An $n$ species stochastic impulsive migration Lotka–Volterra model with Markovian switching in $N$ different patches is presented and studied in this paper. By constructing appropriate Lyapunov functions, some sufficient conditions on the global positivity, the ultimate boundedness in mean and the extinction in mean are established. Real world examples are provided to illustrate the validity of our results. A discussion is given in the end.

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1. Introduction

Owing to natural enemies, competition, seasonal alternatives or deterioration of patches of the environment, species dispersal (or migration) to and from two or more patches is common. In general, dispersal is mainly represented by one or other of the following three categories: (i) Dispersal occurs at any time, simultaneously between any two patches, i.e. continuous bidirectional dispersal, such as activities of some animals on their habitat which is separated by road or stream; activities of some birds between two neighborhood hills and so on. (ii) Dispersal occurs at some fixed time, simultaneously between any two patches, i.e. impulsive bidirectional dispersal. For example, in mating season, some male will search for mate among patches at some transitory time slots; Graziers feed their cattle or sheep group by group in one meadow, etc. All of these cases can be referred as the type of impulsive bidirectional dispersal. (iii) Dispersal is a migration, i.e. impulsive unilateral diffusion. Examples include the migration of some kinds of organisms from cold patches to warm patches with seasonal alternative, migration of some kinds of fish from ocean to their birthplace and so on.
some population dynamical models with impulsive bidirectional dispersal have been proposed and studied (see Wang and Liu, 2007; Jiao et al., 2011, 2009; Shao, 2010 and references cited therein). For instance in Wang and Liu (2007), the authors studied the following autonomous impulsive diffusion single species model:

\[
\begin{align*}
\frac{dY_1(t)}{dt} &= r_1 Y_1(t) \ln \left( \frac{k_1}{Y_1(t)} \right), \quad t \neq n\tau, \\
\frac{dY_2(t)}{dt} &= r_2 Y_2(t) \ln \left( \frac{k_2}{Y_2(t)} \right), \quad t = n\tau,
\end{align*}
\]

\[\Delta Y_1 = d_1 (Y_2(t) - Y_1(t)), \quad \Delta Y_2 = d_2 (Y_1(t) - Y_2(t)), \quad t = n\tau,\]

where \(r_1\) and \(k_i\) (\(i = 1, 2\)) are the intrinsic growth rate and the carrying capacity of the \(i\)th patch, respectively. \(d_i\) (\(i = 1, 2\)) is the dispersal rate in the \(i\)th patch. The pulse diffusion occurs at every \(\tau\) period (\(\tau\) is a positive constant). Obviously, in this model, species \(Y_1\) inhabits two patches and at each pulse disperses from each of the two patches to the other, i.e. impulsive bidirectional dispersal.

However, in all of these investigated dispersal models considered so far, there are few papers to consider the total impulsive migration system, i.e. impulsive unilateral diffusion (type (iii) system). Practically, in the real ecological system, with seasonal alternative, some kinds of organisms will migrate from cold patches (or food resource poor patches) to warm patches (or food resource rich patches) in search of a better habitat to inhabit or breed, fish will go back from ocean to their birthplace to spawn and so on. Therefore, it is a very basic problem to research this kind of impulsive migration systems. In our previous work (Liu et al., 2010, 2011; Zhang et al., 2011), a two-patch impulsive diffusion periodic single-species logistic model (see Liu et al., 2010) and two-patch prey impulsive unilateral diffusion periodic predator–prey models (see Liu et al., 2011; Zhang et al., 2011) have been proposed and studied and some interesting results have been established.

If patches are used for brief stays to supplement energy for the migrating species or the patch that the species moved back to is not where it had just come from, multi-patch dynamical behaviors of the migrating species should be studied. Hence, in this paper, we generalize our previous two-patch impulsive diffusion model to \(N\) patches. When the patch choice of the species is stochastic, it can be regarded as a taking finite values with finite states Markov chain system. Therefore, the purpose of this paper is to research the stochastic impulsive migration model with Markovian switching. We mainly study the effects of the intrinsic growth rate and the inner and outer competitive coefficients to the population's survival and extinction.

This paper is organized as follows. In Section 2, we detail our model and present some preliminaries. In Section 3, we state and prove the global positivity, the ultimate boundedness in mean and the extinction in mean of system (2.2). Real world examples are provided to illustrate the validity of our results in Section 4. Finally, a discussion is given.

2. The model

Before exhibiting our model, we first of all interpret some nomenclature which will be used throughout the paper.

Nomenclature

- \(\mathbb{S}\) a finite positive integer space with \(\mathbb{S} = \{1, 2, \ldots, N\}\)
- \(\mathbb{Z}_+\) positive integer space
- \(\mathbb{R}\) real number space
- \(\mathbb{R}_+\) positive real number space
- \(A\) a time interval with \(A = [t_0^+, \infty)\)
- \(\mathbb{R}^n\) \(n\)-dimensional real number space
- \(x\) \(n\)-dimensional column vector, its transpose is denoted by \(x^T\)
- \(\mathbb{R}_+^n\) a positive cone with \(\mathbb{R}_+^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_m > 0, \text{ for all } 1 \leq m \leq n\}
- \(A = (a_{ij})_{n \times n}\) is an \(n \times n\) real matrix, its transpose is denoted by \(A^T\)
- \(\lambda_{\text{max}}(W)\) the largest eigenvalue for an \(n \times n\) real symmetric matrix \(W\) with \(\lambda_{\text{max}}(W) = \sup_{x \neq 0} \frac{1}{\|x\|} \|\text{Re}(x^T W x)\|\)
- \(\cdot\) the Euclidean norm
- \(I\) unit matrix
- \(\emptyset\) empty set

