist control matrices $F_i$ and $G_i$ such that the following conditions hold,

1. $0 < \Gamma + A < 1$,
2. $0 < \gamma < -(1/\delta) \ln(\Gamma + A)$,

then system (9) is uniformly asymptotically stable at its origin, that is, the zero solution of chaotic system (1) is uniformly asymptotically stable under fuzzy impulsive controller (6).

The proof of Theorem 1 is given in Appendix B.

**Remark 3.** From Theorem 1, we obtain an estimate of the upper bound of the impulsive internals. In fact, it follows from Theorem 1 that

$$
\delta \leq -\frac{1}{\gamma} \ln(\Gamma + A).
$$

In the following, we consider the special case in which the impulsive functions $I_{ik}$ are reduced to linearly impulsive functions, that is,

$$
I_{ik}(x(t), x(t - \tau)) = K_i x(t) + C_i x(t - \tau),
$$

(12)

for $i = 1, 2, \ldots, r$ in Theorem 1, where each $K_i$ and $C_i$ is the $n \times n$ constant matrix. Then system (9) can be rewritten in the following form:

$$
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{r} h_i(z(t))(A_i + F_i)x(t) + \sum_{i=1}^{r} h_i(z(t))G_i x(t - \tau), \quad t \neq t_k, \\
\dot{x}(t_k) & = \sum_{i=1}^{r} h_i(z(t_k^-))[K_i x(t_k^-) + C_i x(t_k^- - \tau)], \quad k \in \mathbb{Z}^+.
\end{align*}
$$

(13)

In this case, we obtain

$$
||I_{ik}(u, v)||^2 = (K_i u + C_i v)^T (K_i u + C_i v) = u^T K_i^T K_i u + 2u^T K_i^T C_i v + v^T C_i^T C_i v
\leq 2u^T K_i^T K_i u + 2v^T C_i^T C_i v \leq \eta_u u^T u + \rho v^T v \leq \eta u^T u + \rho v^T v,
$$

for any $u, v \in \mathbb{R}^n$, where $\eta_i \geq 2\lambda_{\max}(K_i^T K_i), \rho_i \geq 2\lambda_{\max}(C_i^T C_i), \eta = \max_{1 \leq i \leq r}\{\eta_i\}$ and $\rho = \max_{1 \leq i \leq r}\{\rho_i\}$.

In addition, let $P = I$ and $\theta = 1$ in Theorem 1 and denote $\gamma_i$ be the greatest eigenvalue of matrix $(A_i + F_i)^T + (A_i + F_i) + (1/(\eta + \rho))I + G_i G_i^T$ and $\gamma = \max_{1 \leq i \leq r}\{\gamma_i\}$. The following results are directly derived from Theorem 1.
Corollary 1. If there exist control matrices $C_i$, $F_i$, $G_i$ and $K_i$ such that the following conditions hold,

1. $0 < \eta + \rho < 1$,
2. $0 < \gamma < -(1/\delta)\ln(\eta + \rho),$

then system (13) is uniformly asymptotically stable at its origin.

Selecting $F_i = G_i = 0$ and

$$I_{ik}(x(t), x(t - \tau)) = K_i x(t),$$

for $i = 1, 2, \ldots, r$ in (8), where each $K_i$ is an $n \times n$ constant matrix, we can degenerate system (9) to the following form:

$$\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t)) A_i x(t), & t \neq t_k, \\
x(t_k) &= \sum_{i=1}^{r} h_i(z(t_k^-)) K_i x(t_k^-), & k \in \mathbb{Z}^+.
\end{align*}$$

Evidently, $\lambda$ equals to the greatest eigenvalue of matrix $K_i^T K_i$, $\Gamma = \varepsilon x$ and $\beta = \Lambda = 0$ in this case in (11), where $\varepsilon = (\theta^2 + 1)/2\theta \geq 1$ for $\theta > 0$. Furthermore, let $\lambda$ be the greatest eigenvalue of matrix $A_i^T + A_i$. The following statements are easily obtained from Theorem 1.

Corollary 2. If there exist control matrices $K_i$ and a constant $\varepsilon \geq 1$ such that the following conditions hold,

1. $0 < \varepsilon < 1$,
2. $0 < \lambda + 1/\varepsilon x < -(1/\delta) \ln(\varepsilon x),$

then system (15) is uniformly asymptotically stable at its origin.

