Exponential stability of cellular neural networks with time-varying delay

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Abstract: Time-delay appears frequently in the neural network study, and it is often a source of instability and oscillations in a system. It is very important to research the stability of delayed neural networks, especially for neural networks with time-varying delays. In this paper, a novel method is proposed for the exponential stability of cellular neural networks with time-varying delays. New delay-dependent exponential stability conditions of cellular neural network with time-varying delays are presented by constructing Lyapunov function and using linear matrix inequality (LMI). Finally, numerical examples are given to demonstrate the effect of the proposed method.

Keywords: delayed cellular neural networks, DCNNs; Lyapunov functional; linear matrix inequality; LMI; exponential stability; identification and control.


Biographical notes: Xue-li Wu received his BS and MS from Harbin Normal University, Harbin, China and his PhD from Huazhong University of Science and Technology. He is a Professor and Dean of the Electrical Engineering in Hebei University of Science and Technology. His research interests are in non-linear control systems, control systems design over network and intelligent control.
1 Introduction

Neural networks have been extensively studied over the past few decades and have found many applications in a variety of areas, such as associative memory, pattern recognition and combinatorial optimisation. In recent years, considerable effort has been devoted to analysing the stability of neural networks without a time-delay. However, time-delay is frequently encountered in neural networks, and it is often a source of instability and oscillations in a system. As a result, the stability of delayed neural networks has received considerable attention. So far, most of these existing results for the global asymptotic stability of the delayed cellular neural networks (DCNNs) are independent of the delay parameters (Chua and Yang, 1988; Roska et al., 1999; Singh, 2004; Zhou et al., 2003; Arik, 2003a, 2003b; Liao et al., 2002b; Cao, 1999). Recently, delay-dependent asymptotic stability criteria have attracted much attention (Liao et al., 2002a; Zhang et al., 2005; Li et al., 2005; Xu et al., 2006) because delay-dependent criteria make use of information on the length of delays and less conservative than delay-independent ones. However, how to propose a delay-dependent stability criterion for DCNNs is an open problem (Hua et al., 2006; Cao and Wang, 2003; Zhou et al., 2005). In Liao et al. (2004a, 2004b), authors studied the global exponential stability of neural networks, several sufficient conditions guaranteeing the network’s global exponential are established by using Lyapunov method and the technique of linear matrix inequality (LMI) analysis. Our main purpose in this letter is to establish the criteria of matrix form on the global exponential stability for cellular neural networks with time-varying delays.

In this paper, we consider exponential stability problem for time-varying DCNN. By utilising new Lyapunov-Krasovskii functional, we propose the novel sufficient conditions for the time-varying delay neural network. The sufficient conditions obtained in this paper are looser than those in the former literature. Specially, our stability results include the time-delay independent ones in the former literature. The stability conditions obtained in this paper are all in the form of LMIs. Finally, a numerical example will be given to show the effectiveness of the main results.

2 System description

The dynamic behaviour of continuous time-varying DCNNs can be described by the following state equations:

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t))$$

$$+ \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau(t))) + J_i,$$

(1)

or equivalently

$$\dot{y}(t) = -C y(t) + A g(y(t)) + B g(y(t-\tau(t))) + J$$

(2)

where \(n\) corresponds to the number of units in a neural network; \(y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T\) is the state vector at time \(t\); \(g(y(t)) = [g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t))]^T\) denotes the activation function of the neurons; \(C, A, B, J\) are constant matrices; \(C = \text{diag}(c_1, c_2, \ldots, c_n) (c_i > 0)\) (a positive diagonal matrix) represents the rate with which the \(i\)th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, \(A = \{a_{ij}\}\) is referred to as the feedback matrix, \(B = \{b_{ij}\}\) represents the delayed feedback matrix, while \(J = [J_1, J_2, \ldots, J_n]^T\) is the constant external inputs, and the \(\tau(t)\) is the transmission delay at time \(t\) which is time-varying.