Let \((r(t), t \geq 0)\) be a right-continuous Markov chain on the probability space \((\mathcal{Q}, \mathcal{F}, \mathbb{P})\) taking values in the finite space \(\mathbb{S} = \{1, 2, \ldots, N\}\) with generator \(\Pi = (\pi_{ij})_{n \times n}\) given by

\[
\Pi[r(t) + A] = \mathbb{P}[r(t) = i | r(t + \Delta t) = j],
\]

where \(\Delta t > 0, \lim_{\Delta t \to 0} \mathbb{P}[r(t + \Delta t) = i | r(t + \Delta t) = j] = 0\). Here \(\pi_{ij} \geq 0\) \((\forall i, j \in \mathbb{S}, i \neq j)\) is the transition rate from mode \(i\) to mode \(j\), while

\[
\pi_{ii} = - \sum_{j \neq i} \pi_{ij}.
\]

Consider a system of \(n\) species consisting of \(N\) different habitats\((\text{patches})\). Suppose that the \(N\) different patches have similar sizes (which aims to guarantee the population densities do not change instantaneously within the different patches) and the migration of the species to the \(N\) patches is stochastic and instantaneous, our model then takes the form

\[
\begin{align*}
\mathbf{x}(t) &= \mathbf{g}(\mathbf{x}(t)) \mathbf{x}(t), \quad t \in (t_k, t_{k+1}], \\
\mathbf{x}(t_k^+) &= \mathbf{G}(\mathbf{x}(t_k), \mathbf{x}(t_k)) \mathbf{x}(t_k), \\
\mathbf{x}(t_k^-) &= \mathbf{x}(0), \quad t \in (t_k, t_{k+1}).
\end{align*}
\]

where

- \(\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T \in \mathbb{R}^n\), each component \(x_m(t) (m = 1, 2, \ldots, n)\) of \(\mathbf{x}(t)\) represents the population density of the \(m\)th species. \(x_m(t_0^+) = x_m > 0\) is the initial population density of the species \(m\), where the initial time is \(t_0 = 0\).
- For a fixed \(i \in \mathbb{S}\), each element \(b_{mi}(i)\) of \(\mathbf{b}(i) = (b_1(i), b_2(i), \ldots, b_N(i))^T\) represents the intrinsic growth rate of the \(m\)th species in patch \(i\), \(a_{ij}(i)\) are the coefficients which describe the relationships between species \(j\) and species \(i\) in the \(i\)th patch, where \(j = l\) denotes the inner relationship of the \(j\)th patch or \(l\)th patch.
- \(\mathbf{G}(i,j) = \text{diag}(g_{1}(i,j), g_{2}(i,j), \ldots, g_{n}(i,j))\), each component \(g_{mi}(i,j)\) of \(\mathbf{G}(i,j)\) denotes the remaining rate of species \(m\) after the migration from patch \(i\) to patch \(j\).
- \([t_k, k = 1, 2, \ldots]\) is a given positive number sequence satisfying \(0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots\) with \(\lim_{k \to \infty} t_k = \infty\) and denotes the \(k\)th switched time, i.e. the time of transition of the mode from \(r(t_k) = i\) to \(r(t_{k+1}) = j \neq i\), with \(t_k^+ = \lim_{\Delta t \to 0} r(t_k + \Delta t)\)

Remark 2.1. Actually, different values of the components of the matrix \(\mathbf{A}(i) = (a_{ij}(i))_{n \times n}\) imply different relationships between species \(j\) and species \(l\):

1. \(a_{ij} < 0\) and \(a_{lj} < 0\), i.e. one species’ activity has a disadvantageous effect on the other. This implies a competition type relationship between species \(j\) and species \(l\).
2. \(a_{ij} < 0\) but \(a_{lj} > 0\) (or \(a_{lj} < 0\) but \(a_{ij} > 0\)), i.e. one species’ activity has a disadvantageous effect on the other, but an
advantageous effect on itself. This represents a predator–prey type relationship between species $j$ and species $l$.

(3) $a_{ij} > 0$ and $a_{ji} > 0$, i.e. one species’ activity has an advantageous effect on the other. This implies a cooperative relationship between species $j$ and species $l$.

We have supposed that the system is composed of $N$ different patches. When $t \in (t_k, t_{k+1}]$, all the species live in patch $i$, because of the change of the environment, the species will migrate to patch $j$, the remaining rate after the migration is $G(ij)$. Then the species will live in patch $j$ during the period $t \in (t_{k+1}, t_{k+2}]$. When the environment changes again, they will migrate to another patch, which admits a Markov switching property.

The following two definitions are necessary for the study of our main results.

**Definition 2.1.** System (2.2) is said to be ultimately bounded in mean if there is a positive constant $H$ such that for any initial value $x_0 \in \mathbb{R}^n_+$, solution $x(t)$ of system (2.2) has the property that

$$\lim_{t \to \infty} E|x(t)| \leq H.$$  

**Definition 2.2.** System (2.2) is said to be extinct in mean if for any initial value $x_0 \in \mathbb{R}^n_+$, solution $x(t)$ of system (2.2) has the property that

$$\lim_{t \to \infty} E|x(t)| = 0.$$  

Throughout this paper, the following assumption is of major importance.

