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Generalized radial epiderivatives and nonconvex set-valued optimization problems

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In this article, the generalized radial epiderivative for a set-valued map is introduced. An existence result of generalized radial epiderivatives is obtained. Moreover, the relation between these kind of derivatives and radial epiderivatives has been established. Based on the generalized radial epiderivative, necessary and sufficient optimality conditions are derived for nonconvex set-valued optimization problems.

Keywords: nonconvex set-valued optimization problem; generalized radial epiderivative; weakly efficient solution; Benson proper efficient solution; radial cone; optimality condition

AMS Subject Classifications: 90C26; 90C46; 49J52

1. Introduction

Over the past few decades, set-valued optimization problems have been intensively studied by many authors (see [1–14] and references cited there). In set-valued analysis, the notion of a derivative of a set-valued map plays an important role and has been used to derive the optimality condition for set-valued optimization problems. Aubin [15] introduced the notion of the contingent derivative of set-valued maps and used it to obtain the optimality condition for set-valued optimization problems. Several other authors used it to derive the optimality conditions for set-valued optimizations (see [16–20] and the references therein). But it turns out that necessary optimality conditions and sufficient optimality conditions do not coincide under standard assumptions. In order to obtain the satisfying optimality condition, Jahn and Rauh [21] introduced the concept of contingent epiderivatives of a set-valued map and obtained the unified necessary and sufficient optimality conditions for set-valued optimization problems. They gave an existence theorem for contingent epiderivatives in the special case that the range space is the real

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number space, but in general setting the existence of contingent epiderivatives of a set-valued map is still an open problem. For overcoming this difficulty, Chen and Jahn [22] introduced the notion of generalized contingent epiderivatives and obtained a existence result for the generalized contingent epiderivative. They also derived a unified necessary and sufficient optimality conditions in set-valued optimization problems. It is worth mentioning that the convexity plays an important role in proving the sufficiency in [21,22].

Recently, Taa [23] introduced the concept of radial derivatives by replacing the contingent cone in the definition of the contingent derivative with the radial cone [24]. And they also obtained a sufficient optimality condition of weakly efficient solutions for set-valued optimization problems without any convexity assumption. Very recently, Kasimbeyli [25] introduced the notion of radial epiderivatives of a set-valued map which is different from the one proposed by Flore-Bazan [26]. It is important to note that Kasimbeyli [25] introduced a single-valued map as the derivative of a set-valued map. He also derived a unified necessary and sufficient optimality conditions for set-valued optimization problems without convexity conditions.

The purpose of this article is to introduce a new concept of derivative of a set-valued map, which called generalized radial epiderivatives. We would like to mention that the generalized radial epiderivative is a set-valued map, which is similar to the definition of the generalized contingent epiderivative introduced by Chen and Jahn [22]; we use the radial cone instead of the contingent cone. The existence of generalized radial epiderivatives is proved. Moreover, the relation between this kind of derivatives and radial epiderivatives has been established. Finally, by using the new definition, we give some necessary and sufficient optimality conditions of Benson proper and weakly efficient solutions for set-valued optimization problems without convexity conditions. These results extend and improve the corresponding results in [22].

This article is organized as follows. In Section 2, we recall some basic definitions and notations to be needed in subsequent sections. In Section 3, we introduce the notion of the generalized radial epiderivative of a set-valued map and obtain its existence criteria. Also, we study the relation between this kind of derivatives and the radial epiderivative. In Section 4, we establish the necessary and sufficient optimality condition of Benson proper and weakly efficient solutions for nonconvex set-valued optimization problems.

2. Preliminaries

Throughout this article, unless otherwise stated, let $X$ and $Y$ be two real normal spaces, $C$ be a convex cone in $Y$. Let $A$ be a subset of $X$, we denote the topological interior and closure of $A$ by $\text{int}A$ and $\text{cl}A$, respectively. The cone generated by a set $A$ is defined by

$$\text{cone}(A) := \bigcup_{\alpha \geq 0} \alpha A.$$
Let $F: X \to 2^Y$ be a set-valued map. The domain, graph and epigraph of $F$ are, respectively, defined by

$$\text{dom}(F) := \{x \in X: F(x) \neq \emptyset\},$$
$$\text{graph}(F) := \{(x, y) \in X \times Y: x \in \text{dom } F, y \in F(x)\},$$
$$\text{epi}(F) := \{(x, y) \in X \times Y: x \in \text{dom } F, y \in F(x) + C\},$$

where the symbol $\emptyset$ denotes the empty set.

