Robust stabilization of uncertain T–S fuzzy time-delay systems with exponential estimates

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Abstract

This paper deals with the problem of robust stabilization for uncertain Takagi–Sugeno (T–S) fuzzy systems with constant time delays. The purpose is to design a state-feedback fuzzy controller such that the closed-loop system is robustly exponentially stable with a prescribed decay rate. Sufficient conditions for the solvability of this problem are presented in terms of linear matrix inequalities (LMIs). By using feasible solutions of these LMIs, desired fuzzy controllers are designed and their corresponding exponential estimates are given. In addition, the main results of this paper are explicitly dependent on the decay rate. This enables one to design fuzzy controllers by freely selecting decay rates according to different practical conditions. Two numerical examples are provided finally to demonstrate the effectiveness of the proposed design methods.

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1. Introduction

Time delays are unavoidable in various areas such as engineering systems, biology and economics. The existence of time delays is often the main cause of instability and poor performance of a system. For these reasons, the problems of stability analysis and controller design for time-delay systems have been extensively studied; see, for example, [1–6] and the references therein. It should be pointed out that most of the results on time-delay systems are concerned with asymptotic stability. Practically, however, the exponential stability is more important because the transient process of a system can be described more clearly once the decay rate is determined [7]. Therefore, in recent years, the study of exponential stability analysis for time-delay systems has received growing attention. In [8], the problem of robust exponential stability for uncertain time-delay systems was studied by using the properties of matrix measure. In [9], a procedure of constructing quadratic Lyapunov–Krasovskii functionals was described for the exponential estimation of
stable linear delay systems. Very recently, linear matrix inequality (LMI) based conditions for exponential stability of time-delay systems were presented in [10,11], which were further improved in [12].

On the other hand, Takagi–Sugeno (T–S) fuzzy systems have received considerable attention, because they are very effective in representing complex nonlinear systems. This class of systems is described as a weighted sum of some simple linear subsystems, and thus are easily analyzed. Over the past 10 years, there have been a great number of significant results on analysis and synthesis problems of T–S fuzzy systems; see, for instance, [13–21] and the references therein. It is noted that the decay rate for fuzzy systems has been considered in [22,23]. Recently, the T–S fuzzy systems with time delays have been studied. For example, delay-independent stability conditions were derived in [24,25], where an LMI approach was also developed for the design of state-feedback fuzzy controllers. In [26], the decentralized robust control problem for a class of uncertain T–S fuzzy large-scale systems with time delays was studied. The problems of delay-dependent stability and stabilization for T–S fuzzy time-delay systems were addressed in [27–30]. When considering exponential stability, some valuable results related to T–S fuzzy time-delay systems were proposed in [31,32].

In many practical applications, one meaningful task is to find a controller to stabilize a system with a specified exponential decay rate; that is, design a controller such that the state of the closed-loop system converges as a required speed. This problem is referred to stabilization with exponential estimates. To solve such a problem, we must derive design conditions that are explicitly dependent on the decay rate. We note that the approaches developed in [31,32] are effective for the exponential stability analysis, but they cannot be extended to the problem of stabilization with exponential estimates for T–S fuzzy time-delay systems in a straightforward way, because the results proposed there are independent of the decay rates. To the best of our knowledge, so far, the problem of robust stabilization with exponential estimates for uncertain T–S fuzzy time-delay systems has not been addressed in the literature, which is very challenging and practically important. This motivates this study.

In this paper, we investigate the problem of robust stabilization with exponential estimates for T–S fuzzy systems with both time delays and parameter uncertainties. To solve this problem, two approaches are developed, namely, relaxed LMI approach and simplified LMI approach. Based on them, several sufficient conditions for the solvability of the concerned problem are obtained by means of strict LMIs. Different from [31,32], our results are explicitly dependent on the exponential decay rate, which enable one to design fuzzy controllers by freely selecting decay rates according to different practical conditions. As a special case, we also present some sufficient conditions for the solvability of the exponential stabilization problem for T–S fuzzy time-delay systems with no parameter uncertainties. Finally, we provide two examples and several simulation results to show the effectiveness of the proposed design methods.