Remark 4. The asymptotic stability of fuzzy impulsive control system (15) was previously investigated [29] and symmetry of the matrices $K_i$ is required. However, this restriction is removed in the present study. In addition, it follows from condition (2) in Corollary 2 that $\lambda > -1/\varepsilon x < 0$, which is less conservative than the assumption $\lambda > 0$ introduced by Zheng and Chen [29]. Hence, our results are superior and have greater applicability.

4. Fuzzy impulsive synchronization

The synchronization of chaotic systems based on TS fuzzy models has been investigated in previous studies. For instance, Hu et al. [24] and Wang et al. [27] obtained some results on the synchronization of chaotic systems using a fuzzy impulsive control technique. However, to the best of our knowledge, there are no results on the uniformly asymptotical synchronization of chaotic systems based on TS fuzzy models with application of a time-delayed impulsive method to control the driven system. In this section, we consider the time-delayed impulsive synchronization of chaotic systems based on a TS fuzzy model. To deal with the synchronization of chaotic systems, we need to design control inputs so that the driven system achieves asymptotic synchronization with the driving system, provided that the two systems start from different initial conditions. The driving system is given by (2). Suppose that the driven system has the same premise variables as the driving system and that the output states of the driving system are observable. The fuzzy rules of the driven system are as follows.

Rule $i$: If $z_1(t) = M_1^i$, $z_2(t) = M_2^i$, ..., $z_r(t)$ is $M_r^i$, then

$$\dot{y}(t) = A_i y(t) + u_i(t), \quad i = 1, 2, \ldots, r,$$

where $y \in \mathbb{R}^n$ is the state variable of the driven system, $A_i$, $z_i(t)$ and $M_i^i$ are defined in (2), and the control input $u_i(t)$ is designed as

$$u_i(t) = \sum_{k=1}^{\infty} l_k(t)[F_i e(t) + G_i e(t - \tau)] + \sum_{k=1}^{\infty} \delta(t - (t_k - h))[I_{ik}(e(t), e(t - \tau)) - e(t)],$$

where $F_i$, $G_i$ and $I_{ik}(\cdot)$ are the same as in system (6), and the error is $e(t) = y(t) - x(t).$
Similarly, we obtain the defuzzified output of driven system (16) as follows:

\[
\begin{align*}
\dot{y}(t) &= \sum_{i=1}^{r} h_i(z(t))[A_i y(t) + F_i e(t) + G_i e(t - \tau)], \quad t \neq t_k, \\
y(t) &= \sum_{i=1}^{r} h_i(z(t^{-}))[I_{ik}(e(t^{-}), e(t^{-} - \tau)) + x(t^{-})], \quad t = t_k.
\end{align*}
\] (17)

From (3) and (17), the error system can be presented in the following form:

\[
\begin{align*}
\dot{e}(t) &= \sum_{i=1}^{r} h_i(z(t))(A_i + F_i) e(t) + \sum_{i=1}^{r} h_i(z(t))G_i e(t - \tau), \quad t \neq t_k, \\
e(t) &= \sum_{i=1}^{r} h_i(z(t^{-}))I_{ik}(e(t^{-}), e(t^{-} - \tau)), \quad t = t_k.
\end{align*}
\] (18)

Obviously, error system (18) is the same as system (9). Then, similar to Theorem 1, the following results are obtained.

**Theorem 2.** Under assumption (H), if there exist control matrices \( F_i \) and \( G_i \) such that the following conditions hold,

1. \( 0 < \gamma < -(1/\delta) \ln(\Gamma + A) < 1 \),
2. \( 0 < \gamma < -(1/\delta) \ln(\Gamma + A) < 1 \),

then system (18) is uniformly asymptotically stable at its origin, that is, driving system (2) and driven system (16) are uniformly asymptotically synchronized.

The proof of Theorem 2 is the same as for Theorem 1 so it is omitted.

**Remark 5.** Similar to Corollaries 1 and 2, some more easily verified results can be obtained when some special parameters are chosen to ensure the uniformly exponential synchronization of systems (2) and (16).