Now, we recall some definitions.

**Definition 2.1** Let $B \subset Y$ be a nonempty subset.

(i) A point $y_0 \in B$ is said to be an efficient point of $B$, if

$$(B - y_0) \cap (-C) = \{y_0\}. $$

(ii) Let int $C \neq \emptyset$. A point $y_0 \in B$ is said to be a weakly efficient point of $B$, if

$$(B - y_0) \cap (-\text{int } C) = \emptyset. $$

(iii) A point $y_0 \in B$ is said to be a Benson proper efficient point [27] of $B$, if

$$\text{cl cone}(B + C - y_0) \cap (-C) = \{y_0\}. $$

The set of all efficient points, weakly efficient points and Benson proper efficient points of a set $B \subset Y$ with respect to a cone $C$ will be denoted by Min$(D, C)$, WMin$(D, C)$ and BMMin$(D, C)$, respectively.

**Definition 2.2** Let $K$ be a nonempty subset of $X$ and $x_0 \in \text{cl } K$.

(i) The contingent cone [24] $T(K, x_0)$ to $K$ at $x_0$ is the set of all $h \in X$ such that there exist $t_n > 0$ and a sequence $x_n \in K$ with $x_n \to x_0$ and $t_n(x_n - x_0) \to h$.

(ii) The radial cone [24] $R(K, x_0)$ to $K$ at $x_0$ is the set of all $h \in X$ such that there exist $t_n > 0$ and a sequence $x_n \in K$ with $t_n(x_n - x_0) \to h$.

**Remark 2.1** From the above definitions, we have that $T(K, x_0)$ is a closed cone and $T(K, x_0) \subset R(K, x_0) = \text{cl cone}(K - x_0)$. Moreover, If $K$ is convex, then $T(K, x_0) = R(K, x_0)$.

**Remark 2.2** It is not difficult to see that

(i) $h \in T(K, x_0)$ if and only if there exist $t_n > 0$ and $h_n \to h$ with $t_n h_n \to 0$ and $x_0 + t_n h_n \in K$.

(ii) $h \in R(K, x_0)$ if and only if there exist $t_n > 0$ and $h_n \to h$ with $x_0 + t_n h_n \in K$.

**Definition 2.3**[25] Let $(x_0, y_0) \in \text{graph}(F)$. The radial epiderivative $RF(x_0, y_0)$ of $F$ at $(x_0, y_0)$ is a single-valued mapping from $X$ to $Y$ defined by

$$\text{epi}(RF(x_0, y_0)) = R(\text{epi}(F), (x_0, y_0)).$$

**Definition 2.4**[22] Let $(x_0, y_0) \in \text{graph}(F)$. The generalized contingent epiderivative $D_g F(x_0, y_0)$ of $F$ at $(x_0, y_0)$ is the set-valued mapping from $X$ to $Y$ defined by

$$D_g F(x_0, y_0)(x) := \text{Min}(H(x), C),$$

where $H(x) = \{y \in Y: (x, y) \in T(\text{epi}(F), (x_0, y_0))\}$. 
3. Generalized radial epiderivatives

We first introduce the notion of generalized radial epiderivatives for a set-valued map.

**Definition 3.1** Let $F : X \to 2^Y$ be a set-valued map. Let $(x_0, y_0) \in \text{graph}(F)$. The generalized radial epiderivative $R_gF(x_0, y_0)$ of $F$ at $(x_0, y_0)$ is a set-valued map from $X$ to $Y$ defined by

$$R_gF(x_0, y_0)(x) := \text{Min}(G(x), C),$$

where $G(x) = \{y \in Y : (x, y) \in R(\text{epi}(F), (x_0, y_0))\}, \ x \in X.$

**Remark 3.1** If we replace the radial cone by the contingent cone and the Clark tangent cone, then the generalized radial epiderivatives reduces to the definitions of the generalized contingent epiderivatives introduced by Chen and Jahn [22] and the generalized Clark epiderivative introduced by Chen [28], respectively.