Notations: Throughout this paper, for real symmetric matrices $X$ and $Y$, $X \succeq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices. $I$ is an identity matrix with appropriate dimension. The superscript “T” represents the transpose of a matrix. The notation “*” is used as an ellipsis for terms that are induced by symmetry. $\Vert \cdot \Vert$ denotes the Euclidean norm for vectors and $\| \cdot \|$ denotes the spectral norm for matrices. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalue of the corresponding real symmetric matrix, respectively. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem formulation

Consider a continuous-time T–S fuzzy system with a constant state delay, which is described by

**Plant Rule i**: IF $s_1(t)$ is $\mu_{i1}$ and $s_2(t)$ is $\mu_{i2}$ and ... and $s_g(t)$ is $\mu_{ig}$, THEN

$$
\dot{x}(t) = (A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - d) + (B_i + \Delta B_i(t))u(t),
$$

$$
x(t) = \varphi(t), \quad \forall t \in [-d, 0],
$$

where $i \in \mathcal{N} := \{1, 2, \ldots, r\}$, $\mu_{ij}$ is the fuzzy set, $r$ is the number of IF–THEN rules, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $d > 0$ is a constant time delay, and $\varphi(t)$ is the initial condition. In (1), $A_i, A_{di}$ and $B_i$, $i \in \mathcal{N}$, are known matrices, while $\Delta A_i, \Delta A_{di}$ and $\Delta B_i$, $i \in \mathcal{N}$, are real-valued functions representing the time-varying parameter uncertainties that have the following form:

$$
[\Delta A_i(t) \Delta A_{di}(t) \Delta B_i(t)] = C_i F_i(t) [D_{1i} D_{2i} D_{3i}], \quad i \in \mathcal{N},
$$
where $C_i, D_{1i}, D_{2i}$ and $D_{3i}, i \in \mathcal{N}$, are known matrices, and $F_i(t), i \in \mathcal{N}$, are unknown Lebesgue measurable matrix functions satisfying

$$F_i(t)^T F_i(t) \leq I, \quad i \in \mathcal{N}. \tag{4}$$

The parameter uncertainties $\Delta A_i, \Delta A_{di}$ and $\Delta B_i$ are said to be admissible if conditions (3) and (4) hold.

The fuzzy basis functions are given by

$$h_i(s(t)) = \frac{\bar{w}_i(s(t))}{\sum_{j=1}^{r} \bar{w}_j(s(t))}, \quad \bar{w}_i(s(t)) = \prod_{j=1}^{g} \mu_{ij}(s_j(t)).$$

where $\mu_{ij}(s_j(t))$ is the grade of membership of $s_j(t)$ in $\mu_{ij}$. It is easy to see that $\bar{w}_i(s(t)) \geq 0$ and $\sum_{j=1}^{r} \bar{w}_j(s(t)) > 0$.

Hence, we have $h_i(s(t)) > 0$ and $\sum_{i=1}^{r} h_i(s(t)) = 1$. In what follows, we will drop the argument of $h_i(s(t))$ for simplicity; i.e., $h_i = h_i(s(t))$. Given a pair $(x(t), u(t))$, the output of the fuzzy system in (1) is given by

$$
\dot{x}(t) = \sum_{i=1}^{r} h_i[(A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - d) + (B_i + \Delta B_i(t))u(t)]. \tag{5}
$$

In this paper we consider the following state-feedback fuzzy controller:

**Plant Rule i**: IF $s_1(t)$ is $\mu_{i_1}$ and $s_2(t)$ is $\mu_{i_2}$ and ... and $s_g(t)$ is $\mu_{ig}$, THEN

$$u(t) = K_i x(t), \quad i \in \mathcal{N}, \tag{6}$$

where $K_i, i \in \mathcal{N}$, are controller gains to be determined. The overall form of the controller (6) is represented by

$$u(t) = \sum_{i=1}^{r} h_i K_i x(t). \tag{7}$$

From (5) and (7), we obtain the closed-loop system as follows:

$$
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j [(A_{cij} + \Delta A_{cij}(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - d)], \tag{8}
$$

where $A_{cij} = A_i + B_i K_j$, $\Delta A_{cij}(t) = C_i F_i(t) D_{cij}$ and $D_{cij} = D_{1i} + D_{3i} K_j$.

