Controllable quantum state transfer and entanglement generation between distant nitrogen-vacancy-center ensembles coupled to superconducting flux qubits

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We investigate the dissipative dynamics of a hybrid system consisting of two inductively coupled superconducting flux qubits (FQs), each of which couples magnetically with a nitrogen-vacancy-center spin ensemble (NVE). The system displays a series of damped oscillations under various experimental situations, where we obtain analytically the time-dependent populations associated with arbitrary values of the FQ-FQ inductively coupling strength $g$ and the FQ-NVE magnetically coupling strength $G$ in the one-excitation case. Our results show that a reliable high-fidelity quantum state transfer between the two distant NVEs can be realized by modulating the parameters $g$ and $G$. Furthermore, we also present a potentially practical idea to entangle the two separate NVEs, which is within reach due to the currently available technology.

I. INTRODUCTION

As one of the most promising solid-state candidates for quantum information processing (QIP) [1], the nitrogen-vacancy defect center (NV) in diamond is of particular interest because it is optically controllable and has a long coherence time at room temperature [2]. On the other hand, due to the flexibility, stability, and robustness associated with modern solid-state engineering, condensed matter systems offer strong interactions for scalable QIP.

Due to the enhanced magnetic-dipole coupling through the collective excitation of the NV spin ensemble (NVE), much effort has been devoted in recent years to combining the NVE with other solid-state systems [3–8]. The interaction between a spin ensemble and a circuit-QED system [3,4] has been investigated, and it was shown that potential applications can be achieved in QIP, such as quantum memories [5] and continuous variable entanglement [6]. Alternatively, coherent quantum coupling between a NVE and a superconducting gap-tunable flux qubit (FQ) has been explored both theoretically [7] and experimentally [8]. The NVE-FQ magnetic coupling can be stronger by about three orders of magnitude than that between a NVE and a superconducting stripline resonator. Additionally, superconducting FQs have the merits of scalability and flexibility [9]. Various tunable coupling mechanisms have been proposed for inductively coupled flux qubits by dc-pulse control [10,11] or ac-pulse control [12], e.g., inductively coupled flux qubits along with dc superconducting quantum interference devices (DC-SQUIDs) used for tunability, which is theoretically described [10] and then demonstrated experimentally [11].

In the present study, based on previous theoretical and experimental progress, we intend to achieve a scalable hybrid network by investigating the dynamics of a hybrid system consisting of two inductively coupled superconducting FQs, each of which couples magnetically with a NVE. In the presence of decoherence, the system displays a series of damped oscillations under various experimental situations, where we obtain analytically the time-dependent populations associated with the basis states for arbitrary values of the FQ-FQ inductively coupling strength $g$ and FQ-NVE magnetically coupling strength $G$ in the one-excitation case. The remarkable result is that the two tunable parameters $g$ and $G$ influence the dynamics of the total system in different ways. More importantly, our results show that a reliable high-fidelity quantum state transfer (QST) between the two distant NVEs can be realized by modulating the parameters $g$ and $G$. Furthermore, we also present a potentially practical idea in this work to entangle these two separate NVEs.

The present system has distinct advantages, such as the tunability and scalability of the FQ and the high magnetic coupling between the NVE and the FQ. The strong spin-field coupling of the NVE and the FQ benefits from the collective enhancement of the coupling by a factor $\sqrt{N}$, with $N$ denoting the number of NVs in the ensemble. The FQ can be tuned into resonance with the spin ensemble in situ, keeping the FQ at its optimum flux bias (degeneracy point) [8]. On the other hand, QST and entanglement generation between distant NVs are prerequisites for the realization of large-scale spin-based quantum networks [13,14]. So our scheme may lead to promising perspectives for building a distributed QIP architecture, where each FQ-NVE unit acts as a quantum node, and quantum information is transferred and processed among the NVEs at spatially different locations. This may be a significant step toward a future full-scale quantum-information processor based on the increasingly developed nanoscale solid-state technology.

In the next section, we present the model of the combined NVE-FQ system using an effective Hamiltonian based on the Holstein-Primakoff transformation [15] and the rotating-wave approximation (RWA). Section III is devoted to the dynamics of the whole system under different conditions, and we discuss

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the possibility of realizing a high-fidelity QST between these two indirectly interacting NVEs in a deterministic way. In Sec. IV, we investigate the preparation of entanglement and the implementation of the SWAP gate between distant NVEs. We present our conclusions in Sec. V.

II. MODEL AND HAMILTONIAN

As schematically shown in Fig. 1, we consider a setup consisting of two indirectly coupled FQs, which are now coupled through a circuit that allows us to modulate the coupling constant \( g \). Each FQ is coupled magnetically to a NVE in a diamond crystal. The combined FQ-NVE system we study is governed by the Hamiltonian

\[
H_{\text{tot}} = H_{\text{NVE}} + H_{\text{FQ}} + H_{\text{FQI}} + H_{\text{FN}},
\]

in which \( H_{\text{NVE}} \) and \( H_{\text{FQ}} \) are for the NVE and the FQ, respectively, \( H_{\text{FQI}} \) corresponds to the interaction between the FQs, and \( H_{\text{FN}} \) represents the coupling between the NVE and the FQ.