(H) The positive remaining rate $G(t_{k+1}, t_{k+2}, r_{k+1})$ of the migration satisfies

$$g_m(r_{k+1}, r_{k+2}) \leq \beta_{m}^k$$  

and

$$\prod_{k=1}^{\infty} \beta_{m}^k \leq M < \infty,$$

for any $k \in \mathbb{Z}^+$, $m = 1, 2, \ldots, n$ and $\beta_{m}^k = \max_{1 \leq m \leq n} \beta_{m}^k$. Especially, $G(\cdot) = I$ is defined.

**Remark 2.2.** Under normal circumstances, the remaining rate $G(ij)$ of the species is less than unity after migration from patch $i$ to patch $j$ because of the death during the migration caused by starvation, tiredness, hunting or other disadvantageous factors. However, in this paper, we allow that the remaining rate $g_m(ij)$ of some species $m$ is larger than unity, which may be caused by human intervention. We only need that the assumption $\prod_{k=1}^{\infty} \beta_{m}^k \leq M < \infty$ is satisfied.

3. Main results

In this section, we study the global positivity, the ultimate boundedness in mean and the extinction in mean of system (2.2).

Since the $m$th state $x_m(t)$ of system (2.2) is the population density of the $m$th species, it should be nonnegative. Moreover, the coefficients of system (2.2) do not satisfy the linear growth condition though they are locally Lipschitz continuous, so the solutions of system (2.2) may explode to infinity at a finite time (Mao, 1994, 1999). Therefore, it is necessary to establish some conditions under which the solutions of system (2.2) are not only positive but also not explode to infinity at any finite time.

**Theorem 3.1.** Assume that assumption (H) holds. In addition, if there are positive numbers $c_1(i), c_2(i), \ldots, c_d(i), i \in \mathbb{S}$ such that

$$(3.1) \quad z_{\max}^{++} C(i) A(i) + A(i) C(i) \leq 0,$$

where $C(i) = \text{diag}(c_1(i), c_2(i), \ldots, c_d(i))$. Then, for any system parameter $\mathbf{b}(\cdot)$ and any given initial value $x_0 \in \mathbb{R}^n_+$, there is a unique solution $x(t)$ of system (2.2) defined on $t \in \Lambda$, and the solution will remain in $\mathbb{R}^n_+$ with probability one, namely $x(t) \in \mathbb{R}^n_+$ for all $t \in \Lambda$ almost surely.

**Proof.** Since the coefficients of the system are locally Lipschitz continuous in $x$, it is known (see e.g. Mao, 1999, Theorem A2) that for any given initial value $x_0 \in \mathbb{R}^n_+$, there is a unique maximal local solution $x(t)$ on $t \in (t_0^+, \rho_x)$, where $\rho_x$ is the explosion time. To show this solution is global, we need only to prove that $\rho_x = \infty$ a.s.

From system (2.2), we know that the solution $x(t)$ is positive for any given initial value $x_0 \in \mathbb{R}^n_+$ on $t \in [t_0^+, t_1]$, hence $x(t)$ is also positive. Therefore, we can obtain that the solution $x(t)$ of system (2.2) is positive for all $t \in [t_0^+, \rho_x]$ by deduction.

Now, we will prove that $\rho_x = \infty$ a.s. Let $k_0 > 0$ be sufficiently large such that each component of $x_0$ is no larger than $k_0$. For each integer $k \geq k_0$, define the stopping time

$$\rho_k = \rho_x \wedge \inf\{t \in [t_0^+, \rho_x) : x_m(t) \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Obviously, $\rho_k$ is increasing as $k \to \infty$. Set $\rho_\infty = \lim_{k \to \infty} \rho_k$ hence $\rho_\infty \leq \rho_x$ a.s.

Define a function $V : \mathbb{R}^n_+ \times \mathbb{S} \to \mathbb{R}_+$ by

$$V(x, i) = \sum_{m=1}^{n} c_m(i) x_m.$$  

For any $k \geq k_0$, there exists $i_k$ such that $\rho_k \in (t_{k-1}, t_k]$. By the generalized Itô formula, we get for any $t \in (t_{k-1}, t_k]$

$$EV(x(t \wedge \rho_k), t(t \wedge \rho_k)) = EV(x(t \wedge \rho_k^+, t(t \wedge \rho_k^+)) + E \int_{t \wedge \rho_k}^{t \wedge \rho_k^+} LV(x(s), r(s)) ds.$$  

(3.4)

Here $LV$ is a mapping from $\mathbb{R}^n_+ \times \mathbb{S}$ to $\mathbb{R}$ defined by

$$LV(x, i) = \mathbf{x}^T C(i) \mathbf{b}(i) + \mathbf{x}^T A(i) x(i) + \mathbf{x}^T A(i) C(i) \mathbf{x} + \sum_{j=1}^{n} \pi_j V(x, j).$$  

(3.5)

Setting $\pi = \max_{i \in \mathbb{S}} (\sum_{j \in \mathbb{S}} |\pi_j| : j \in \mathbb{S})$. It follows from condition (3.1) that there is a constant $K_1 > 0$ such that

$$LV(x, i) \leq K_1 |x| + \pi \sum_{j=1}^{n} V(x, j).$$

Let

$$K_2 = \max\{c_m(i)/c_m(j) : 1 \leq m \leq n \text{ and } i, j \in \mathbb{S}\}$$

and

$$K_3 = \min\{c_m(i) : 1 \leq m \leq n, i \in \mathbb{S}\}.$$  

Then, by the definition of $V$, for any $i, j \in \mathbb{S}$, we have

$$K_2 V(x, i) = \sum_{m=1}^{n} K_2 c_m(i) x_m \geq \sum_{m=1}^{n} c_m(j) x_m = V(x, j),$$  