In this paper, we will assume that the activation functions \(g_i(i = 1, 2, \ldots n)\) satisfy the following conditions:

A1 for each \(i\), \(g_i\) is bounded on \(R\), \(|g_i(x)| \leq M\), without generalities, we choose the \(M = 1\)

A2 for each \(i\), there exist real number \(l_i > 0\) such that

\[0 \leq g_i(x) - g_i(y) \leq l_i (x - y) \quad \forall x, y \in R,\]

and \(L = \text{diag}(l_1, l_2, \ldots, l_n)\), without generalities, we choose the \(l_i = 1\)

A3 \(g(0) = 0\).

In the simulation, the activation functions \(g_i\) are given by

\[g_i(y_i) = 0.5[|y_i + 1| - |y_i - 1|], \quad i = 1, 2, \ldots n.\]  

(3)
Lemma 1: (Cao, 1999) Assume the function $g_i$ satisfies the hypotheses above then there exists an equilibrium point for equation (1).

According to Lemma 1, assume $y^* = [y^*_1, y^*_2, \cdots, y^*_n]^T$ is an equilibrium point for the equation (1), using the transformation $x(t) = y(t) - y^*$, System (1) or (2) is transformed to

$$
\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t - \tau(t))\right)
$$

where

$$
f_j(x_i(\cdot)) = g_i(x_i(\cdot) + y^*_i) - g_i(y^*_i),
$$
or equivalently

$$
\dot{x}(t) = -C x(t) + A f(x(t)) + B f(x(t - \tau(t)))
$$

where $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))]^T$ represents the activation vector of transformed system. Note that since each function $g_k(\cdot)$ satisfies the hypotheses above, each $f_k(\cdot)$ satisfies

$$
f^*_i(x_i(\cdot)) \leq x_i(\cdot) f_i(x_i), \quad f_i(0) = 0 \quad i = 1, 2, \cdots, n.
$$

And the time delay, $\tau(t)$ is a time-varying differentiable function that satisfies

$$
0 \leq \tau(t) \leq \tau
$$

and

$$
|\tau(t)| \leq d
$$

where $\tau > 0$ and $0 < d < 1$ are constants.

Lemma 2: (Gu, 2000) For any constant matrices $Z \in \mathbb{R}^{m \times n}$, $Z^T > 0$, scalar $b > a \geq 0$, vector function $w : [a, b] \to \mathbb{R}^m$ such that the integrations in the following are well defined, then

$$
(b - a) \int_{a}^{b} w^T(s) Zw(s) \, ds \geq \int_{a}^{b} w^T(s) \, ds \otimes \left(\int_{a}^{b} w(s) \, ds\right)^T
$$

Lemma 3: (Boyd et al., 1994) The LMI

$$
\begin{bmatrix}
E(x) & H(x) \\
H(x)^T & F(x)
\end{bmatrix} > 0
$$

where $> 0$ denotes that the matrix is positive definite,

$$
E(x) = E(x)^T, \quad F(x) = F(x)^T,
$$

and $H(x)$ depend affinely on $x$, is equivalent to

$$
F(x) < 0, \quad E(x) - H(x) F(x)^{-1} H(x)^T < 0
$$

Proof: See Boyd et al. (1994).

From above analysis, we can see that the stability problem of system (1) on equilibrium $y^*$ is changed into the zero stability problem of system (5). Therefore, in the following part, we will investigate the stability analysis problem for system (5). Based on Lyapunov-Krasovskii functional method, new stability conditions will be proposed.

3 New stability results

In this section, aimed at the system of cellular neural network with time-varying delay, we give one theorem and the responding process of proof.