**Remark 3.2** It is noted that the domain of $R_gF(x_0, y_0)$ may not be the whole space $X$. Moreover, if there exists an $x \in X$ such that the set $G(x)$ is empty, then $R_gF(x_0, y_0)(x) = \emptyset$.

We now discuss the existence of the generalized radial epiderivative for which we recall the following definitions.

**Definition 3.2** [9, 17] Let $B \subseteq Y$ be a subset.

(i) A cone $C$ is said to be Daniell, if any decreasing sequence in $Y$ having a lower bound converges to its infimum.

(ii) A set $B$ is said to satisfy the domination property if $B \subseteq \text{Min} B + C$.

(iii) The space $Y$ is said to be boundedly order complete if every bounded decreasing sequence has an infimum.

**Lemma 3.1** [17, 29] Let $C \subseteq Y$ be a closed convex cone and $A \subseteq Y$ be a nonempty subset. If $A$ is closed and bounded, $C$ is Daniell and $Y$ is boundedly order complete, then $\text{Min}(A, C) \neq \emptyset$.

Now we give an existence theorem for generalized radial epiderivative. We remark that our theorem is a simple generalization of the existence theorem for the generalized contingent epiderivatives, formulated by Chen and Jahn [22].

**Theorem 3.1** Let $C \subseteq Y$ be a closed convex cone and $(x_0, y_0) \in \text{graph}(F)$. If $C$ is Daniell and $Y$ is boundedly order complete, and for every $x \in \text{dom} G$, the set $G(x)$ is bounded, then $R_gF(x_0, y_0)(x)$ exists for every $x \in \text{dom} G$.

**Proof** The conclusion follows directly from Lemma 3.1 since $G(x)$ is closed for every $x \in \text{dom} G$. This completes the proof.

The following theorem establishes the relationship between the generalized radial epiderivative and the radial epiderivative.

**Theorem 3.2** Let $(x_0, y_0) \in \text{graph}(F)$. If the radial epiderivative $RF(x_0, y_0)$ exists and the set $G(x)$ satisfies the domination property for every $x \in \text{dom} G$, then

$$\text{epi} R_gF(x_0, y_0) = \text{epi} RF(x_0, y_0).$$
**Proof** By using the similar argument as in the proof of Theorem 4 in [22], one can easily prove the conclusion. This completes the proof.

**Theorem 3.3** Let \( (x_0, y_0) \in \text{graph}(F) \). If \( R_g F(x_0, y_0)(x) \) exists for every \( x \in \text{dom} \, G \), then

\[
R_g F(x_0, y_0)(X) \subseteq R(F(X) + C, y_0),
\]

where \( R_g F(x_0, y_0)(X) = \bigcup \{ R_g F(x_0, y_0)(x) \mid x \in \text{dom} \, G \} \) and \( F(X) = \bigcup \{ F(x) \mid x \in \text{dom} \, F \} \).

**Proof** Let \( x \in \text{dom} \, G \) and \( y \in R_g F(x_0, y_0)(x) \). Then \( (x, y) \in R(\text{epi}(F), (x_0, y_0)) \). By the definition of radial cone, there exist a sequence \( \{(x_n, y_n)\} \subseteq \text{epi} \, F \) and a sequence \( t_n > 0 \) such that

\[
(x, y) = \lim_{n \to +\infty} t_n(x_n - x_0, y_n - y_0).
\]

It follows that \( y = \lim_{n \to +\infty} t_n(y_n - y_0) \). Since \( \{(x_n, y_n)\} \subseteq \text{epi} \, F \), there exists \( y'_n \in F(x_n) \) and \( c_n \in C \) such that \( y_n = y'_n + c_n \). Thus \( y = \lim_{n \to +\infty} t_n(y'_n + c_n - y_0) \). By the definition of radial cone and \( y'_n + c_n \in F(X) + C \), we have \( y \in R(F(X) + C, y_0) \). This completes the proof.

### 4. Optimality conditions

In this section we apply the presented generalized radial epiderivatives concept to obtain an unified necessary and sufficient optimality condition for the set-valued optimization problems without any convexity assumption. Let \( F : X \to 2^Y \) be a set-valued map.

We consider the following set-valued optimization problem:

\[
(P) \quad \min F(x), \quad \text{subject to } x \in K,
\]

where \( K = \text{dom}(F) \).