(3.6)

and

$$|x| \leq \sum_{m=1}^{n} x_m = \sum_{m=1}^{n} c_m(i) x_m / c_m(i) \leq V(x, i) / K_3.$$  

Hence, there is a constant $K_4 > 0$ such that

$$LV(x(t \wedge \rho_k), t(t \wedge \rho_k)) \leq LV(x(t \wedge \rho_k^+, t(t \wedge \rho_k^+)) + K_4 \int_{t \wedge \rho_k}^{t \wedge \rho_k^+} LV(x(s \wedge \rho_k), t(s \wedge \rho_k)) ds.$$  

(3.7)

Therefore, we have

$$EV(x(t \wedge \rho_k), t(t \wedge \rho_k)) \leq EV(x(t \wedge \rho_k^+, t(t \wedge \rho_k^+)) + K_4 \int_{t \wedge \rho_k}^{t \wedge \rho_k^+} EV(x(s \wedge \rho_k), t(s \wedge \rho_k)) ds.$$  

(3.8)
Using the well-known Gronwall inequality, we get
\[ EV(x(t) : \rho_k) = EV(x(t_{\tau_k}^-)) \leq EV(x(t_{\tau_k}^-), r(t_{\tau_k}^-)) \exp(K_d(t_{\tau_k}^-)) \]
\[ \leq \beta^{k-1} EV(x(t_{\tau_{k-1}}), r(t_{\tau_{k-1}})) \exp(K_d(t_{\tau_{k-1}})). \]

(3.9)

Setting \( q(t) = EV(x(t), r(t)) \) and taking the Dini-right-upper derivative of \( q(t) \), we have
\[ D^+ q(t) \leq K d q(t) \quad \text{for} \quad t \in (\tau_{k-1}, \tau_k], \quad k = 1, 2, \ldots. \]
\[ q(t_{\tau_k}^-) \geq \beta^{k-1} q(t_{\tau_{k-1}}), \]
\[ q(t_{\tau_k}) = EV(x_0, r_0). \]

Eq. (3.10) implies that
\[ q(t) \leq \beta^{k-1} q(t_{\tau_k^-}), \quad t \in (\tau_{k-1}, \tau_k], \quad k = 1, 2, \ldots. \]

Eqs. (3.9), (3.11) and assumption (H) yield
\[ EV(x(t) : \rho_k) = \sum_{i=1}^{k-1} \beta^i EV(x(t_{\tau_i}), r(t_{\tau_i})) \exp(K_d(t_{\tau_i})) \leq M \cdot EV(x_0, r_0) \exp(K_d(t)) \]
\[ \cdot EV(x_0, r_0) \exp(K_d(t)). \]

(3.12)

For all \( t \in A \). Hence we have
\[ M \cdot EV(x_0, r_0) \exp(K_d(t)) \geq EV(x(t), \rho_k), \quad t \in A, \quad \rho_k \leq \rho_k \leq \rho_k \leq t(K_d k). \]

(3.13)

Let \( k \to \infty \), then \( t \to \infty \), one can obtain that
\[ \rho_k = \infty \quad \text{a.s.} \]

(3.14)

That is, \( \rho_k = \infty \quad \text{a.s.} \). This completes the proof. \( \square \)

It turned out from Theorem 3.1 that under assumption (H) and condition (3.1), the solution of system (2.2) will be positive with probability one. These properties of positivity and nonexplosion are essential for a population system. On the other hand, due to the limit of the resources, the property of the ultimate boundedness is more desired, which can be guaranteed by the following result.

**Theorem 3.2.** Assume that assumption (H) holds. In addition, condition (3.1) is strengthened by
\[ -\lambda \leq \max_{i \in S} \lambda_{\max}^+ C(i) + A^T(i) C(i) < 0. \]

Then there exists a positive number \( H > 0 \) such that
\[ \sup_{t \to \infty} EV(x(t)) \leq H, \]
for any solution \( x(t) \) of system (2.2) with the initial value \( x(0) \in \mathbb{R}_0^+ \).

**Proof.** By Theorem 3.1, the unique solution \( x(t) \) of system (2.2) will remain in \( \mathbb{R}_0^+ \) for all \( t \in A \) with probability one. Define
\[ V(x(t), i) = e^{C(i) X}, \quad (x(t), i) \in \mathbb{R}_0^+ \times A \times S, \]

(3.17)

where \( C(i) = (c_1(i), c_2(i), \ldots, c_n(i))^T \). For any \( t \in A \), there exists a positive integer \( k \in Z_+ \) such that \( t \in (\tau_k, \tau_{k+1}) \). By the generalized Itô formula, we have
\[ EV(x(t), r(t)) = EV(x(\tau_{k+1}^-), r(\tau_{k+1}^-)) + E \int_{\tau_{k+1}^-}^t LV(x(s), r(s)) ds, \]

(3.18)

where \( LV \) is a mapping from \( \mathbb{R}_0^+ \times A \times S \) to \( \mathbb{R} \) defined by
\[ LV(x, i) = V(x, i) + e^{C(i) X} I(x, b(i)) + x^T C(\iota X) A(i) x + \sum_{j=1}^n \pi_j V(x, j). \]

By condition (3.15), there is
\[ x^T C(\iota X) A(i) x = x^T C(\iota X) A(i) x + A^T(i) C(i) x / 2 \leq -\lambda_x x^2 / 2. \]

Therefore,
\[ LV(x, i) \leq e^{C(i) X} + \pi_j C(\iota X) + \pi_j C(i) x / 2 \leq K_c e^{x^2 / 2}. \]

(3.19)

where \( \pi_j = \sum_{i \neq j} \pi(i, j) \) and \( K_c = \max \{ C(i) + C(i) b(i) + \pi_j x^2 / (2\lambda) \} \).