Theorem 1: Given scalars $\tau > 0$ and $0 < d < 1$, the system (5) with a time-varying delay $\tau(t)$ is exponential stability, if there exist positive definite symmetric matrices $P$, $Z_1$, $Z_2$, $Q$, $R$, positive definite diagonal matrices $D$, $D_1$, $D_2$, $D_3$, $D_4$, $D_5$, $D_6$, where

$$
Q = \begin{bmatrix} Q_{11} & Q_{12} \\
Q_{12}^T & Q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & R_{12} \\
R_{12}^T & R_{22} \end{bmatrix},
$$

such that the following LMI holds,

$$
\Pi = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \frac{1}{\tau} Z_2 & 0 \\
* & \Pi_{22} & \Pi_{23} & \Pi_{24} & 0 & 0 \\
* & * & \Pi_{33} & \Pi_{34} & 0 & 0 \\
* & * & * & \Pi_{44} & 0 & 0 \\
-\tau^2 Z_2 & -\tau^2 Z_2 & -\tau^2 Z_2 & -\tau^2 Z_2 & -\tau^2 Z_2 & -\tau^2 Z_2 \\
+2D_5 & -\frac{1}{\tau} Z_2 & -D_5 & D_6 & -\tau^2 D_6 & -\tau^2 D_6 \\
-\frac{1}{\tau} Z_2 & -\frac{1}{\tau} Z_2 & -\frac{1}{\tau} Z_2 & -\frac{1}{\tau} Z_2 & -\frac{1}{\tau} Z_2 & -\frac{1}{\tau} Z_2
\end{bmatrix} < 0
$$

where

$$
\Pi_{11} = 2kP - 2PC + \frac{\varepsilon^{2k\tau}}{2k} - \frac{1}{\tau} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{12} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{13} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{14} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{22} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{23} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{24} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{33} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{34} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

$$
\Pi_{44} = \frac{\varepsilon^{2k\tau}}{2k} C^T (Z_1 + Z_2) C - \frac{1-d}{\tau} Z_1 - \frac{1}{\tau} Z_2 + Q_{11} + R_{11} + 2D_1
$$

Proof: See Boyd et al. (1994).
\[ \Pi_{12} = 2kD + PA - \frac{e^{2k \tau}}{4k} C^T (Z_1 + Z_2) A \\
- \frac{e^{2k \tau}}{4k} A^T (Z_1 + Z_2) C - C^T D + Q_{12} + R_{12} \\
- D_1 + D_2 \]

\[ \Pi_{13} = \frac{1-d}{\tau} Z_1 \]

\[ \Pi_{14} = PB - \frac{e^{2k \tau}}{4k} C^T (Z_1 + Z_2) B \\
- \frac{e^{2k \tau}}{4k} B^T (Z_1 + Z_2) C \]

\[ \Pi_{22} = \frac{e^{2k \tau}}{2k} A^T (Z_1 + Z_2) A + 2A^T D + Q_{22} + R_{22} - 2D_2 \]

\[ \Pi_{24} = \frac{e^{2k \tau}}{4k} A^T (Z_1 + Z_2) B + \frac{e^{2k \tau}}{4k} B^T (Z_1 + Z_2) A \\
+ B^T D \]

\[ \Pi_{33} = \frac{1-d}{\tau} Z_1 - (1-d)e^{-2k \tau} Q_{11} + 2D_3 \]

\[ \Pi_{34} = -(1-d)e^{-2k \tau} Q_{12} - D_3 + D_4 \]

\[ \Pi_{44} = \frac{e^{2k \tau}}{2k} B^T (Z_1 + Z_2) B - (1-d)e^{-2k \tau} Q_{22} - 2D_4 \]

and * denotes the symmetric term in a symmetric matrix.

**Proof:** Choose the following positive definite Lyapunov functional:

\[ V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \quad (10) \]

in which

\[ V_1(t) = e^{2u(t)} x^T(t) Px(t) \quad P = P^T > 0 \]

\[ V_2(t) = \int_{t-\tau(t)}^{t} \int_{t-\tau(t)-\sigma}^{t} e^{2k(\sigma+\tau)} x^T(s) Z_1 \dot{x}(s) ds d\sigma \]

\[ + \int_{t-\tau(t)}^{t} \int_{t-\tau(t)-\sigma}^{t} e^{2k(\sigma+\tau)} \dot{x}^T(s) Z_2 \dot{x}(s) ds d\sigma \]

\[ Z_1 = Z_1^T > 0, \quad Z_2 = Z_2^T > 0 \]