The ground state of a NV is a spin triplet with a zero-field splitting, \( D_\theta = 2.87 \, \text{GHz} \). between \( m_S = 0 \) and the nearly degenerate sublevels \( m_S = \pm 1 \) in the absence of external magnetic field [16]. In our proposal, the logic states \(|g\rangle \) and \(|e\rangle \) of the NVE are denoted by the states \( |3A, m_s = 0 \rangle \), and \( |3A, m_s = -1 \rangle \), respectively. So the Hamiltonian of a single NVE reads

\[
H_{\text{NVE}} = \hbar \delta J_z,
\]

where \( \hbar \delta \approx \hbar D_\theta - \gamma_e B_0 \) is the energy gap between the ground state sublevels \( m_s = 0 \) and \( -1 \), under the external magnetic field \( B_0 \), with \( \hbar \) denoting the Planck constant and \( \gamma_e \) denoting the gyromagnetic ratio of an electron. \( J_j = (1/\sqrt{N}) \sum_{i=1}^N T_{j}^i \) is the collective spin operator for the NVE with \( T_j = |e\rangle \langle e| - |g\rangle \langle g| \), \( T_z = |g\rangle \langle g| + |e\rangle \langle e| \), \( T_+ = |e\rangle \langle g| \), and \( T_- = |g\rangle \langle e| \). The collective spin operator \( J_\pm \) can create symmetric Dicke excitation states \( |n\rangle_{\text{NV}} \), and we encode the qubit of the NVE as

\[
|0\rangle_{\text{NV}} = |g_1 g_2 \ldots g_N\rangle,
\]

\[
|1\rangle_{\text{NV}} = J_+ |0\rangle_{\text{NV}} = (1/\sqrt{N}) \sum_{k=1}^N T_{1}^k |g_1 g_2 \ldots g_N\rangle.
\]

When biasing the main loop close to one-half of a flux quantum, the gap-tunable FQ can be treated as an effective two-level system described by the Hamiltonian

\[
H_{\text{FQ}} = \hbar \Delta \sigma_x + \varepsilon \sigma_z,
\]

which is given on the basis of the tunnel splitting and \( \varepsilon = 2I_\text{FQ}(\Phi_\text{ext} - \Phi_0/2)/\hbar \) is the energy bias, with \( I_\text{FQ} \) the persistent current in the qubit, \( \Phi_\text{ext} \) the external flux threading the qubit loop, and \( \Phi_0 = h/2e \) the flux quantum. \( \sigma_x \) and \( \sigma_z \) are the usual Pauli spin-half operators. Note that \( \Delta \) and \( \varepsilon \) are controlled by the external magnetic flux threading the loops [8,17].

The FQ-FQ interaction Hamiltonian can be cast on the basis of the energy eigenstates in the following form:

\[
H_{\text{FQI}} = \hbar g \sigma_1^x \sigma_2^z,
\]

where \( g \) corresponds to the coupling between FQ1 and FQ2, tunable by extra circuits [10,12].

On the other hand, there exists a magnetic transition between the dressed states of the FQ coupled to the electronic spins of NVE. We describe the near-resonance interaction between the NVE and the FQ by

\[
H_{\text{FN}} = \Sigma_{i=1}^N \hbar G_i J_i^x \sigma_i^z,
\]

where \( G_i \) is the coupling strength between the NVE and the FQ, and different NVEs are denoted by the superscript \( i \). Here we have made several assumptions about the nature of the coupling between the NVE and the FQ. In our scheme, the antialigned magnetic fields created by the circular persistent current of the flux qubit are along the \( z \) axis, which makes it dominant for the coupling \( J_i^x \sigma_i^z \) with \( k = x, y, z \). Following the idea in [8], the NVE is supposed to be orientated along the \( z \) axis, and thereby \( J_i^x \sigma_i^z \) is negligible. Furthermore, considering the homogeneous coupling between the NVE and the FQ as well as the symmetry of the NVE, we have the dominant term \( J_i^x \sigma_i^z \) left in the interaction.