Denote
\[ \tilde{R} = \max \left\{ \sup_{k \geq 1} \sum_{i \in S} \beta^i, 1 \right\}. \]

it follows from (3.18) and (3.19) that
\[ EV(x(t), r(t)) \leq EV(x(\tau_k^-), r(\tau_k^-)) + E \int_{\tau_k^-}^t K_c e^{x^2 / 2} ds \]
\[ \leq \beta^{k-1} EV(x(t_{\tau_{k-1}}), r(t_{\tau_{k-1}})) + E \int_{\tau_{k-1}}^t K_c e^{x^2 / 2} ds \]
\[ \leq \beta^{k-1} EV(x(t_{\tau_k^-}), r(t_{\tau_k^-})) + E \int_{\tau_{k-1}}^t K_c e^{x^2 / 2} ds \]
\[ + E \int_{\tau_{k-1}}^t K_c e^{x^2 / 2} ds \]
\[ \leq \beta^{k-1} EV(x(t_{\tau_k^-}), r(t_{\tau_k^-})) + K \tilde{R} \int_{t_{\tau_{k-1}}}^t K_c e^{x^2 / 2} ds \]
\[ \leq M \cdot EV(x_0, r_0) + K \tilde{R} \int_{t_{\tau_{k-1}}}^t K_c e^{x^2 / 2} ds \]
\[ \leq M \cdot EV(x_0, r_0) + \tilde{R} K_c e^{x^2 / 2}. \]

(3.20)

Noting that
\[ \sup_{t \to \infty} EV(x(t)) \leq \sup_{t \to \infty} \sup_{s \to \infty} EV(x(s), r(t)) \]
\[ \leq K_c e^{x^2 / 2}, \]

therefore we obtain
\[ \lim_{t \to \infty} EV(x(t)) \leq \lim_{t \to \infty} \sup_{s \to \infty} EV(x(s), r(t)) \]
\[ \leq \frac{\tilde{R} K_c e^{x^2 / 2}}{K_c}, \]

which means (3.16) holds. The proof of Theorem 3.2 is completed. \( \square \)

Theorem 3.2 shows that the property of the ultimate boundedness of the species is because of the limit of the resources. But species may also become extinct under some special circumstances such as resource shortages or major environmental changes. Hence, for system (2.2), we have the following result that all the species will be extinct in mean.

**Theorem 3.3.** Assume that assumption (H) and condition (3.1) hold. In addition, condition
\[ -\gamma \leq \max_{i \in S} \left( \max_{1 \leq m \leq n} \{ b_0(i) + \pi_0(1 - K_d) \} \right) < 0. \]

is satisfied. Then any solution \( x(t) \) of system (2.2) will be extinct in mean with the initial value \( x(0) \in \mathbb{R}_0^+ \).

**Proof.** It follows from Theorem 3.1 that the unique solution \( x(t) \) of system (2.2) will remain in \( \mathbb{R}_0^+ \) for all \( t \in A \) with probability one. Define
\[ V(x, i) = \sum_{m=1}^n c_m(i) x_m, \quad (x, i) \in \mathbb{R}_0^+ \times S. \]

(3.21)

For any \( t \in A \), there exists a positive integer \( k \in Z_+ \) such that \( t \in (\tau_k, \tau_{k+1}) \). By the generalized Itô formula, we have
\[ EV(x(t), r(t)) = EV(x(\tau_{k+1}^-), r(\tau_{k+1}^-)) + E \int_{\tau_{k+1}^-}^t LV(x(s), r(s)) ds. \]

(3.23)
Here \( LV \) is a mapping from \( \mathbb{R}_+^n \times \mathbb{S} \) to \( \mathbb{R} \) defined by (3.5). By conditions (2.1), (3.1) and (3.21), for any \( t \in (\tau_{k-1}, \tau_k) \), we have the following estimation:

\[
LV(\mathbf{x}, t) = \mathbf{x}^T C(\mathbf{i}) b(\mathbf{i}) + \mathbf{x}^T [C(\mathbf{i}) A(\mathbf{i}) + A^T(\mathbf{i}) C(\mathbf{i})] \mathbf{x} / 2 + \sum_{j=1}^{N} \pi_j V(\mathbf{x}_j) 
\]

\[
\leq \sum_{m=1}^{n} b_m(i) y_m(i) x_m + \sum_{j=1}^{N} \pi_j V(\mathbf{x}_j) + \pi_i V(\mathbf{x}_i) 
\]

\[
\leq \max \{b_m(i)\} V(\mathbf{x}_i) + \sum_{j=1}^{N} \pi_j K_2 V(\mathbf{x}_j) + \pi_i V(\mathbf{x}_i) 
\]

\[
\leq \max \{b_m(i)\} V(\mathbf{x}_i) - \pi_i K_2 V(\mathbf{x}_j) + \pi_i V(\mathbf{x}_i) 
\]

\[
\leq -\gamma V(\mathbf{x}_i). 
\]