Throughout this paper, we shall adopt the following definitions.

**Definition 1.** The uncertain fuzzy time-delay system in (8) is said to be robustly exponentially stable with a decay rate $\lambda$ if there exist scalars $\sigma \geq 1$ and $\lambda > 0$ such that the following inequality holds for all admissible uncertainties:

$$|x(t)| \leq \sigma \exp(-\lambda t)|\varphi|_d, \tag{9}$$

where $|\varphi|_d = \sup_{-d \leq x \leq 0} |\varphi(x)|$. When (9) is satisfied, $\lambda$, $\sigma$ and $\sigma \exp(-\lambda t)|\varphi|_d$ are called the decay rate, the decay coefficient and an upper bound of the state trajectories, respectively.

**Definition 2.** The uncertain fuzzy time-delay system (5) is called to be robust stabilization with exponential estimates if there exists a state-feedback fuzzy controller in the form of (7) such that the closed-loop system in (8) is robustly exponentially stable with a decay rate $\lambda$ and the decay coefficient $\sigma$ can be exactly estimated.

The objective of this paper is to first derive sufficient conditions for robust stabilization with exponential estimates for system (5). Then, using such conditions, we will further provide a design procedure for desired fuzzy controllers and give the estimates of the decay rate and the decay coefficient.
3. Stability, exponential estimates and stabilization

In this section, we solve the robust stabilization problem formulated in the previous section by using two approaches, namely, the relaxed LMI approach and simplified LMI approach. We will first derive sufficient conditions that guarantee the robust exponential stability with a decay rate for closed-loop system (8). Then, we will give the design results and exponential estimates. We will also discuss the conservatism and complexity of the results obtained by the two approaches.

3.1. Relaxed LMI approach

**Theorem 1.** For given scalars \( \lambda > 0 \) and \( d > 0 \), system (8) is robustly exponentially stable with a decay rate \( \lambda \), if there exist matrices \( P > 0, Q > 0, R > 0, \{ M_i \}_{i \in \mathbb{R}}, \{ N_i \}_{i \in \mathbb{R}}, \{ \Phi_i \}_{i \in \mathbb{R}}, \{ \Psi_{ij} \} \) such that the following conditions hold:

\[
\begin{bmatrix}
\Phi_1 & \Psi_{12} & \cdots & \Psi_{1r} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
* & * & \cdots & \Phi_r
\end{bmatrix} < 0,
\]

(10)

\[
\begin{bmatrix}
\mathcal{W}_{ii} - \Phi_i & C_i & e_{ij}D_{ij} \\
* & -e_{ii}I & 0 \\
* & * & -e_{ii}I
\end{bmatrix} < 0, \quad i \in \mathbb{R},
\]

(11)

\[
\begin{bmatrix}
\mathcal{W}_{ij} + \mathcal{W}_{ji} & -\Psi_{ij} - \Psi_{ij}^T & C_i & C_j & e_{ij}D_{ij} & e_{ji}D_{ji} \\
* & -e_{ij}I & 0 & 0 \\
* & * & -e_{ij}I & 0 \\
* & * & * & -e_{ij}I \\
* & * & * & * & -e_{ij}I
\end{bmatrix} < 0, \quad 1 \leq i < j \leq r.
\]