Based on currently experimental conditions [8], we may neglect the terms regarding the strain and the external magnetic field. As a result, the total Hamiltonian can be simplified by using the new basis \( \{|E_i\} = \cos(\eta/2)|L\rangle + \sin(\eta/2)|R\rangle \), \( |G_i\rangle = -\sin(\eta/2)|L\rangle + \cos(\eta/2)|R\rangle \) in the following form:

\[
H_{\text{tot}} = \hbar \Sigma_{i=1}^N \left( \frac{1}{2} g_1 \sigma_i^x \sigma_i^z + \frac{1}{2} \Delta \delta J_i^x + g_2 \sigma_i^x \sigma_i^z + G_i J_i^x \sigma_i^z \right),
\]

FIG. 1. (Color online) Schematic of the hybrid system consisting of two superconducting flux qubits, which are now coupled through a circuit (i.e., DC-SQUID) that allows us to modulate the coupling constant \( g \). The top of each flux qubit is glued by a diamond crystal, doped with a NVE. Inset: Energy diagram of the NV. We encode two indirectly interacting NVEs in a deterministic way.
where \( \eta = \tan^{-1}(2\varepsilon/\Delta) \), and \( \omega_r = h\sqrt{\varepsilon^2 + \Delta^2} \) represents the splitting energy of the \( i \)th gap-tunable FQ. Under the RWA, the effective Hamiltonian in the interaction picture is (in units of \( h = 1 \))

\[
H_{\text{int}} = \sum_{i=1}^{2} \left( g\sigma_i^+ \sigma_i^- + G_i J_i^+ \sigma_i^+ + \text{H.c.} \right).
\]  

(7)

In addition, we consider the decay of the FQs and the NVEs. Before the photon due to decay is detected, the dissipative evolution of the system can be effectively described by employing the quantum trajectory method [18,19] with the conditional Hamiltonian [20].

\[
H_D = \sum_{j=1}^{2} \left[ -\frac{\gamma_j}{2} |e_j\rangle\langle e_j| - i \frac{\Gamma_j}{2} |E_j\rangle\langle E_j| \right].
\]  

(8)

where \( \gamma_j \) is the decay rate from the \( |e\rangle \) of the NVE, and \( \Gamma_j \) is the decay rate from the effective excited state \( |e\rangle \) of the \( j \)th FQ [21].

In the interaction picture, the total Hamiltonian of the combined FQ-NVE system is

\[
H_{\text{eff}} = H_{\text{int}} + H_D.
\]  

(9)

III. QUANTUM STATE TRANSFER BETWEEN TWO DISTANT NVEs

In QIP, the transfer of the quantum state from one location to another is an important task. It refers not only to large-scale quantum computing, but also quantum communication. In this section, we investigate how to realize a high-fidelity QST between these two distant and indirectly interacting NVEs. Our results indicate that some special relations between FQ-FQ and FQ-NVE coupling strengths should be satisfied to implement the QST.

For our purpose, we first focus on the time evolution of the total system. For simplicity, the analysis is restricted to the dynamics of the one-excitation subspace spanned by the basis vectors \( \{ |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle \} \) with

\[
|\phi_1\rangle = |1\rangle_{\text{NV1}}|00\rangle_{i,F_Q}|0\rangle_{\text{NV2}},
|\phi_2\rangle = |0\rangle_{\text{NV1}}|10\rangle_{i,F_Q}|0\rangle_{\text{NV2}},
|\phi_3\rangle = |0\rangle_{\text{NV1}}|01\rangle_{i,F_Q}|0\rangle_{\text{NV2}},
|\phi_4\rangle = |0\rangle_{\text{NV1}}|00\rangle_{i,F_Q}|1\rangle_{\text{NV2}},
\]

where \( |i,F_Q\rangle \) is the FQ state, and \( |0\rangle_{\text{NVj}} \) and \( |1\rangle_{\text{NVj}} \) \( (j = 1,2) \) are the symmetric Dicke excitation states of the NVE.

The time evolution of the total system is governed by the Schrödinger equation

\[
i \frac{\partial}{\partial t} |\Psi(t)\rangle = H_{\text{eff}} |\Psi(t)\rangle.
\]  

(10)

Suppose that all the FQs are initially nonexcited and the first NVE is prepared in the one-excitation state. We can expand the time-dependent state vector \( |\Psi(t)\rangle \) of the entire system in terms of the basis vectors \( \{|\phi_j\rangle\} \) as

\[
|\Psi(t)\rangle = \sum_{j=1}^{4} C_j(t) |\phi_j\rangle.
\]  

(11)

The definition of the related parameters and some other details of our deductions are presented in Appendix. In the weak decay case of \( \{\gamma, \Gamma\} \ll \{g, G\} \), we can obtain the approximate forms of the four parameters \( \lambda_j \) as \( \lambda_j^\pm \approx 2g - \sqrt{16G^2 + 4g^2} \), \( \lambda_3^\pm \approx -2g + \sqrt{16G^2 + 4g^2} \), and \( \lambda_4^\pm \approx -2g - \sqrt{16G^2 + 4g^2} \). (See the detailed definition in Appendix.) Substituting these expressions into Eqs. (10) and (11) yields

\[
|\Psi(t)\rangle = U(t)|1\rangle_{\text{NV1}}|00\rangle_{i}|0\rangle_{\text{NV2}} = C_1(t)|\phi_1\rangle + C_2(t)|\phi_2\rangle + C_3(t)|\phi_3\rangle + C_4(t)|\phi_4\rangle,
\]  