**Remark 6.** The stability of chaotic systems based on a TS fuzzy model has previously been considered under impulsive control [24, 25, 28–30]. Our approach differs in that we use time-delayed control and impulsive control to derive some new and less conservative criteria to ensure the stability of the origin for system (1). In fact, our results and means are also applicable and effective for stabilizing any unstable fixed point \( \bar{x} \) of a nonlinear system. Moreover, the stability problem for \( \bar{x} \) can be converted to the stability of the origin using the coordinate transformation \( y = x - \bar{x} \).

**Remark 7.** Guan et al. proposed a full delayed feedback controller design method to stabilize the unstable fixed points for continuous chaotic systems and the upper bound of time delay \( \tau \) in the controller was derived [22]. However, it is evident that our results are independent of the feedback time delay \( \tau \). In practice, it can be arbitrarily determined or chosen according to specific requirements. Evidently, our results extend and generalize previous studies.

5. Numerical simulations

In this section, based on the results obtained in the previous sections, some numerical simulations are presented to show the effectiveness of our results.

Consider Chen’s system [34] described by

\[
\begin{align*}
\dot{x}_1 &= -35x_1 + 35x_2, \\
\dot{x}_2 &= -7x_2 + 28x_1 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - 3x_3.
\end{align*}
\] (19)

System (19) can be exactly represented as a TS fuzzy model described by

Rule 1 : If \( x_1 \) is about \( M_1 \), then \( \dot{x} = A_1 x(t) \),

Rule 2 : If \( x_1 \) is about \( M_2 \), then \( \dot{x} = A_2 x(t) \),

(20)
where $x = (x_1, x_2, x_3)^T$, $M_1 = -30$, $M_2 = 30$ and

$$A_1 = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & -30 \\ 0 & 30 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 30 \\ 0 & -30 & -3 \end{pmatrix},$$

and the membership functions are $M_1(x_1) = 1/2 - x_1/2\sigma$ and $M_2(x_1) = 1/2 + x_1/2\sigma$ with $\sigma = 30$.

The overall output of Chen’s system based on TS fuzzy model (20) is inferred as follows and its chaotic behavior is shown in Fig. 1:

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_1(t))A_i x(t),$$

where $\mu_i(x_1(t)) = M_i(x_1(t))/\sum_{i=1}^{2} M_i(x_1(t))$.

First, we design a controller to ensure the stability of the origin for system (19) based on TS fuzzy model (20). Assume that the control roles are as follows.

Control Rule $i$: If $x_1$ is about $M_i$, then $\dot{x}(t) = A_i x(t) + u_i(t)$ and

$$u_i(t) = \sum_{k=1}^{\infty} l_k(t)[F_i x(t) + G_i x(t - \tau)] + \sum_{k=1}^{\infty} \delta(t - (t_k - h))[K_i x(t) + C_i x(t - \tau) - x(t)],$$

for $i = 1, 2$, where $F_1 = F_2 = \text{diag}(-16, -25, -30)$, $G_1 = G_2 = \text{diag}(2, -2, 1)$, $K_1 = K_2 = \text{diag}(0.2, 0.2, 0.2)$ and $C_1 = C_2 = \text{diag}(0.3, 0.3, 0.3)$, $\tau = 1$.

The defuzzified output of (21) is

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_1(t))(A_i + F_i)x(t) + \sum_{i=1}^{2} \mu_i(x_1(t))G_i x(t - \tau), \quad t \neq t_k,$$

$$x(t) = \sum_{i=1}^{2} \mu_i(x_1(t^-))[K_i x(t^-) + C_i x(t^- - \tau)], \quad t = t_k.$$

Let $P = I$ and $\theta = 1$ in Theorem 1. Then, for the above parameters, we obtain $\eta = 0.08$, $\rho = 0.18$ and $\gamma = 20.6738$ in Corollary 1. In addition, from condition (2) in Corollary 1, we have

$$0 < \delta < -\frac{1}{\gamma} \ln(\eta + \rho) = 0.0652.$$
For convenience, we assume that the interval of the impulsive control is a constant $\delta$ and choose $\delta = 0.06$. Hence, from Corollary 1, we know that the origin of system (22) is uniformly asymptotically stable. The simulation results for stabilizing chaotic system (19) based on a TS fuzzy model for $\tau = 1$ and $\delta = 0.06$ are shown in Figs. 2 and 3.

In the second simulation, we study time-delayed impulsive synchronization of the chaotic system (19) based on a TS fuzzy model. The driving system is given in (20). Assume that the state of the driven system is $y(t) = (y_1(t), y_2(t), y_3(t))^T$ and the fuzzy rules of the driven system are represented as follows.