(12)

where

\[
\mathcal{W}_{ij} = \begin{bmatrix}
\mathcal{W}^{(11)} & \mathcal{W}^{(12)} \\
* & \mathcal{W}^{(22)}
\end{bmatrix} = \begin{bmatrix}
dM_i & d(A_{cij} + \lambda I)^T R \\
* & -dN_i
\end{bmatrix},
\]

\[
C_i = [C_i^T P 0 0 dC_i^T R]^T, \quad D_{ij} = [D_{cij} \exp(\lambda d)D_{2ij} 0 0]^T,
\]

\[
\mathcal{W}^{(11)} = P(A_{cij} + \lambda I) + (A_{cij} + \lambda I)^T P + Q + M_i + M_i^T,
\]

\[
\mathcal{W}^{(12)} = \exp(\lambda d)PA_{di} - M_i + N_i^T, \quad \mathcal{W}^{(22)} = -Q - N_i - N_i^T.
\]

In this case, the corresponding exponential decay coefficient \( \sigma \) is given by

\[
\sigma = \max \left\{ \sqrt{\frac{\mu_4(\lambda)}{\mu_3(\lambda)}}, (\mu_3(\lambda)d + 1) \exp((\mu_3(\lambda) + \lambda) d) \right\},
\]

(13)

where

\[
\mu_1(\lambda) = \max_{i,j \in \mathbb{R}} \{ \| A_{cij} + \lambda I \| + \| C_i \| \cdot \| D_{cij} \| \},
\]

(14)

\[
\mu_2(\lambda) = \max_{i \in \mathbb{R}} \{ \| A_{di} \| + \| C_i \| \cdot \| D_{2ij} \| \} \cdot \exp(\lambda d), \quad \mu_3(\lambda) = \mu_1(\lambda) + \mu_2(\lambda),
\]

(15)

\[
\mu_4(\lambda) = (\lambda_{\max}(P) + d \cdot \lambda_{\max}(Q) + \mu_1(\lambda) \lambda d^2 \cdot \lambda_{\max}(R)) (\mu_3(\lambda)d + 1)^2 \exp(2\mu_3(\lambda)d)
+ 2\mu_1(\lambda) \mu_2(\lambda) d^2 \cdot \lambda_{\max}(R)(\mu_3(\lambda)d + 1) \exp(\mu_3(\lambda)d) + \mu_2(\lambda) d^2 \cdot \lambda_{\max}(R).
\]

(16)
Based on these inequalities and recalling (10), we obtain
\[
\mathcal{A}(h) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j [A_{cij} + \Delta A_{cij}(t) + \lambda I],
\]
\[
\mathcal{A}_d(h) = \sum_{i=1}^{r} h_i [A_{di} + \Delta A_{di}(t)] \exp(\lambda d).
\]

Then, system (8) with (2) is equivalently transformed to
\[
\dot{y}(t) = \mathcal{A}(h)y(t) + \mathcal{A}_d(h)y(t - d),
\]
\[
y(t) = \phi(t) = \exp(\lambda t)\phi(t), \quad \forall t \in [-d, 0].
\]

Under the conditions of the theorem, we first prove the asymptotic stability of system (19). To this end, we employ the following Lyapunov functional candidate for (19):
\[
V(y_t) = y(t)^T P y(t) + \int_{-d}^{t} y(z)^T Q y(z) \, dz + \int_{-d}^{t} \int_{t+\beta}^{t} \dot{y}(z)^T R \dot{y}(z) \, dz \, d\beta.
\]