(12)

in which

\[
C_1(t) = \frac{r+1}{2r} \left[ \cos \left( \frac{r-1}{2} gt \right) + \cos \left( \frac{r+1}{2} gt \right) \right],
\]  

(13)

\[
C_2(t) = -\frac{r^2-1}{2r} \left[ \sin \left( \frac{r-1}{2} gt \right) + \sin \left( \frac{r+1}{2} gt \right) \right],
\]  

(14)

\[
C_3(t) = \frac{r^2-1}{2r} \left[ \cos \left( \frac{r-1}{2} gt \right) - \cos \left( \frac{r+1}{2} gt \right) \right],
\]  

(15)

and

\[
C_4(t) = \frac{i}{2r} \left[ \left( r+1 \right) \cos \left( \frac{r-1}{2} gt \right) - \left( r-1 \right) \sin \left( \frac{r+1}{2} gt \right) \right].
\]  

(16)

with \( r = \sqrt{1 + 4G^2/g^2} \). The populations associated with the basis states \( \{|\phi_j\rangle\} \) are given by \( P_y = P_j = |C_j(t)|^2 \).

Note that the decay effects have been removed from Eqs. (13)–(16) due to the approximation, which helps us to better understand the physics, such as the balance and the competition between different couplings. However, our numerical treatment will still consider the decay effects by using Eq. (A8) in Appendix.

Since this hybrid system contains two different kinds of couplings, i.e., the inductive-coupling between FQ-FQ and the magnetic-coupling between the NVE and the FQ, the one-excitation dynamics of the system reflects the intricate balance and competition between these two kinds of couplings. On the other hand, from Eqs. (13)−(16), we find that the time evolution of all the one-excitation states fully depends on the time-dependent superposition of the four harmonic wave functions. So one can predict that the overall quantum dynamics may exhibit some complex interferences or competing effects, which also exhibit the oscillation damping due to decoherence.

In what follows, we will briefly illustrate the related time evolution in three different cases, namely the large inductive-coupling case \( g \gg G \), the large magnetic-coupling case \( g \ll G \), and the equilibrium case \( g \approx G \). We first focus on the large inductive-coupling case \( g \gg G \), namely \( r \to 1 \), where the total system is broken into the subsystem 1 (i.e., NVE1 + FQ1) and another subsystem 2 (i.e., NVE2 + FQ2), and the excitation transfer between these two subsystems dominates the dynamics. In this extreme case of \( r \to 1 \), we obtain a
simplified wave function as

$$|\Psi(t)\rangle_G \approx \cos \left(\frac{G^2}{g} t\right) |\phi_1\rangle + i \sin \left(\frac{G^2}{g} t\right) |\phi_2\rangle. \quad (17)$$

From Eqs. (14) and (15), one can find that the contribution from the states $|\phi_2\rangle$ and $|\phi_3\rangle$ is negligible, which implies that the excitation mainly transfers between the two distant NVEs with a relatively higher speed, and the populations with respect to the excitation residing in the FQs, such as $|C_2(t)|^2$ and $|C_3(t)|^2$, will be very low in the limit of $r \to 1$, as shown in Figs. 2(a) and 2(b). Another remarkable characteristic during the evolution is that there exist two distinct oscillating frequencies, i.e., a slow oscillation period $\sim 1/G$ and a fast oscillation period $\sim 1/g$. So the dynamics of the system is a linear superposition of two such oscillations, as illustrated in Figs. 2(a) and 2(b), where the ratio between the slow and fast oscillation periods is $G/g$.

In the large magnetic-coupling case $g \ll G$, namely $r \gg 1$, we can obtain the following approximate amplitudes of the state of the system:

$$|\Psi(t)\rangle_g \approx \cos(Gt) \cos(Gt)|\phi_1\rangle - i \sin(Gt) \cos(Gt)|\phi_2\rangle
+ \sin(Gt) \sin(Gt)|\phi_3\rangle - i \cos(Gt) \sin(Gt)|\phi_4\rangle, \quad (18)$$

in which the interface between the NVE and the FQ dominates the single-excitation evolution. So the magnitude of the large coupling strength $G$ determines the relatively large speed of the energy exchange between the NVE and the FQ. As shown in Figs. 2(c) and 2(d), the overall dynamics of the system is characterized by a fast oscillation with the period $T_f = 2\pi/G$ accompanied by a slow oscillation envelope, whose period is $T_s = 2\pi/g$. That is to say, the excitation will be ultimately transferred between the two subsystems with a relatively slower speed than the rate regarding excitation exchange in each subsystem. This implies that these two subsystems interact weakly with each other through the inductive coupling between FQs.