Therefore, we have

\[
EV(\mathbf{x}(t), r(t)) \leq EV(\mathbf{x}(t_{\tau_{k-1}^-}), r(\tau_{k-1}^-)) + \int_{\tau_{k-1}^-}^{t} EV(\mathbf{x}(s), r(s)) \, ds, 
\]

(3.25)

by inequality (3.24). Using the well-known Gronwall inequality, we obtain

\[
EV(\mathbf{x}(t), r(t)) \leq EV(\mathbf{x}(t_{\tau_{k}^+}), r(\tau_{k}^+)) \exp \{-\gamma (t - \tau_{k-1}^-)\} 
\]

\[
\leq \beta^{k-1} EV(\mathbf{x}(t_{\tau_{k-1}^-}), r(\tau_{k-1}^-)) \exp \{-\gamma (t - \tau_{k-1}^-)\}. 
\]

(3.26)

Setting \( q(t) = EV(\mathbf{x}(t), r(t)) \) and taking the Dini-right-upper derivative of \( q(t) \), we have

\[
D^+ q(t) \leq -\gamma q(t) \quad \text{for} \quad t \in (\tau_{i-1}, \tau_i), \quad i = 1, 2, \ldots, 
\]

\[
q(\tau_{i-1}^+) \leq \beta i q(\tau_i^-), 
\]

\[
q(\tau_0^-) = EV(\mathbf{x}_0, r_0). 
\]

(3.27)

Eq. (3.27) implies that

\[
q(t) \leq \beta^{k-1} q(\tau_{i-1}^-) \exp \{-\gamma (t - \tau_{i-1}^-)\}, \quad t \in (\tau_{i-1}, \tau_i), \quad k = 1, 2, \ldots. 
\]

(3.28)

It follows from (3.26), (3.28) and assumption (H) that

\[
EV(\mathbf{x}(t), r(t)) \leq \sum_{k=1}^{n} \beta^k EV(\mathbf{x}_0, r_0) \exp \{-\gamma t\} \leq M 
\]

\[
\cdot EV(\mathbf{x}_0, r_0) \exp \{-\gamma t\}, 
\]

(3.29)

for all \( t \in A \). Noting that

\[
|x(t)| \leq V(x(i)/K_3, 
\]

therefore we have

\[
\limsup_{t \to \infty} EV(\mathbf{x}(t)) \leq \limsup_{t \to \infty} M \cdot EV(\mathbf{x}_0, r_0) \exp \{-\gamma t\} / K_3 = 0. 
\]

This completes the proof. \( \square \)

**Remark 3.1.** It can be seen from system (2.2) that the coefficients \( b(r(t)) \) and \( A(r(t)) \) are only associated with the changing state \( r(t) = i \in \mathbb{S} \), but not related to the changing time \( t \). However, for a fixed patch \( i \), the intrinsic growth rate \( b(i) \) and the competitive coefficients \( A(i) \) may not always be invariable with the changing time \( t \). Hence, considering the coefficients like \( b(\tau r(t)) \) and \( A(\tau r(t)) \) will be more close to the real world, which will be studied in our future work.

4. Examples

Bird migrations are very common and are made in response to changes in food availability, habitat or weather. The most common pattern involves flying north in the spring to breed in the temperate or Arctic summer and returning in the fall to wintering grounds in warmer regions to the south. Lots of kinds of birds have such behaviors, for example barn swallow, cuckoo, mallard, swan, goose and so on.

Many types of fish also have migratory behaviors in nature. Fish usually migrate because of diet or reproductive needs or some other unknown reasons. Examples include forage fish, which often make great migrations between their spawning, feeding and nursery grounds; six species of Pacific salmon: chinook, coho, sockeye, chum, pink, and Cherry, which are the best-known anadromous fish; a number of large marine fishes, such as the tuna, migrate north and south annually, following temperature variations in the ocean and so on.

Mathematically, all of these species' dynamical behaviors with migration mentioned above can be modeled simply by a class of differential equations with impulsive diffusion, for example the model (2.2) investigated in the present paper. If the migratory species have three or more patches that can be taken (for a brief stay to supplement energy or inhabitation), dynamical behaviors of the species should be considered in \( N (N \geq 3) \) patches, and the choice of patches will be stochastic. As far as the number of the species is concerned, two or more species are also involved in our model. Single species impulsive migration phenomenon (such as barn swallow) and two species competition impulsive migration phenomenon (such as chub and Bighead Carp Aristichthys nobilis, which have similar foods, habitats and migration time) and such species' dynamics can be modeled as follows.