The time derivative of \(V(y_t)\) along the trajectories of (19) is obtained as
\[
\dot{V}(y_t) = 2y(t)^T P \dot{y}(t) + y(t)^T Q y(t) - y(t-d)^T Q y(t-d) + \dot{y}(t)^T R \dot{y}(t) - \int_{-d}^{t} \dot{y}(z)^T R \dot{y}(z) \, dz
\]
\[
= 2y(t)^T P [\mathcal{A}(h)y(t) + \mathcal{A}_d(h)y(t - d)] + y(t)^T Q y(t) - y(t-d)^T Q y(t-d)
\]
\[
+ d[\mathcal{A}(h)y(t) + \mathcal{A}_d(h)y(t - d)]^T R [\mathcal{A}(h)y(t) + \mathcal{A}_d(h)y(t - d)] - \int_{-d}^{t} \dot{y}(z)^T R \dot{y}(z) \, dz
\]
\[
+ 2[y(t)^T M(h) + y(t-d)^T N(h)] \left[ y(t) - y(t-d) - \int_{-d}^{t} \dot{y}(z) \, dz \right]
\]
\[
= \frac{1}{d} \int_{-d}^{t} \dot{\zeta}(t, z)^T \mathcal{Y}(h) \dot{\zeta}(t, z) \, dz,
\]
where
\[
\mathcal{Y}(h) = \begin{bmatrix}
P \mathcal{A}(h) + \mathcal{A}(h)^T P + Q + M(h) + M(h)^T & P \mathcal{A}_d(h) - M(h) + N(h)^T & -d M(h) \\
* & -Q - N(h) - N(h)^T & -d N(h) \\
* & * & -d R
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
\mathcal{A}(h)^T \\
\mathcal{A}_d(h)^T \\
0
\end{bmatrix} (dR) \begin{bmatrix}
\mathcal{A}(h)^T \\
\mathcal{A}_d(h)^T \\
0
\end{bmatrix}^T.
\]

From (11) and (12), there always exists a scalar \(\delta > 0\) such that
\[
\mathcal{W}_{ii} + \mathcal{W}_{ii}^{-1} \mathcal{C}_i C_i^T + \mathcal{W}_{ii} \mathcal{D}_{ij} \mathcal{D}_{ij}^T \leq \Phi_i - \delta \begin{bmatrix}
I_n & 0 \\
0 & 0
\end{bmatrix}, \quad i \in \mathfrak{N}
\]
and
\[
\mathcal{W}_{ij} + \mathcal{W}_{ji} + \mathcal{W}_{ij}^{-1} \mathcal{C}_j C_j^T + \mathcal{W}_{ji} \mathcal{D}_{ij} \mathcal{D}_{ij}^T \leq \Psi_{ij} + \Psi_{ij}^T - 2\delta \begin{bmatrix}
I_n & 0 \\
0 & 0
\end{bmatrix}, \quad 1 \leq i < j \leq r.
\]

Based on these inequalities and recalling (10), we obtain
\[
\begin{bmatrix}
P \mathcal{A}(h) + \mathcal{A}(h)^T P + Q + M(h) + M(h)^T & P \mathcal{A}_d(h) - M(h) + N(h)^T & -d M(h) & d \mathcal{A}(h)^T R \\
* & -Q - N(h) - N(h)^T & -d N(h) & d \mathcal{A}_d(h)^T R \\
* & * & -d R & 0
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
\mathcal{A}(h)^T \\
\mathcal{A}_d(h)^T \\
0
\end{bmatrix} (dR) \begin{bmatrix}
\mathcal{A}(h)^T \\
\mathcal{A}_d(h)^T \\
0
\end{bmatrix}^T.
\]
We observe that
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j (W_{ij} + C_i F_i(t) D_{ij}^T + D_{ij} F_i(t)^T C_i^T)
\]
\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j (W_{ij} + e_i^{-1} C_i^T + e_j D_{ij}^T)
\]
\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \left( \Phi_i - \delta \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right) + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} h_i h_j \left( \Psi_{ij} + \Psi_{ij}^T - 2\delta \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right)
\]
\[
= \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix} \begin{bmatrix} \Phi_1 & \Psi_{12} & \cdots & \Psi_{1r} \\ \Phi_2 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \Phi_r \\ \Phi_1 & \cdots & \cdots & \Phi_r \end{bmatrix} \begin{bmatrix} h_1 I \\ h_2 I \\ \vdots \\ h_r I \end{bmatrix} - \delta \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}
\]
\[
\leq -\delta \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}.
\]

This, by using the Schur complement equivalence, implies \( V(h) \leq -\delta I \). By this and (22), it is easy to have
\[
V(y(t)) \leq -\delta |y(t)|^2.
\]

(23)

Therefore, system (19) is asymptotically stable.