The third case is the equilibrium case, in which the values of $g$ are comparable to $G$, namely $r \to \sqrt{3}$. As expected, the situation becomes very abstruse when these two crucial parameters $g$ and $G$ take arbitrary values. So the results lead to complicated time-evolution curves in Figs. 2(e) and 2(f). However, there also exist some particular parameters $g$ and $G$ for QST, which will be discussed later.

Based on the above-mentioned quantum dynamics, we focus below on the possibility of realizing the QST between the two distant NVEs in a deterministic way. Suppose that the system is initially prepared in an arbitrary superposition state $(\alpha|0\rangle_{NV1} + \beta|1\rangle_{NV1})|00\rangle_{f,1}|0\rangle_{NV2}$ with $\alpha$ and $\beta$ being the normalized complex numbers. As stated above, the time evolution of the total system is governed by Eq. (12), and the state of the system evolves as $U(t)(\alpha|0\rangle_{NV1} + \beta|1\rangle_{NV1})|00\rangle_{f,1}|0\rangle_{NV2} = U(t)\alpha|0\rangle_{NV1}|00\rangle_{f,1}|0\rangle_{NV2} + U(t)\beta|1\rangle_{NV1}|00\rangle_{f,1}|0\rangle_{NV2}$, where $U(t) = \exp(-iH_{Eff}t)$. 

![Figure 2](image-url)
is matched, where we can set \( k = 0 \) and \( k' \) takes non-negative integers. It is clear that the initial state \( |\alpha|0_{NV1} + \beta|1_{NV1}\rangle|01_{f,i}f_i0_{NV2}\) evolves to the final state \( |0_{NV1}|00_{f,i}f_i0_{NV2} - i|01_{NV1}|00_{f,i}f_i1_{NV2}\) [or \( |0_{NV1}|00_{f,i}f_i(0|0_{NV2} + i|1_{NV2})\rangle\] at \( t = \pi/(r - 1)g \) [or \( t = \pi r/(r - 1)g\)] if the parameter \( r \) fulfills the condition \( r = 2(k' + 1)/(2k' + 1) \) [or \( r = 2(k' + 2)/(2k' + 1)\)]. Once the condition with respect to these coupling constants can be well satisfied, the ideal QST can be implemented in a deterministic way.

In practice, considering the mismatch between the coupling strengths, we define the fidelity of the QST as \( F = [\langle\Psi(t)|\Psi_T\rangle]^2 = (1 + |C_4|^2)/2\), where \( |\Psi_T\rangle \) is the target state of the transfer. We investigate in Fig. 3 the fidelity for the state transfer \((0|_{NV1} + |1_{NV1}\rangle|00_{f,i}f_i0_{NV2} - \sqrt{2} \rightarrow |0_{NV1}|00_{f,i}f_i|0_{NV2} + 1_{NV2}\rangle)/\sqrt{2}\), with different mismatches around the parameter \( r = 2\). One can find the nearly perfect state transfer under the mismatching conditions.

**IV. ENTANGLEMENT GENERATION BETWEEN TWO DISTANT NVEs**

In this section, we investigate the preparation of the entangled states between two NVEs in this hybrid system. Two different cases are included: the equilibrium case \( g \approx G \) and the large magnetic-coupling case \( g \ll G \). The key idea of the two cases is to control the dynamics of the total system by meeting specific conditions between the tunable parameters \( g \) and \( G \). First, for comparable \( g \) and \( G \), we suppose there exists an excitation in the first NVE when the other physical qubits are not excited, i.e., the initial state of the system as \( |\phi_i\rangle = |1_{NV1}|00_{f,i}f_i0_{NV2}\). Then the evolution of the system is governed by Eq. (12), as we discussed in the section above. If the parameters fulfill the relations

\[
\begin{align*}
\left( \frac{r - 1}{2} \right) g &= \left( 2k + \frac{3}{4} \right) \pi, \\
\left( \frac{r + 1}{2} \right) g &= \left( 2k' + \frac{5}{4} \right) \pi
\end{align*}
\]

(20)

we obtain \( |C_4| = |C_1| = 1/\sqrt{2}, |C_2| = 0, \) and \( |C_3| = 0 \) from Eqs. (13)–(16). So the system evolves to \( |\psi_E\rangle = -(|1_{NV1}|00_{f,i}f_i0_{NV2} - i|0_{NV1}|00_{f,i}f_i1_{NV2})/\sqrt{2} \) or \( |\psi_E\rangle = (|1_{NV1}|00_{f,i}f_i0_{NV2} + i|0_{NV1}|00_{f,i}f_i1_{NV2})/\sqrt{2} \) if \( r = 4(k' + 1)/(2k' + 1) \) or \( r = 4(k' + 1)/(2k' + 3) \) with \( k = 0 \) and \( k' \) non-negative integers. Thus, the EPR-type entangled state between two separated NVEs can be generated as \((-|00_{f,i}f_i0_{NV2} - i|0_{NV1}|00_{f,i}f_i1_{NV2})/\sqrt{2} \) or \((|1_{NV1}|00_{f,i}f_i0_{NV2} + i|0_{NV1}|00_{f,i}f_i1_{NV2})/\sqrt{2}\).