**Definition 4.1** Denote \( F(K) = \bigcup_{x \in K} F(x) \). Let \( x_0 \in K \) and \( y_0 \in F(x_0) \). (i) Let \( \text{int} \, C \neq \emptyset \). A pair \( (x_0, y_0) \) is said to be a weakly efficient solution of the problem \( P \) if \( F(K) \cap (y_0 - \text{int} \, C) = \emptyset \). (ii) A pair \( (x_0, y_0) \) is said to be a Benson proper efficient solution [27] of the problem \( P \) if \( \text{cl} \, \text{cone}(F(K) + C - y_0) \cap (-C) = \{0\} \).

**Definition 4.2** Let \( K = X, x_0 \in K \) and \( y_0 \in F(x_0) \). Assume that the generalized radial epiderivative \( R_g F(x_0, y_0) \) exists. The set \( \partial F(x_0, y_0) \) defined by

\[
\partial F(x_0, y_0) = \{ T \in L(X, Y) : 0 \in B \text{Min}[(RF(x_0, y_0) - T)(X), C] \}
\]

is called Benson generalized gradient of \( F \) at \( (x_0, y_0) \), where \( L(X, Y) \) denotes the set of all continuous linear operators from \( X \) to \( Y \).

The following lemma plays an important role in proving the sufficient optimality condition.

**Lemma 4.1** Let \( x_0 \in K, y_0 \in F(x_0) \). If the generalized radial epiderivative \( R_g F(x_0, y_0) \) exists and the set \( G(x - x_0) \) satisfies the domination property for every \( x \in K \), then

\[
F(x) - y_0 \subseteq R_g F(x_0, y_0)(x - x_0) + C, \quad \forall x \in K.
\]
Proof Let \( x \in K \) and \( y \in F(x) \). Set \( t_n = 1 \), \( x_n = x \), \( y_n = y \). It follows that \((x_n, y_n) \in \text{epi } F\) and

\[
\lim_{n \to +\infty} t_n(x_n - x_0, y_n - y_0) = (x - x_0, y - y_0).
\]

Thus,

\[(x - x_0, y - y_0) \in R(\text{epi } F, (x_0, y_0)),\]

which implies,

\[y - y_0 \in G(x - x_0) = \{ y \in Y : (x - x_0, y) \in R(\text{epi } F, (x_0, y_0)) \}.
\]

By the definition of \( R_g F(x_0, y_0) \) and domination property, one has

\[G(x - x_0) \subset R_g F(x_0, y_0)(x - x_0) + C.
\]

Therefore, \( F(x) - y_0 \subset R_g F(x_0, y_0)(x - x_0) + C \). This completes the proof. \( \blacksquare \)

Remark 4.1 It is important to note that the above lemma holds without any convexity assumption. However, a similar lemma for the generalized contingent epiderivative was given by Chen and Jahn [22] under the assumption of convexity.

Theorem 4.1 Let \( x_0 \in K \), \( y_0 \in F(x_0) \). If the generalized radial epiderivative \( R_g F(x_0, y_0) \) exists and the set \( G(x - x_0) \) satisfies the domination property for every \( x \in K \), then \((x_0, y_0)\) is a weakly efficient solution of the problem (P) if and only if

\[R_g F(x_0, y_0)(x - x_0) \cap - \text{int } C = \emptyset, \quad \forall x \in K.
\]

Proof Let \( x_0 \in K \), \( y_0 \in F(x_0) \). Suppose by contradiction that there exists an element \( x \in K \) such that

\[R_g F(x_0, y_0)(x - x_0) \cap - \text{int } C \neq \emptyset.
\]

Then there exists a \( y' \in R_g F(x_0, y_0)(x - x_0) \) such that

\[y' \in - \text{int } C.
\]

(4.2)

Since \( y' \in R_g F(x_0, y_0)(x - x_0) \), one has

\[(x - x_0, y') \in R(\text{epi } F, (x_0, y_0)),
\]

which implies that there exist a sequence \( \{(x_n, y_n)\} \subset \text{epi } F \) and a sequence \( t_n > 0 \) such that

\[(x - x_0, y) = \lim_{n \to +\infty} t_n(x_n - x_0, y_n - y_0).
\]

(4.3)

It follows from (4.2) and (4.3) that there exists an \( N \in \mathbb{N} \) such that

\[t_n(y_n - y_0) \in - \text{int } C, \quad \forall n \geq N
\]

and so

\[y_n \in \{ y_0 \} - \text{int } C, \quad \forall n \geq N.
\]

(4.4)
Therefore, \((x_n, y_n)\) is a weakly efficient solution of the problem (P). This completes the proof.