\[ V_3(t) = 2e^{2k \tau} \sum_{i=1}^{n} d_i \int_{0}^{\tau} f_i(x(s)) ds \]

\[ D = \text{diag} \left( d_1, d_2, \ldots, d_n \right) \quad (d_i > 0) \]

\[ V_4(t) = \int_{t-\tau(t)}^{t} e^{2k \tau} \left( x^T(s) f^T(x(s)) \right) \left[ \begin{array}{c} x(s) \\ f(x(s)) \end{array} \right] ds \]

\[ + \int_{t-\tau}^{t} e^{2k \tau} \left( x^T(s) f^T(x(s)) \right) R \left[ \begin{array}{c} x(s) \\ f(x(s)) \end{array} \right] ds \]

\[ Q = Q^T = \left[ \begin{array}{c} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{array} \right], \quad R = R^T = \left[ \begin{array}{c} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{array} \right] \]

The time derivative of \( V(t) \) along the system (5) is

\[ \dot{V}_1(t) = 2k e^{2k \tau} x^T(t) Px(t) + 2e^{2k \tau} x^T(t) P \dot{x}(t) \]

\[ = e^{2u(t)} x^T(t) (2Px(t) - x^T(t) 2PCx(t)) \]

\[ + x^T(t) 2PBf(x(t)) + x^T(t) 2PBf(x(t) - \tau(t))) \]

\[ \dot{V}_2(t) = \int_{t-\tau(t)}^{t} e^{2k(\sigma+\tau)} x^T(t) Z_1 \dot{x}(t) d\sigma \]

\[ - (1-\tau(t)) \int_{t-\tau(t)}^{t} e^{2k(\tau(t)+\sigma)} x^T(t) Z_1 \dot{x}(s) ds \]

\[ + \int_{t-\tau(t)}^{t} e^{2k(\sigma+\tau)} \dot{x}^T(t) Z_1 \dot{x}(s) ds \]

\[ = e^{2u(t)} \left( \frac{1}{2k} e^{2k \tau} - e^{2k(\tau(t)+\sigma)} \right) x^T(t) Z_1 \dot{x}(t) \]

\[ + e^{2u(t)} \left( \frac{1}{2k} e^{2k \tau} - e^{2k(\tau(t)-\sigma)} \right) \dot{x}^T(t) Z_2 \dot{x}(t) \]

\[ - (1-\tau(t)) e^{2k \tau} x^T(t) Z_1 \dot{x}(s) ds \]

\[ + e^{2k \tau} x^T(t) Z_2 \dot{x}(s) ds \]

\[ \leq e^{2u(t)} \left( \frac{1}{2k} e^{2k \tau} - 1 \right) x^T(t) (Z_1 + Z_2) \dot{x}(t) \]

\[ + (1-d) e^{2k \tau} \int_{t-\tau(t)}^{t} \dot{x}^T(t) Z_1 \dot{x}(s) ds \]

\[ + e^{2k \tau} \int_{t-\tau}^{t} \dot{x}^T(t) Z_2 \dot{x}(s) ds \]

\[ \dot{V}_3(t) = 2e^{2k \tau} \sum_{i=1}^{n} d_i \int_{0}^{\tau} f_i(x(s)) ds \]

\[ + 4ke^{2k \tau} \sum_{i=1}^{n} d_i \int_{0}^{\tau} f_i(x(s)) ds \]