In practical situations, the ratio of the coupling constants required for the entangled state generation may not be satisfied exactly. To visualize the effects of the imperfection in the coupling constants, we define the fidelity of the entangling gate: \( F = [\langle\Psi(t)|\psi_E\rangle]^2 = (|C_1|^2 + |C_3|^2)/2 \) with \( i = 1.2 \). In Fig. 4(a), we plot the fidelity as a function of the dimensionless time \( t \) for different values of mismatch in the equilibrium case \( G = \sqrt{15}g/2 \). It is seen that the fidelity is remarkably robust.
to the deviation of the coupling constants from the values for the perfect entangling generation.

Next we describe another case, $g \ll G_1, G_2$, for which the normal operators $\tau^1$ and $\tau^2$ are introduced for the first and the second FQ, for convenience of treatment of the FQs, with the transformation $\sigma^j = \frac{1}{\sqrt{2}}(\tau^j + \tau^j)$ and $\sigma^j = \frac{1}{\sqrt{2}}(\tau^j - \tau^j)$ ($j = x, y, z$). Treating the spin ensembles as generalized harmonic oscillators strongly coupled to the FQs, we rewrite the effective Hamiltonian Eq. (9) in terms of the new operators,

$$H_{\text{int}} = g(\tau^1_+ \tau^1_+ - \tau^2_+ \tau^2_+ \sqrt{2} \frac{1}{\sqrt{2}}G_1 J_1((\tau^1_+ + \tau^1_+) + \frac{1}{\sqrt{2}}G_2 J_2(\tau^1_+ - \tau^2_+) + \text{H.c.} \ (21)$$

Making use of the transformation $R = e^{-i \sigma^1_1 \tau^1_+ \tau^1_+ - \tau^2_+ \tau^2_+ \sqrt{2}}$ to simplify the dynamics of the system, the Hamiltonian of the composite system in the interaction picture becomes

$$H_{\text{int}} = \frac{1}{\sqrt{2}}G_1 J_1(\tau^1_+ e^{i \sigma^1_1} + \tau^2_+ e^{i \sigma^2_1}) + \frac{1}{\sqrt{2}}G_2 J_2(\tau^1_+ e^{i \sigma^1_1} + \tau^2_+ e^{i \sigma^2_1}) + \text{H.c.} \ (22)$$

Under the limit $g \gg G_1, G_2$, we can further achieve an effective interaction Hamiltonian,

$$H_{\text{int}} = -\lambda (J^1_+ J^2_+ + J^2_+ J^1_+), \ (23)$$

with $\lambda = G_1 G_2/g$, which implies that the FQs are only virtually excited. From Eq. (23), the state evolutions from different initial states of the two NVEs are given by

$$|1\rangle_{\text{NV1}}|0\rangle_{\text{NV2}} \rightarrow \cos(\lambda t)|1\rangle_{\text{NV1}}|0\rangle_{\text{NV2}} + i \sin(\lambda t)|0\rangle_{\text{NV1}}|1\rangle_{\text{NV2}},$$

$$|0\rangle_{\text{NV1}}|1\rangle_{\text{NV2}} \rightarrow \cos(\lambda t)|0\rangle_{\text{NV1}}|1\rangle_{\text{NV2}} + i \sin(\lambda t)|1\rangle_{\text{NV1}}|0\rangle_{\text{NV2}}. \ (24)$$

Therefore, if there exists an excitation initially in one of the two NVEs, e.g., $|\Psi\rangle = |1\rangle_{\text{NV1}}|0\rangle_{\text{NV2}}$ or $|\Psi\rangle = |0\rangle_{\text{NV1}}|1\rangle_{\text{NV2}}$, the system will evolve into an entangled state as $|\Psi(\lambda t = \pi / 4)\rangle = |\Psi(\lambda t = \pi / 4)\rangle = 1/\sqrt{2}(|1\rangle_{\text{NV1}}|0\rangle_{\text{NV2}} + i |0\rangle_{\text{NV1}}|1\rangle_{\text{NV2}})$. Compared with the previous entanglement generation in the equilibrium case, the FQs in our case are only virtually excited, and thereby the excitations of the FQs are highly suppressed, i.e., there is a negligible influence from the decoherence of the FQs. However, the implementing time may be longer due to the effective coupling strength $\lambda = G_1 G_2/g$ under this limit, which decreases the fidelity. As a result, we conclude in Fig. 4(b) the fidelity of the entangling gate with exact time evolutions of the system in Eq. (12). Our method works well for different coupling strengths. Additionally, the above result can also be applied to quantum SWAP between the two separated NVEs following the ideas in Ref. [22].