**Theorem 4.2** Let \(x_0 \in K\), \(y_0 \in F(x_0)\). If the generalized radial epiderivative \(R_g F(x_0, y_0)\) exists and the set \(G(x-x_0)\) satisfies the domination property for every \(x \in K\), then \((x_0, y_0)\) is a Benson proper efficient solution of the problem (P) if and only if

\[
\text{cl cone}(R_g F(x_0, y_0)(K-x_0) + C) \cap -C \neq \emptyset.
\]

**Proof** Let \((x_0, y_0)\) be a Benson proper efficient solution of the problem (P). Then

\[
\text{cl cone}(F(K) + C - y_0) \cap (-C) = \{0\}.
\]

If (4.5) does not hold, then there exists a \(c \in Y\) such that

\[
c \in \text{cl cone}(R_g F(x_0, y_0)(K-x_0) + C) \cap -C \setminus \{0\}.
\]

It follows that there exist \(\lambda_n > 0\), \(x_n \in K\), \(y_n \in R_g F(x_0, y_0)(x_n-x_0)\) and \(c_n \in C\) such that

\[
\lim_{n \to +\infty} \lambda_n (y_n + c_n) = c \in -C \setminus \{0\}.
\]

By the definition of \(R_g F(x_0, y_0)\), we have

\[
(x_n - x_0, y_n) \in R(\text{epi}(F), (x_0, y_0)).
\]

Then there exist a sequence \(\{(x_n^i, y_n^i)\} \subseteq \text{epi } F\) and a sequence \(\iota_n > 0\) such that

\[
(x_n - x_0, y_n) = \lim_{i \to +\infty} \iota_n (x_n^i - x_0, y_n^i - y_0).
\]

Since \(\{(x_n^i, y_n^i)\} \subseteq \text{epi } F\), \(\forall i \in \mathbb{N}\), there exist \(\bar{y}_n^i \in F(x_n^i)\) and \(\bar{c}_n^i \in C\) such that \(y_n^i = \bar{y}_n^i + \bar{c}_n^i\). Note that

\[
\lim_{i \to +\infty} \iota_n (y_n^i - y_0) = \lim_{i \to +\infty} \iota_n (\bar{y}_n^i + \bar{c}_n^i - y_0) = y_n, \quad \forall n \in \mathbb{N},
\]

which implies that there exists \(i(n) \in \mathbb{N}\) such that

\[
\|\iota_n (\bar{y}_n^i + \bar{c}_n^i - y_0) - y_n\| < \frac{1}{n\lambda_n}, \quad \forall i \geq i(n).
\]
It follows that
\[ \| t_n^{\alpha(n)}(\gamma_n^{\alpha(n)} + c_n^{\alpha(n)} - y_0) - y_n \| < \frac{1}{n \lambda_n}, \quad \forall n \in \mathbb{N}. \]

Let \( v_n = \lambda_n t_n^{\alpha(n)} \). Then \( v_n > 0 \) and
\[
\left\| v_n \left( \gamma_n^{\alpha(n)} + c_n^{\alpha(n)} - y_0 + \frac{c_n}{t_n^{\alpha(n)}} - c \right) = \| \lambda_n t_n^{\alpha(n)}(\gamma_n^{\alpha(n)} + c_n^{\alpha(n)} - y_0) + \lambda_n c_n - c \| 
\leq \frac{1}{n} + \| \lambda_n (c_n + y_n) - c \|. \]

Note that \( \lim_{n \to +\infty} \lambda_n (y_n + c_n) = c \) and \( \lim_{n \to +\infty} \lambda_n = 0 \). It follows that
\[
bc = \lim_{n \to +\infty} v_n \left( \gamma_n^{\alpha(n)} + c_n^{\alpha(n)} - y_0 + \frac{c_n}{t_n^{\alpha(n)}} \right). \]

Since \( v_n(\gamma_n^{\alpha(n)} + c_n^{\alpha(n)} - y_0 + \frac{c_n}{t_n^{\alpha(n)}}) \in \text{cone}(F(x_n^{\alpha(n)}) - y_0 + C) \subset \text{cone}(F(K) - y_0 + C) \), one has
\[
c \in \text{cl cone}(F(K) + C - y_0) \cap (-C \setminus \{0\}), \]
which contradicts (4.6).