\[ \leq 2e^{2k \tau} x^T(t) Df(x(t)) + 4ke^{2k \tau} x^T(t) Df(x(t)) \]
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\[ \dot{V}_4(t) = e^{2t} \left[ x^T(t) f^T(x(t)) \right] (Q + R) \left[ x(t) \right] \\
- (1 - \tau(t)) e^{2t(\tau(t))} \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times \left[ x(t - \tau(t)) \right] \\
- e^{2t(\tau(t))} \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times R \left[ x(t - \tau(t)) \right] \\
\leq e^{2t} \left[ x^T(t) f^T(x(t)) \right] (Q + R) \left[ x(t) \right] \\
- (1 - \tau(t)) e^{2t(\tau(t))} \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times \left[ x(t - \tau(t)) \right] \\
- e^{2t(\tau(t))} \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times R \left[ x(t - \tau(t)) \right] \\
\begin{align}
\leq 2e^{2t} & \left[ x^T(t) f^T(x(t)) \right] (Q + R) \left[ x(t) \right] \\
- (1 - \tau(t)) e^{2t(\tau(t))} & \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times \left[ x(t - \tau(t)) \right] \\
- e^{2t(\tau(t))} & \left[ x^T(t - \tau(t)) f^T \left( x(t - \tau(t)) \right) \right] \\
\times R \left[ x(t - \tau(t)) \right] \\
\end{align}
\]

according the system formula (5), we know the following formula comes into existence

\[ x^T(t) = -x^T(t)C + f^T(x(t))A^T + \int_{t-\tau(t)}^t f^T(x(t-\tau(s)))ds \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ + f^T(x(t-\tau(t)))B^T \]
\[ \begin{align}
\int_{t-\tau(t)}^t & x^T(s)Z_1 x(s)ds \\
\geq & \frac{1}{\tau(t)} \left[ \int_{t-\tau(t)}^t x(s)ds \right] Z_1 \left[ \int_{t-\tau(t)}^t x(s)ds \right] \\
\geq & \frac{1}{\tau(t)} \left[ x^T(t) - x^T(t - \tau(t)) \right] \\
Z_1 & \left[ x(t) - x(t - \tau(t)) \right] \\
\int_{t-\tau(t)}^t & x^T(s)Z_2 x(s)ds \\
\geq & \frac{1}{\tau(t)} \left[ \int_{t-\tau(t)}^t x(s)ds \right] Z_2 \left[ \int_{t-\tau(t)}^t x(s)ds \right] \\
\geq & \frac{1}{\tau(t)} \left[ x^T(t) - x^T(t - \tau(t)) \right] \\
Z_2 & \left[ x(t) - x(t - \tau(t)) \right] \\
\end{align} \]

Furthermore, we known that there exist positive diagonal matrices \( D_i, i = 1, 2, 3, 4, 5, 6 \) such that the following three inequalities hold based on \( f_i^2(x_i) \leq x_i f_i(x_i) \).

\[ \psi_i = e^{2t} x_i(t) \Theta \zeta(t) \geq 0 \]

where

\[ \Theta = \begin{bmatrix} 2D_1 & D_2 - D_1 & 0 & 0 & 0 & 0 \\
* & -2D_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2D_3 & D_4 - D_3 & 0 & 0 \\
0 & 0 & 0 & -2D_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2D_5 & D_6 - D_5 \\
0 & 0 & 0 & 0 & 0 & -2D_6 \end{bmatrix} \]

\[ \zeta(t) = \begin{bmatrix} x^T(t) & f^T(x(t)) & x^T(t - \tau(t)) & f^T \left( x(t - \tau(t)) \right) \end{bmatrix} \begin{bmatrix} x^T(t - \tau(t)) & f^T \left( x(t - \tau(t)) \right) \end{bmatrix} \]

substituting (5), (11), (12), (13), (14) into (10), then the following inequality is obtained.

\[ \dot{V}(t) \leq \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \psi_i(t) \leq e^{2t} \zeta(t) \Pi \zeta(t) \]

if \( \Pi < 0 \), which is equivalent to \( \dot{V}(t) \leq 0 \), that indicates the system (5) is exponential stability. Thus, the proof is complete.