To illustrate the validity of the results from our mathematical model, here we consider a three species predator–prey system with Markovian switching in two different patches; in which, two prey species and one predator species are denoted by \( x_1(t), x_2(t), \) and \( x_3(t) \), respectively. Species \( x_1 \) prey on species \( x_2 \) and \( x_3 \), however, species \( x_1 \) and \( x_2 \) are reciprocal. Let \( \mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T \). Then the impulsive migration system with Markovian switching can be written as

\[
\mathbf{x}(t) = \text{diag}(x_1(t), x_2(t), x_3(t)) [b(r(t)) + A(r(t)) \mathbf{x}(t)]. 
\]

\[
\mathbf{x}(\tau_k^+) = G(r(\tau_k), \tau_k^+ \mathbf{x}(\tau_k^-), 
\]

\[
\mathbf{x}(\tau_k^+) = x_0 > 0, \quad r(\tau_k^+) = r_0 \in \mathbb{S}, 
\]

(4.1)

where \( r(t) \in \mathbb{S} = \{1, 2\} \) is a right-continuous Markov chain generated by \( \Pi = \left( \begin{array}{cc} 0.6 & 0.4 \\ 0.4 & 0.6 \end{array} \right) \). Hence we have the transfer matrix \( P = \left( \begin{array}{cc} 0.4 & 0.6 \\ 0.6 & 0.4 \end{array} \right) \). Choose \( \tau_{k-1} - \tau_{k-1} = 0.05 \times 10 \text{ years} = 0.5 \text{ year} \) (10 years is the time unit we choose in this example and there are two migrations by the birds every year) and

\[
A(1) = \left( \begin{array}{ccc} -0.5 & 0.4 & -0.4 \\ 0.3 & -0.4 & -0.3 \\ 0.3 & 0.2 & -0.2 \end{array} \right), \quad A(2) = \left( \begin{array}{ccc} -0.6 & 0.3 & -0.5 \\ 0.2 & -0.5 & -0.4 \\ 0.3 & 0.2 & -0.2 \end{array} \right). 
\]

(4.2)

From the selection of the coefficients \( A(1) \) and \( A(2) \) we note that the relationship between \( x_1 \) species and \( x_2 \) species is reciprocal coexistence because the effects of these two species are positive \( a_{12}(1) = 0.4 > 0 \) and \( a_{21}(1) = 0.3 > 0 \), \( a_{12}(2) = 0.3 > 0 \) and \( a_{21}(2) = 0.2 > 0 \). Also note that \( g(x_1, 1) = 1.3 > 1 \), which shows that the number of \( x_1 \) species is increased after the migration from patch 1 to patch 2, \( G(1, 2) > G(2, 1) \) tells us that the danger of the migration from patch 1 to patch 2 is less than one from patch 2 to patch 1.
If we further choose $C(1)=C(2)=\text{diag}(5, 10, 10)$, then we have

$$
\lambda^+_{\text{max}}(C(1)A(1)+A^T(1)C(1)) = -1.2633,
$$

and

$$
\lambda^+_{\text{max}}(C(2)A(2)+A^T(2)C(2)) = -2.6236.
$$

Hence, $-\lambda = \max_{i \in \{1,2\}} \lambda^+_{\text{max}}(C(i)A(i)+A^T(i)C(i)) = -1.2633 < 0$, which satisfies conditions (3.1) and (3.15). Suppose the species live in patch 1 as the initial state and have a migration in every impulsive time $t_k$, because the transfer probability is 0.6, then for the remaining rate of migration $G(ij)$, $i,j \in \{1,2\}$, we have $\prod_{k=1}^{\infty} G(ij) \leq \prod_{k=1}^{\infty} (1.3 \times 0.7)^{1/2} < \infty$, assumption (H) is satisfied. Hence, all the conditions of Theorems 3.1 and 3.2 are satisfied, then for any system parameter $b(\cdot)$ and any given initial value $x_0 \in \mathbb{R}^n_+$, system (4.1) has a unique solution $x(t)$ on $t \in \mathbb{R}$ and $x(t)$ will be ultimately bounded in mean. See Fig. 1 (a) and (b).

Note that the system parameter $b(\cdot)$ can be chosen as an arbitrary number in Theorem 3.2. It is easy to understand that system (4.1) is ultimately bounded if the intrinsic growth rate $b(\cdot) < 0$. However, if $b(\cdot) > 0$, because of the existence of the predator, prey species $x_1$ and $x_2$ are bounded; the loss during the migration and the inner competition guarantees the boundedness of species $x_3$. Consequently, the selection of the intrinsic growth rate $b(\cdot)$ does not affect the boundedness of the three species, but it affects the size of every species.

On the extinction in mean of system (4.1), we choose $A(1)$, $A(2)$, $G(1,2)$, $G(2,1)$, $G(1,1)$, and $G(2,2)$ as above and

$$
b(1) = \begin{pmatrix} -0.05 \\ -0.02 \\ -0.01 \end{pmatrix}, \quad b(2) = \begin{pmatrix} -0.03 \\ -0.02 \\ -0.01 \end{pmatrix}.
$$

From the definition of $K_2$ we get $K_2 = 1$. Therefore, we have

$$
-\gamma = \max_{i \in \{1,2\}} \max_{1 \leq m \leq 3} \left\{ b_m(i) + \pi(i)(1-K_2) \right\} = -0.01 < 0,
$$

then all the conditions of Theorem 3.3 are satisfied. Hence, system (4.1) will be extinct in mean. See Fig. 2 (a) and (b).

Fig. 2 verifies the validity of Theorem 3.3. Meanwhile, it follows from Fig. 2 that even if there is an increase of the number of the species after the migration (for example, the number of $x_1$ species is increased after the migration from patch 1 to patch 2), it also admits to be extinct as time goes if all the intrinsic growth rates of the species are negative.