Rule $i$: If $x_1$ is about $M_i$, then

$$\dot{y}(t) = A_i y(t) + u_i(t),$$

$$i = 1, 2, 3.$$ 

\[ (23) \]

Fig. 2. Time response of Chen’s system under controller (21).

Fig. 3. Phase portrait of Chen’s system under controller (21).
where

\[ u_i(t) = \sum_{k=1}^{\infty} l_k(t) [F_i e(t) + G_i e(t - \tau)] + \sum_{k=1}^{\infty} \delta(t - (t_k - h)) [K_i e(t) + C_i e(t - \tau) - e(t)], \]

(24)

Fig. 4. Time response of error system (25).

Fig. 5. Time response of \( x_1(t) \) and \( y_1(t) \) under controller (24).
here \( e(t) = y(t) - x(t) \), \( M_i \) and \( A_i \) are defined in (20), \( F_i, G_i, K_i \) and \( C_i \) are chosen in (21), and the feedback delay is \( \tau = 1 \). The error system can be expressed as follows:

\[
\begin{align*}
\dot{e}(t) &= \sum_{i=1}^{2} \mu_i(x_1(t))[A_i + F_i]e(t) + \sum_{i=1}^{2} \mu_i(x_1(t))G_i e(t - \tau), \quad t \neq t_k, \\
e(t) &= \sum_{i=1}^{2} \mu_i(x_1(t^-))[K_i e(t^-) + C_i e(t^- - \tau)], \quad t = t_k.
\end{align*}
\]  

(25)

Fig. 6. Time response of \( x_2(t) \) and \( y_2(t) \) under controller (24).

Fig. 7. Time response of \( x_3(t) \) and \( y_3(t) \) under controller (24).
Let $P = I$, $\theta = 1$. From Theorem 2, the origin of error system (25) is uniformly asymptotically stable when $\delta = 0.06$, which is shown in Fig. 4. The synchronization between driving system (20) and the driven system (23) is shown in Figs. 5–7.

**Remark 8.** Zheng and Chen studied the stability of Chen’s system and derived the upper bound of the impulsive interval $\delta = 0.0389$ [29]. In this paper, $\delta$ can be increased to 0.0652. In other words, their results are invalid when $\delta = 0.06$. In fact, the stability and synchronization of Chen’s system are not ensured for $\delta = 0.06$ based on the control parameters given in [29].

![Fig. 8. Time response of the controlled Chen’s system for $\delta = 0.06$ based on the control parameters given in [29].](image1)

![Fig. 9. Time response of the synchronized errors for $\delta = 0.06$ based on the control parameters given in [29].](image2)
6. Conclusion

The stability and synchronization of chaotic systems based on a TS fuzzy model were proposed. Some sufficient conditions that lead to the uniformly asymptotical stability and synchronization of chaotic systems were obtained by constructing time-delayed impulsive controllers. It is noted that the impulsive functions are related to the time delay because a DFC technique was applied, which extends previous results. Finally, some numerical simulations were presented to show the effectiveness and feasibility of the methods developed.

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Appendix A. Preliminaries

Consider the following impulsive functional differential equations in which the state variables for the impulses are related to the time delay:

\[
\begin{cases}
    \dot{x}(t) = f(t, x_t), & t \geq t_0, \ t \neq t_k, \\
    x(t_k) = I_k(x(t_{k-}^-)) + J_k(x(t_{k-}^- - \tau)), & k \in \mathbb{Z}^+,
\end{cases}
\]

(A.1)

where \( \tau > 0, x \in R^n, f \in C[R^+ \times D, R^n], I_k, J_k \in C[R^n, R^n], D \) is an open set in \( PC([-\tau, 0], R^n) \) and \( PC([-\tau, 0], R^n) \) denotes the set of piecewise right continuous functions \( \phi : [-\tau, 0] \to R^n \) with the sup-norm \( \| \phi \| = \sup_{-\tau \leq s \leq 0} \| \phi(s) \| \), where \( \| \cdot \| \) is a norm in \( R^n \). For each \( t \geq t_0, x_t \in PC([-\tau, 0], R^n) \) is defined by \( x_t(s) = x(t + s) \) for \( -\tau \leq s \leq 0 \), and the sequence \( \{t_k\} \) satisfies \( 0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots \) and \( \lim_{k \to \infty} t_k = \infty \).