**Theorem 2:** Given scalars \( \tau > 0 \) and \( 0 < d < 1 \), the system (5) with a time-varying delay \( \tau(t) \) satisfying (7) and (8) is exponential stability if there exist positive definite symmetric matrices \( P, Q, R, Z \), positive definite diagonal matrices \( D, D_1, D_2, D_3, D_4, D_5, D_6 \) and appropriate dimensions matrices \( N_i, P_{ij}, P_{jk}, P_{kl}, P_{lm}, P_{mn} \) and \( P_i \) such that the following LMI holds:

\[ \begin{bmatrix} \Xi & \tau(1-d)N \end{bmatrix} \begin{bmatrix} \tau(1-d)N \end{bmatrix} < 0 \]
Proof: Choose the following Lyapunov functional

\[ V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \]

where

\[ V_2(t) = \int_{t/\tau}^{t} \int_{\sigma}^{\tau} e^{2k(t-\tau)} x^T(s) Z(s) ds d\sigma, \]

\[ Z = Z^T > 0, V_1(t), V_2(0) \text{ and } V_4(t) \text{ are defined in the proof of Theorem 1.} \]

We can see that \( V(t) \) is positive definite, the time derivative of \( V_2(t) \) along the system (5) is

\[
\dot{V}_2(t) = \frac{e^{2k(t-\tau)} - e^{2k(t-\tau)/\tau}}{2k} \dot{x}^T(t) Z\dot{x}(t) - (1 - t/\tau) \int_{t/\tau}^{t} \dot{x}^T(s) Z\dot{x}(s) ds \\
\leq e^{2k\tau} \frac{1}{2k} \dot{x}^T(t) Z\dot{x}(t) - e^{2k\tau} (1 - d) \int_{t/\tau}^{t} \dot{x}^T(s) Zs(s) ds
\]

Now we define

\[ \psi_2 = 2e^{2k\tau} \left[ \dot{x}^T(t) P_{11} + x^T(t) P_{21} + x^T(t - \tau(t)) P_{22} \right] \\
+ x^T(t - \tau(t)) f^T(x(t)) P_{51} + f^T(x(t)) f(t - \tau(t)) \right] P_{6} \\
+ f^T(x(t - \tau(t))) P_{51} \left[ -\dot{x}(t) - Cx(t) \right] \\
+ Af(x(t)) + Bf(x(t - \tau(t))) \]

According to the system formulation (5), we know the following formulation comes into existence

\[ \left[ -\dot{x}(t) - Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \right] = 0 \]

so \( \psi_2(t) = 0 \).
From the Leibniz-Newton formula (Hua et al., 2006; Cao and Wang, 2003; Zhou et al., 2005), the following equations are true for any matrices $N$ with appropriate dimensions

$$2\xi^T(t)N\left[ x(t) - x(t - d(t)) - \int_{t-\tau(t)}^t \dot{x}(s) ds \right] = 0$$

$\xi(t)$ is a row vector with appropriate dimensions.

Moreover, we let

$$\psi_3 = 2e^{2\xi(t)(1-d)}\xi^T(t)N$$

$$\times \left[ x(t) - x(t - \tau(t)) - \int_{t-\tau(t)}^t \dot{x}(s) ds \right]$$

Where $\xi(t) = [\xi(t), \dot{\xi}(t)]^T$, $\zeta(t)$ is defined in the proof of Theorem 2.

Now we add the positive term $\psi_1(t)$ along with the zero terms $\psi_2(t)$ and $\psi_3(t)$ into $V(t)$, then the following inequality is obtained.

$$V'(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t)$$

$$\leq \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \psi_1(t) + \psi_2(t) + \psi_3(t)$$

$$\leq e^{2\|\dot{\xi}(t)\|}\Xi + (1-d)NZ^{-1}N^T\xi(t)$$

$$-e^{2\|\dot{\xi}(t)\|}\int_{t-\tau(t)}^t \dot{\xi}(s) ds$$

$$Z^{-1}[Z\dot{\xi}(s) + N^T\dot{\xi}(s)]ds$$

Since $Z > 0$ and $0 < d < 1$, then the last parts in the above formula are all less than 0. So, if $\Xi + (1-d)NZ^{-1}N^T < 0$, which is equivalent to (16) by Lemma 3. Thus, the proof is complete.

4 Numerical examples

In this section, we use two examples to show the effectiveness of our results.