V. EXPERIMENTAL FEASIBILITY

For experimental feasibility, we list the following experimental conditions required by our scheme: (i) The gap-tunable flux qubits should have a long coherence time of the order of a microsecond; (ii) the density of the NVE should be high enough to realize the required coupling strength between the NVE and the SQ; (iii) to obtain the high degree of symmetry, the orientation of the NVEs should be along the y axis so that the interaction term $J^j_i \tau^j_i$ in Eq. (5) dominates the total dynamical process.

Experimentally, it is possible to realize a coupled FQ-NVE system [8] with $3.1 \times 10^7$ NVs in an ensemble, where the magnetic coupling strength $g \approx 70$ MHz has been achieved. Here the value of $G$ is much larger than the decoherence rate $\sim 1$ MHz of the FQ. With technical advances, the increasing number of the NV in each ensemble or the larger size of the FQ will further enhance the coupling strength. On the other hand, the flux qubits should be tuned to be in resonance with the NVEs. Experimentally, there are many tunable coupling methods for coupling the FQs. For instance, dc pulse control is a widely used technique for the coupling strength modulation between the FQs in which the coupling strength can be varied continuously from zero to a large positive value of hundreds of MHz [10,11].

In current experiments, the electron spin relaxation time $T_1$ of the NV ranges from 6 ms at room temperature [23] to 1 s at low temperature. In addition, the dephasing time $T_2 = 350 \mu s$ induced by the nearby nuclear-spin fluctuation has been reported [24]. However, the most recent experimental progress [25] with an isotopically pure diamond sample has demonstrated a longer $T_2$ with $T_2 = 2$ ms. In addition, the current $T_2$ time in FQs at best takes a few microseconds. In our case, we consider the coherent time $T_2 = 1.8 \mu s$ with a FQ of the size of $L = 5 \mu m$ and a NV density $n = 10^{17} \text{ cm}^{-3}$ [7]. Taking $r = 4$ as an example, we set $G \approx 70$ MHz and the inductive-coupling rate to be $g = \sqrt{35}G/2 \approx 135$ MHz, which yield the operating time $t_{\text{op}} = 3 \pi/(2g) \approx 30$ ns for QST between two separated NVEs and $t_{\text{op}} = 3 \pi/(2g) \approx 15$ ns for the entanglement generation of the two NVEs. These are sufficiently short for infidelity induced by the paramagnetic impurities and FQs, in both the transfer from the FQ to the NVE and the entanglement generation between two distant NVEs. The recent experimental achievement in a superconducting quantum device consisting of four coupled flux qubits [26] gives us hope that our ideas can be used in the near future to realize a scalable quantum hybrid network.

For our schemes to be achievable, we should try to suppress the decoherence to our model. One of the typical decoherences is from the nuclear spins of $^{13}$C defects in the NVs, which are detrimental to the coherence of QST and the entanglement generation. Fortunately, this can be alleviated by using isotopically purified $^{12}$C diamond through the purification technique [7,25]. Another typical decoherence comes from the dipole interaction between the redundant nitrogen spins and the NVs, which is due to the low nitrogen-to-NV conversion rate. One of the solutions is to improve the conversion rate, which would reduce the linewidth of the NVs while maintaining the large collective coupling constants [3]. Alternatively, this problem could be overcome by applying the external driving field to the electron spins on the redundant nitrogen atoms, leading to an increased coherence time of the NVE if the nitrogen spins are flipped by the spin-echo pulses on a time scale much faster than the flip-flop processes [7].
VI. CONCLUSION

In conclusion, we have investigated the one-excitation dynamics of a hybrid system consisting of two inductively coupling superconducting FQs, each of which couples magnetically with a NVE. The system displays a series of damped oscillations under various experimental situations. The remarkable result is that the two tunable parameters $g$ and $G$ influence the dynamics of the total system in different ways. Our results have shown a reliable high-fidelity QST between the two distant NVEs by modulating the parameters $g$ and $G$. In addition, we have also discussed the possibility of entangling these two separate NVEs and implementing a swap gate between them. Moreover, we have justified the experimental feasibility and challenges using currently available technology. We believe that our present investigation would be useful for the coherent transfer and processing of quantum information in future quantum networks.

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APPENDIX: DERIVATION OF Eqs. (13)–(16)

We give detailed deductions for Eqs. (13)–(16) below. We can expand the time-dependent state vector $|Ψ_i⟩$ of the entire system in terms of the basis vectors $|φ_i⟩$ as [27]

$$|Ψ_i⟩ = Σ_{j=1}^4 C_i(t)|φ_j⟩. \tag{A1}$$

Substitution of Eq. (A1) into the Schrödinger equation [Eq. (10)] leads to

$$i\frac{∂}{∂t} C_i(t) = H_{\text{eff}} Σ_j C_j(t)|Ψ_i⟩, \tag{A2}$$

with

$$H_{\text{eff}} = \begin{pmatrix}
\frac{-iγ}{2} & G & 0 & 0 \\
G & \frac{-iγ}{2} & 0 & 0 \\
0 & 0 & \frac{-iγ}{2} & G \\
0 & 0 & G & \frac{-iγ}{2}
\end{pmatrix}, \tag{A3}$$

where $G_1 = G_2 = G$, $Γ_1 = Γ_2 = Γ$, and $γ_1 = γ_2 = γ$ have been assumed.