A function \( x(t) \) is called a solution of (A.1) with the initial condition \( x_0 = \varphi \) if it satisfies (A.1) and this initial value, where \( \sigma \geq t_0 \) and \( \varphi \in PC([-\tau, 0], R^n) \).

For system (A.1), we let the following hypotheses hold:

\((H_1)\) For \( t \in [\sigma - \tau, \sigma] \), the solution \( x(t, \sigma, \varphi) \) coincides with the function \( \varphi(t - \sigma) \).

\((H_2)\) For each function \( x(s) : [\sigma - \tau, \infty) \to R^n \), which is continuous everywhere except at the points \( \{t_k\} \) at which \( x(t_k^-), x(t_k^+) \) exist and \( x(t_k^+) = x(t_k^-) \), \( f(t, x_t) \) is continuous for almost all \( t \in [\sigma, \infty) \) and at the discontinuous points \( f \) is right continuous.

\((H_3)\) \( f(t, \phi) \) is Lipschitzian in \( \phi \) in each compact set in \( PC([-\tau, 0], R^n) \).

\((H_4)\) If \( x \in D, I_k \neq 0 \) and \( J_k \neq 0 \), then \( I_k(x) + J_k(x(t - \tau)) \in D \).

\((H_5)\) \( f(t, 0) \equiv 0, I_k(0) \equiv 0 \) and \( J_k(0) \equiv 0 \).

For convenience, we introduce the following notation and definitions:

\[
S(\rho) = \{ x \in R^n : \| x \| < \rho \};
\]

\[
PC(\rho) = \{ \phi \in PC([-\tau, 0], R^n) : \| \phi \| < \rho \};
\]

\[
K = \{ \omega \in C(R^+, R^+) : \text{strictly increasing and } \omega(0) = 0 \};
\]

\[
K_1 = \{ \omega \in C(R^+, R^+) : \text{increasing and } \omega(s) < s \text{ for } s > 0 \};
\]

\[
K_2 = \{ \omega \in C(R^+, R^+) : \text{increasing} \}.
\]

**Definition 1** (Zhang and Sun [9]). The function \( V : [t_0, \infty) \times S(\rho) \to R^+ \) is said to belong to class \( \nu_0 \), if

(1) The function \( V \) is continuous on each of the sets \( [t_{k-1}, t_k) \times S(\rho) \) and \( V(t, 0) \equiv 0 \) for all \( t \geq t_0 \).

(2) \( V(t, x) \) is locally Lipschitzian in \( x \in S(\rho) \).
Then independent impulsive controllers \[2,3,5,6,10–14,24,25,27–31\] or the original delay-dependent impulsive controllers are reduced to delay-independent functions using inequality techniques\[4,7,8\]. Hence, Lemma 1 extends many previous results.

**Definition 2 (Zhang and Sun \cite{9}).** Let \( V \in v_0 \) define the right upper derivative as follows:

\[
D^+ V(t, x(t)) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, x(t + h)) - V(t, x(t)) \right].
\]

To further the study, the following results are necessary.

**Lemma 1 (Zhang and Sun \cite{9}).** Assume that there exist functions \( a, b \in K, V \in v_0, g_1, g_2 \in K_2, g = g_1 + g_2 \) and \( g \in K_1 \) such that

1. \( a(||x||) \leq V(t, x) \leq b(||x||) \) for all \( (t, x) \in [t_0, \infty) \times S(p); \)
2. \( D^+ V(t, x) \leq p(t)c(V(t, x(t))) \) for all \( t \neq t_k \) whenever \( g^{-1}(V(t, x(t))) \geq V(t + s, x(t + s)) \) for all \( s \in [-\tau, 0] \), where \( p \) and \( c : [t_0 - \tau, \infty) \to R^+ \) are locally integrable, and \( g^{-1} \) is the inverse function of \( g; \)
3. \( V(t_k, h_k(x(t_k^-))) \leq g_1(V(t_k^-, x(t_k^-))) + g_2(V(t_k^- - \tau, x(t_k^- - \tau))) \); let \( r = \sup \{k \in Z^+: [t_k - t_{k-1}] < \infty, M_1 = \sup_{s \geq 0} \int_{t_k^+}^{t_k^+ + r} p(s) \, ds < \infty \) and \( M_2 = \inf_{t = 0}^{r} \int_{g(t)}^{g(t)} ds/c(s) > M_1. \)

Then the zero solution of system (A.1) is uniformly asymptotically stable.