But, if we change the negative intrinsic growth rates $b(1)$ and $b(2)$ chosen above to positive, i.e.

$$
b(1) = \begin{pmatrix} 0.5 \\ 0.2 \\ 0.1 \end{pmatrix} \quad \text{and} \quad b(2) = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.1 \end{pmatrix}.
$$

Then we have

$$
-\gamma = \max_{i \in \{1,2\}} \max_{1 \leq m \leq 3} \left\{ b_m(i) + \pi(i)(1-K_2) \right\} = 0.5 > 0,
$$

![Fig. 1](image1.png)

Fig. 1. (a) The ultimate boundedness in mean of the time series of system (4.1) for any $b(\cdot) \in \mathbb{R}^n$, $A(1)$, $A(2)$ taken from (4.2). (b) The ultimate boundedness in mean of the phase of system (4.1) for any $b(\cdot) \in \mathbb{R}^n$ and $A(1)$, $A(2)$ taken from (4.2).

![Fig. 2](image2.png)

Fig. 2. (a) The extinction in mean of the time series of system (4.1) for $b(1) = (-0.05, -0.02, -0.01)^T$, $b(2) = (-0.03, -0.02, -0.01)^T$ and $A(1)$, $A(2)$ taken from (4.2). (b) The extinction in mean of the phase of system (4.1) for $b(1) = (-0.05, -0.02, -0.01)^T$, $b(2) = (-0.03, -0.02, -0.01)^T$ and $A(1)$, $A(2)$ taken from (4.2).
which does not satisfy condition (3.21) of Theorem 3.3. We also illustrate the trends of the solutions of system (4.1), see Fig. 3. While, from Fig. 3(a) and (b), we observe that the densities of the three species will tend to zero as $t \to \infty$, i.e. all the three species will be extinct in mean as time goes despite the positive intrinsic growth rates of the species.

However, if we keep the negative growth rates $b(1)$ and $b(2)$ unchanged and transform the competitive coefficients $A(1)$ and $A(2)$ to

$$A(1) = \begin{pmatrix} -0.5 & 1.4 & -0.4 \\ 1.2 & -0.2 & -0.3 \\ 0.3 & 0.2 & -0.1 \end{pmatrix} \quad \text{and} \quad A(2) = \begin{pmatrix} -0.5 & 1.3 & -0.5 \\ 1.1 & -0.4 & -0.4 \\ 0.2 & 0.1 & -0.1 \end{pmatrix}. \quad (4.3)$$

Other coefficients $G(1,2), G(2,1), C(1)$ and $C(2)$ are not changed, then we have

$$\lambda_{\max}(C(1)A(1) + A^T(1)C(1)) = 14.5066,$$

and

$$\lambda_{\max}(C(2)A(2) + A^T(2)C(2)) = 11.3751.$$ 

Thereby, $-\lambda = \max_{i=1,2}\lambda_{\max}(C(i)A(i) + A^T(i)C(i)) = 14.5066 > 0$, which does not satisfy condition (3.1) of Theorem 3.1. But the illustration in Fig. 4(a) and (b) shows that the solutions of system (4.1) are also extinct in mean.

Comparing the choice of $A(1)$ and $A(2)$ here to the previous example, we note that the reciprocal relationship between $x_1$ species and $x_2$ species is strengthened and the inner competition of all the three species is weakened. But Fig. 4 tells us all the three species will also be extinct in mean as time goes despite these. Figs. 3 and 4 show that even if the conditions of Theorem 3.3 are not satisfied, three species will also be extinct, which bring forward a further question for us to search for some better and exact conditions that guarantee the validity of the Theorem.

5. Discussion

In this paper we have established an $n$ species impulsive migration model with Markovian switching in $N$ different patches. By constructing appropriate Lyapunov functions, we have analyzed the global positivity, the ultimate boundedness and the extinction of the species. This paper tells us that under what environment conditions the species are not only positive but also not explode to infinity at any time and under what environment conditions the species will be extinct as time goes.

It is worth noting that the model (stochastic dispersal) studied in this paper is new and firstly presented. In most dispersal models that have been investigated so far, the choice of the species to the patches is fixed, and unalterable. However, many facts show that due to the deterioration of patch of the environment or human intervention, the migratory species move from one patch to another, will not go back to the previous one when natural enemy or winter comes. Or the migratory species find another more suitable habitat than the previous one during the
migration, they will also not go back to the previous one. All of these indicate that, in a long time, the choice of the species to the patches during the species diffusion (or migration) is alterable. The model presented in this paper just makes up this point. Besides, Remark 2.2 tells us another message that we allow the remaining rate \( g_{\text{res}} \) of some species \( m \) is larger than unity, but only need the assumption \( \prod_{i=1}^{m} \beta_i \leq M < \infty \) is satisfied, which has not been involved in the previous literature. These two points show that our model presented in this paper is valuable and worthy of studied further. But unfortunately, due to shortage of the analysis techniques on the stochastic model, the permanence of the species has not been studied at the present paper. This is still an open problem.

Meanwhile, we have mainly considered the case that the coefficients \( A \) and \( b \) are constants in a given patch, which do not change with time \( t \) as well as the population densities \( x \). This assumption is a little rough. We have noted that the coefficients \( A \) and \( b \) may also be affected by the changing time (Remark 3.1) and population densities \( x \). Then the first equation of our model will take the form

\[
x(t) = \text{diag}(x_1(t), x_2(t), \ldots, x_m(t)) \begin{bmatrix} b(t, x, r(t)) + A(t, x, r(t))x(t) \end{bmatrix}.
\]

This is also still an open problem and we will study this model in our future work.

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