Example 1: We consider the system of cellular neural networks with time-varying delay (5) with the following parameters:

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4.6 & 0 \\ 0 & 5 \end{bmatrix}$$

$$k = 0.1 \quad \tau = 0.8 \quad d = 0.5$$

The method in He et al. (2004) does not reach time-delay 0.8. We can get the matrices through LMI toolbox such that

$$P = \begin{bmatrix} 34.4580 & -8.2068 \\ -8.2068 & 18.2688 \end{bmatrix} \quad D = \begin{bmatrix} 15.0500 & 0 \\ 0 & 7.0590 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 3.7945 & -2.5687 \\ -2.5687 & 1.8047 \end{bmatrix} \quad Z_2 = \begin{bmatrix} 3.7306 & -2.5242 \\ -2.5242 & 1.7741 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 13.0654 & 0 \\ 0 & 6.2013 \end{bmatrix} \quad D_2 = \begin{bmatrix} 69.1469 & 0 \\ 0 & 6.2013 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 9.8703 & 0 \\ 0 & 4.1780 \end{bmatrix} \quad D_4 = \begin{bmatrix} 25.9029 & 0 \\ 0 & 11.6800 \end{bmatrix}$$

$$D_5 = \begin{bmatrix} 5.7602 & 0 \\ 0 & 3.7947 \end{bmatrix} \quad D_6 = \begin{bmatrix} 6.3431 & 0 \\ 0 & 3.7947 \end{bmatrix}$$

$$Q = \begin{bmatrix} 61.8259 & 0.3769 \\ 0.3769 & 29.5541 \end{bmatrix} \quad -6.1164$$

$$R = \begin{bmatrix} 30.1307 & 0.2193 \\ 0.2193 & -6.0765 \end{bmatrix} \quad -13.6234$$

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.4 \end{bmatrix}$$

$$k = 0.1 \quad \tau = 1.4 \quad d = 0.5$$

We can get the matrices through LMI toolbox such that

$$P = \begin{bmatrix} 0.8286 & 0.2326 \\ 0.2326 & 0.5397 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.7876 & 0.0175 \\ 0.0175 & 0.8053 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.1935 & 0 \\ 0 & 0.3769 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3.6276 & -0.0279 \\ -0.0279 & 0.5542 \end{bmatrix} \quad -0.3187$$

$$R = \begin{bmatrix} * & 3.1629 \\ 3.1629 & -0.3324 \end{bmatrix} \quad 0.4690$$

Example 2: Consider the system of cellular neural networks with time-varying delay (5) with the following parameters:

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3.5 & 0 \\ 0 & 3.4 \end{bmatrix}$$

$$k = 0.1 \quad \tau = 1.4 \quad d = 0.5$$

We can get the matrices through LMI toolbox such that

$$P = \begin{bmatrix} 0.8286 & 0.2326 \\ 0.2326 & 0.5397 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.7876 & 0.0175 \\ 0.0175 & 0.8053 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.1935 & 0 \\ 0 & 0.3769 \end{bmatrix}$$

$$Q = \begin{bmatrix} 3.6276 & -0.0279 \\ -0.0279 & 0.5542 \end{bmatrix} \quad -0.3187$$

$$R = \begin{bmatrix} * & 3.1629 \\ 3.1629 & -0.3324 \end{bmatrix} \quad 0.4690$$

From the simulation result, we can see that the matrices which are mentioned in the Theorem 1 satisfied supposed conditions, therefore the exponential stability of cellular neural network with time-varying delay is tested.
A neural network with time-varying delay is tested. Therefore, the exponential stability of cellular neural networks with time-varying delay is presented by constructing a new Lyapunov functional and using LMI method. The sufficient conditions obtained in this paper are looser than those in the former literature. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.

5 Conclusions

In this note, new classes of the Lyapunov functional were introduced to study the exponential stability of cellular neural networks with time-varying delay. A new exponential stability condition of cellular neural network with time-varying delay is presented by constructing Lyapunov functional and using LMI method. The sufficient conditions obtained in this paper are looser than those in the former literature. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.

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References