**Remark 9.** Lemma 1 provides a novel method for studying the stability of the equilibrium of a generally impulsive system. To the best of our knowledge, many previous studies on the stability of impulsive systems are limited to delay-independent impulsive controllers \[2,3,5,6,10–14,24,25,27–31\] or the original delay-dependent impulsive controllers are reduced to delay-independent functions using inequality techniques \[4,7,8\]. Hence, Lemma 1 extends many previous results to some extent. In addition, it follows from the proof of Lemma 1 \cite{9} that the time-delayed impulsive functions \( I_k(x(t_k^-)) + J_k(x(t_k^- - \tau)) \) in system (A.1) can be generalized as \( H_k(x(t_k^-), x(t_k^- - \tau)) \), where \( H_k \in C[R^n \times R^n, R^n] \) satisfying \( H_k(0, 0) = 0 \) for all \( k \in Z^+ \). In other words, Lemma 1 is also suitable for the following more general impulsive system:

\[
\begin{align*}
\dot{x}(t) &= f(t, x_t), & t \geq t_0, t \neq t_k, \\
x(t_k) &= H_k(x(t_k^-), x(t_k^- - \tau)), & k \in Z^+.
\end{align*}
\]

**Lemma 2 (Huang \cite{32}).** If \( P \in R^{n \times n} \) is a positive-definite matrix and \( Q \in R^{n \times n} \) is a symmetric matrix, then

\[
\lambda_{\min}(P^{-1}Q)x^TPx \leq x^TQx \leq \lambda_{\max}(P^{-1}Q)x^TPx, \quad \forall x \in R^n.
\]

**Lemma 3 (Shen et al. \cite{33}).** Let \( D, S \) and \( P \) be real matrices of appropriate dimensions and \( P \) be positive-definite. Then, for any vectors \( x, y \) with appropriate dimensions, the following inequality holds:

\[
2x^TDy \leq x^TDPSx + y^TS^TP^{-1}Sy.
\]

**Appendix B. Proof of Theorem 1.**

We construct the Lyapunov function as follows:

\[
V(t, x(t)) = x^T(t)P x(t).
\]

It is easy to see that

\[
\dot{\lambda}_1||x(t)||^2 \leq V(t, x(t)) \leq \dot{\lambda}_2||x(t)||^2,
\]

for all \( t \in [-\tau, +\infty) \), which implies that condition (1) in Lemma 1 is satisfied.

For \( t = t_k \) \( \in Z^+ \), we obtain

\[
V(t_k, x(t_k)) = \left[ \sum_{i=1}^{r} h_i(z(t_k^-))I_{i_k}(x(t_k^-), x(t_k^- - \tau)) \right]^T P \left[ \sum_{i=1}^{r} h_i(z(t_k^-))I_{i_k}(x(t_k^-), x(t_k^- - \tau)) \right].
\]
It is easy to see that
\[
\begin{align*}
&\sum_{i=1}^{r} h_i(z(t_k^-)) I_{ik}(x(t_k^-), x(t_k^- - \tau)) \\ &\leq \lambda_2 \left[ \sum_{i=1}^{r} h_i(z(t_k^-)) I_{ik}(x(t_k^-), x(t_k^- - \tau)) \right]^T \times \left[ \sum_{i=1}^{r} h_i(z(t_k^-)) I_{ik}(x(t_k^-), x(t_k^- - \tau)) \right] \\ &= \lambda_2 \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t_k^-)) h_j(z(t_k^-)) [I_{ik}(x(t_k^-), x(t_k^- - \tau))]^T \times [I_{jk}(x(t_k^-), x(t_k^- - \tau))] \\ &\leq \lambda_2 \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t_k^-)) h_j(z(t_k^-)) \left[ \frac{\theta}{2} \|I_{ik}(x(t_k^-), x(t_k^- - \tau))\|^2 + \frac{1}{2\theta} \|I_{jk}(x(t_k^-), x(t_k^- - \tau))\|^2 \right] \\ &= \frac{\theta^2 + 1}{2\theta} \lambda_2 \sum_{i=1}^{r} h_i(z(t_k^-)) [z_{ik} \|x(t_k^-)\|^2 + \beta_{ik} \|x(t_k^- - \tau)\|^2] \\ &\leq \lambda_2 \left( \frac{\theta^2 + 1}{2\theta} \right) \lambda_2 \sum_{i=1}^{r} h_i(z(t_k^-)) [z_{ik} \|x(t_k^-)\| + \beta_{ik} \|x(t_k^- - \tau)\|] \\ &\leq \lambda_2 \left( \frac{\theta^2 + 1}{2\theta} \right) \lambda_2 \sum_{i=1}^{r} h_i(z(t_k^-)) [z_{ik} x(t_k^-) + \beta_{ik} x(t_k^- - \tau)] \\ &\leq \Gamma V(t_k^-), x(t_k^-)) + \Lambda V(t_k^- - \tau, x(t_k^- - \tau)). \quad \text{(B.2)}
\end{align*}
\]