Conversely, let (4.5) hold. By Lemma 4.1,
\[
F(K) - y_0 \subset R_x F(x_0, y_0)(K - x_0) + C. \]

It follows that
\[
\text{cl cone}(F(K) + C - y_0) \subset \text{cl cone}(R_x F(x_0, y_0)(K - x_0) + C) \subset \text{cl cone}(R_x F(x_0, y_0)(K - x_0) + C), \]
which together with (4.5) yields
\[
\text{cl cone}(F(K) + C - y_0) \cap (-C \setminus \{0\}) = \emptyset. \]

Thus, \((x_0, y_0)\) is a Benson proper efficient solution of the problem (P). This completes the proof.

**Remark 4.1** It is worth mentioning that Theorems 4.1 and 4.2 show that the importance of the concept of the generalized radial epiderivative introduced in this article. Chen and Jahn [22] gave a characterization of weakly efficient solutions for the generalized contingent epiderivative under convexity conditions, meanwhile our results hold without any convexity assumption.

**Theorem 4.3** Let \( x_0 \in K = X, \ y_0 \in F(x_0) \). If the generalized radial epiderivative \( R_x F(x_0, y_0) \) exists and the set \( G(x - x_0) \) satisfies the domination property for every \( x \in X \), then \((x_0, y_0)\) is a Benson proper efficient solution of the problem (P) if and only if \( 0 \in \partial F(x_0, y_0) \).

**Proof** Suppose that \((x_0, y_0)\) is a Benson proper efficient solution of the problem (P). Then
\[
\text{cl cone}(F(X) + C - y_0) \cap (-C) = \{0\}. \quad (4.7) \]
We now prove that
\[ R_g F(x_0, y_0)(X) \cap (-C) = \{0\}. \]
In fact, if there exist \( x \in X, y \in R_g F(x_0, y_0)(x) \cap (-C) \) with \( y \neq 0 \), then \( (x, y) \in R(\text{epi}(F), (x_0, y_0)) \). It follows that there exist a sequence \( \{(x_n, y_n)\} \subseteq \text{epi} F \) and a sequence \( t_n > 0 \) such that
\[ (x, y) = \lim_{n \to +\infty} t_n(x_n - x_0, y_n - y_0). \]
Thus, \( y = \lim_{n \to +\infty} t_n(y_n - y_0) \). Since \( \{(x_n, y_n)\} \subseteq \text{epi} F \), there exist \( y'_n \in F(x_n) \) and \( c_n \in C \) such that \( y_n = y'_n + c_n \). Hence, \( y = \lim_{n \to +\infty} t_n(y'_n + c_n - y_0) \) and so
\[ y \in \text{cl cone}(F(X) + C - y_0) \cap (-C), \]
which contradicts (4.7). Therefore,
\[ \text{cl cone}(R_g F(x_0, y_0)(X) + C) \cap (-C) = \{0\}. \]
This implies that \( 0 \in \partial F(x_0, y_0) \).
Conversely, let \( 0 \in \partial F(x_0, y_0) \). Then \( \text{cl cone}(R_g F(x_0, y_0)(X) + C) \cap (-C) = \{0\} \). By Lemma 3.1, for any \( x \in X \),
\[ F(x) - y_0 + C \subseteq R_g F(x_0, y_0)(x - x_0) + C + C \]
\[ \subseteq R_g F(x_0, y_0)(x - x_0) + C \]
\[ \subseteq R_g F(x_0, y_0)(X) + C \]
It follows that
\[ \text{cl cone}(F(X) + C - y_0) \cap (-C) \subseteq \text{cl cone}(R_g F(x_0, y_0)(X) + C) \cap (-C) = \{0\}, \]
which implies that \( (x_0, y_0) \) is a Benson proper efficient solution of the problem (P). This completes the proof.

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