Now, we are in a position to show the exponential stability of system (8). From (19) we have
\[
|y(t)| \leq |y(0)| + \int_0^t \|A(h(s(x)))\| \cdot |y(x)| \, dx + \int_0^t \|A_d(h(s(x)))\| \cdot |y(x - d)| \, dx.
\]

(24)

Observe that
\[
\|A(h(s(x)))\| = \left\| \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(s(x)) h_j(s(x)) [A_{cij} + C_i F_i(x) D_{cij} + \lambda I] \right\|
\]
\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(s(x)) h_j(s(x)) \left\| [A_{cij} + \lambda I] + \|C_i\| \cdot \|D_{cij}\| \right\|
\]
\[
\leq \mu_1(\lambda)
\]

and
\[
\|A_d(h(s(x)))\| = \left\| \sum_{i=1}^{r} h_i(s(x)) [A_{di} + C_i F_i(x) D_{2i}] \exp(\lambda d) \right\|
\]
\[
\leq \exp(\lambda d) \sum_{i=1}^{r} h_i(s(x)) \left\| [A_{di} + \|C_i\| \cdot \|D_{2i}\|] \right\|
\]
\[
\leq \mu_2(\lambda),
\]

where \( \mu_1(\lambda) \) and \( \mu_2(\lambda) \) are given in (14) and (15), respectively. Then, it follows
\[
|y(t)| \leq |y(0)| + \mu_1(\lambda) \int_0^t |y(x)| \, dx + \mu_2(\lambda) \int_0^t |y(x - d)| \, dx
\]
\[
\leq |y(0)| + \mu_3(\lambda) \int_{-d}^t |y(x)| \, dx,
\]
where $\mu_3(\lambda)$ is given in (15). When $0 \leq t \leq d$, we have
\[
|y(t)| \leq |y(0)| + \mu_3(\lambda) \int_{-d}^{0} |y(x)| \, dx + \mu_3(\lambda) \int_{0}^{t} |y(x)| \, dx
\]
\[
\leq (\mu_3(\lambda)d + 1)|\phi|_d + \mu_3(\lambda) \int_{0}^{t} \sup_{0 \leq \beta \leq x} |y(\beta)| \, dx.
\]
This implies that, for any $t$ satisfying $0 \leq t \leq d$,
\[
\sup_{0 \leq \beta \leq d} |y(\beta)| \leq (\mu_3(\lambda)d + 1)|\phi|_d + \mu_3(\lambda) \int_{0}^{t} \sup_{0 \leq \beta \leq x} |y(\beta)| \, dx.
\]
Applying the Gronwall–Bellman Lemma to this inequality yields
\[
\sup_{0 \leq \beta \leq d} |y(\beta)| \leq (\mu_3(\lambda)d + 1)|\phi|_d \exp(\mu_3(\lambda)d), \tag{25}
\]
which further implies
\[
\sup_{0 \leq \beta \leq d} |x(\beta)| \leq (\mu_3(\lambda)d + 1) \exp(\mu_3(\lambda)d + \lambda d) \exp(-\lambda d)|\phi|_d. \tag{26}
\]
The inequality in (25) also guarantees
\[
\sup_{0 \leq \beta \leq d} |y(\beta)|^2 \leq (\mu_3(\lambda)d + 1)^2 \exp(2\mu_3(\lambda)d)|\phi|_d^2.
\]
By considering this and using (21), we obtain
\[
V(y_d) \leq \lambda_{\max}(P)|y(d)|^2 + \lambda_{\max}(Q) \int_{0}^{d} |y(x)|^2 \, dx + d \cdot \lambda_{\max}(R) \int_{0}^{d} |y(x)|^2 \, dx
\]
\[
\leq (\lambda_{\max}(P) + d \cdot \lambda_{\max}(Q)) \sup_{0 \leq \beta \leq d} |y(\beta)|^2 + d \cdot \lambda_{\max}(R) \int_{0}^{d} (\mu_1(\lambda)|y(x)| + \mu_2(\lambda)|y(x - d)|)^2 \, dx