Let
\[
\begin{align*}
g_1(u) &= \Gamma u, \quad g_2(u) = Au \quad \text{for all } u \in R^+. \quad \text{(B.3)}
\end{align*}
\]

Then we have
\[
\begin{align*}
g(u) &= g_1(u) + g_2(u) = (\Gamma + A)u \quad \text{for all } u \in R^+ \quad \text{(B.4)}
\end{align*}
\]

and
\[
\begin{align*}
g^{-1}(u) &= \frac{1}{\Gamma + A} u \quad \text{for all } u \in R^+.
\end{align*}
\]

It is easy to see that \(g_1, g_2 \in K_2\) and from condition (1) in Theorem 1, we know that \(g \in K_1\).

For \(t \neq k\), the derivative of \(V(t)\) along the solution of system (9) is
\[
D_+ V(t, x(t)) = \sum_{i=1}^{r} h_i(z(t))x^T(t) [(A_i + F_j)^T P + P(A_i + F_j)]x(t) + 2 \sum_{i=1}^{r} h_i(z(t))x^T(t - \tau)G_i^T P x(t).
\]

From Lemma 3, we obtain
\[
2x^T(t - \tau)G_i^T P x(t) \leq x^T(t - \tau)Px(t - \tau) + x^T(t)PG_iP^{-1}G_i^T P x(t).
\]

For any solution of system (9), if
\[
g^{-1}(V(t, x(t))) \geq V(t + s, x(t + s)),
\]

for \(s \in [-\tau, 0]\), especially for \(s = -\tau\), if
\[
V(t - \tau, x(t - \tau)) \leq \frac{1}{\Gamma + A} V(t, x(t)),
\]

then, from Lemma 2, we have
\[
D_+ V(t, x(t)) \leq \sum_{i=1}^{r} h_i(z(t))x^T(t) [(A_i + F_j)^T P + P(A_i + F_j)]x(t)
\]
\[
+ \sum_{i=1}^{r} h_i(z(t)) \left[ \frac{1}{\Gamma + A} x^T(t)Px(t) + x^T(t)PG_iP^{-1}G_i^T P x(t) \right].
\]
\[ = \sum_{i=1}^{r} h_i(z(t))x^T(t) \left[ (A_i + F_i)^T P + P(A_i + F_i) + \frac{1}{I + A} P + PG_iP^{-1}G_i^TP \right] x(t) \leq \gamma V(t, x(t)). \] (B.5)

Let \( p(s) \equiv 1 \) and \( c(s) = \gamma s \). Then condition (2) in Lemma 1 holds.

In addition, from the definitions of \( p(s) \), \( c(s) \) and \( g(s) \), we have

\[ M_1 = \sup_{t \geq 0} \int_{t+\delta}^{t+\delta} p(s) ds = \delta \]

and

\[ M_2 = \inf_{q > 0} \int_{g(q)}^{q} \frac{1}{c(s)} ds = -\frac{1}{\gamma} \ln(I + A). \]

It follows from condition (2) in Theorem 1 that \( M_2 > M_1 \), which implies that condition (3) in Lemma 1 is also satisfied.

From the above discussion and Lemma 1, the trivial solution \( x(t) \equiv 0 \) of system (9) is uniformly asymptotically stable. This completes the proof of Theorem 1. \( \square \)

References