With $η_± = \sqrt{16G^2 + (2g ± iχ ±)}$ and $χ_± = γ ± Γ$, the Hamiltonian matrix (A3) has four eigenvalues,

$$E_{1,2} = (-2g - iχ_± ± η_±)/4, \tag{A4}$$

$$E_{3,4} = (2g - iχ_± ± η_+) /4.$$  

The corresponding eigenvectors are

$$|φ_1⟩ = |φ_1⟩ + \frac{2g + iχ_+ ± η_+}{4G}|φ_2⟩ + \frac{2g + iχ_+ ± η_+}{4G}|φ_3⟩ + |φ_4⟩,$$

$$|φ_2⟩ = |φ_1⟩ + \frac{2g + iχ_+ ± η_+}{4G}|φ_2⟩ + \frac{2g + iχ_+ ± η_+}{4G}|φ_3⟩ + |φ_4⟩.$$

From the above eigenvectors, we deduce the unitary matrix $S$ that diagonalizes the Hamiltonian matrix (A3),

$$S = \begin{pmatrix}
1 & 2g + iχ_+ ± η_+ & 2g + iχ_+ ± η_+ \\
2g + iχ_+ ± η_+ & 2g + iχ_+ ± η_+ & 2g + iχ_+ ± η_+ \\
-1 & -2g - iχ_± ± η_± & -2g - iχ_± ± η_± \\
-1 & -2g - iχ_± ± η_± & -2g - iχ_± ± η_± \\
\end{pmatrix}, \tag{A5}$$

which allows Eq. (A2) to be rewritten as the compact form

$$i\frac{∂}{∂t} SC = SHS^{-1} SC, \tag{A6}$$

with $C = [C_1, C_2, C_3, C_4]^T$. Since the matrix $SHS^{-1}$ is diagonal, a general solution of Eq. (A6) is given by

$$C_j(t) = \sum_{k=1}^{4} [S^{-1}]_{jk}[SC(0)]_k e^{-iE_k t}. \tag{A7}$$

Considering the initial condition $C(0) = [1, 0, 0, 0]^T$, namely, all the FQs initially nonexcited and only the first NVE prepared in the one-excitation state, we have

$$C_1(t) = \{η_+(λ_1 e^{-iλ_1^2 t/4} + λ_2 e^{-iλ_2^2 t/4}) + η_-(λ_3 e^{-iλ_3^2 t/4} + λ_4 e^{-iλ_4^2 t/4})/4η_+, \tag{A8}$$

$$C_2(t) = \{Gη_+(e^{-iλ_1^2 t/4}(e^{iγt/2} - 1) - 1)/η_+ - η_-(e^{iγ t/2})/4η_+, \tag{A9}$$

$$C_3(t) = \{Gη_+(e^{-iλ_1^2 t/4}(e^{iγt/2} - 1) - 1)/η_+ - η_-(e^{iγ t/2})/4η_+, \tag{A10}$$

$$C_4(t) = \{±η_+(λ_1 e^{-iλ_1^2 t/4} + λ_2 e^{-iλ_2^2 t/4}) + η_-(λ_3 e^{-iλ_3^2 t/4} + λ_4 e^{-iλ_4^2 t/4})/4η_+, \tag{A11}$$

with $λ_± = 2g - iχ_± ± η_±$, $λ_± = ±2g ± iχ_± ± η_±$, $λ_± = ±2g ± iχ_± ± η_±$, and $η_± = \sqrt{16G^2 + (2g ± iχ_±)}$ and $χ_± = γ ± Γ$. We have assumed in the above equations that $G_1 = G_2 = G$, $Γ_1 = Γ_2 = Γ$, and $γ_1 = γ_2 = γ$. In the weak decay case of $|γ, Γ⟩ ≪ |g, G⟩$, we can obtain the approximate forms of the four parameters $λ_j$ as $λ_1^+ ≈ 2g + \sqrt{16G^2 + 4g^2}$, $λ_2^+ ≈ 2g - \sqrt{16G^2 + 4g^2}$, $λ_3^− ≈ -2g + \sqrt{16G^2 + 4g^2}$, and $λ_4^− ≈ -2g - \sqrt{16G^2 + 4g^2}$. Substituting these expressions into Eqs. (A1)–(A7) yields Eqs. (12)–(16).