\]
\[
= (\lambda_{\max}(P) + d \cdot \lambda_{\max}(Q)) \sup_{0 \leq \beta \leq d} |y(\beta)|^2 + \mu_1(\lambda)^2d \cdot \lambda_{\max}(R) \int_{0}^{d} |y(x)|^2 \, dx
\]
\[
+ 2\mu_1(\lambda)\mu_2(\lambda)d \cdot \lambda_{\max}(R) \int_{0}^{d} |y(x)||y(x - d)| \, dx + \mu_2(\lambda)^2d \cdot \lambda_{\max}(R) \int_{0}^{d} |y(x - d)|^2 \, dx
\]
\[
\leq (\lambda_{\max}(P) + d \cdot \lambda_{\max}(Q) + \mu_1(\lambda)^2d^2 \cdot \lambda_{\max}(R)) \sup_{0 \leq \beta \leq d} |y(\beta)|^2
\]
\[
+ 2\mu_1(\lambda)\mu_2(\lambda)d^2 \cdot \lambda_{\max}(R)|\phi|_d \sup_{0 \leq \beta \leq d} |y(\beta)| + \mu_2(\lambda)^2d^2 \cdot \lambda_{\max}(R)|\phi|_d^2
\]
\[
\leq (\lambda_{\max}(P) + d \cdot \lambda_{\max}(Q) + \mu_1(\lambda)^2d^2 \cdot \lambda_{\max}(R)(\mu_3(\lambda)d + 1)^2|\phi|_d^2 \exp(2\mu_3(\lambda)d)
\]
\[
+ 2\mu_1(\lambda)\mu_2(\lambda)d^2 \cdot \lambda_{\max}(R)(\mu_3(\lambda)d + 1)|\phi|_d^2 \exp(\mu_3(\lambda)d) + \mu_2(\lambda)^2d^2 \cdot \lambda_{\max}(R)|\phi|_d^2
\]
\[
= \mu_4(\lambda)|\phi|_d^2, \tag{27}
\]
where $\mu_4(\lambda)$ is given in (16). It is not difficult to see that, for $t \geq d$,
\[
\lambda_{\min}(P)|y(t)|^2 \leq V(y_t) \leq V(y_d),
\]
which, together with (27), implies that, for $t \geq d$,
\[
|y(t)|^2 \leq \frac{\mu_4(\lambda)}{\lambda_{\min}(P)}|\phi|_d^2.
\]
This further implies that, for $t \geq d$,
\[
|x(t)| \leq \sqrt{\frac{\mu_4(\lambda)}{\lambda_{\min}(P)}} \exp(-\lambda t)|\phi|_d.
\]
By considering this and (26), we have that, for all \( t \geq 0 \),
\[
|\dot{x}(t)| \leq \sigma \exp(-\lambda t)|\varphi|_d,
\]
where \( \sigma \) is given in (13). Therefore, by Definition 1, the uncertain fuzzy time-delay system in (8) is robustly exponentially stable with a decay rate \( \lambda \). \( \square \)

**Remark 1.** The relaxed LMI approach was used in the proof of Theorem 1. This approach has been used in papers related to T–S fuzzy systems, such as \([21,33,34]\), and has been proved to be very effective in reducing the conservatism of results on fuzzy controller design. Moreover, in the proof of Theorem 1, the fuzzy-basis-dependent free weighting matrices \( M \) and \( N \) are introduced so as to further reduce the conservatism of the corresponding results.

Based on Theorem 1, we have the following results on the solvability of the problem of robust stabilization with exponential estimates for systems (5).

**Theorem 2.** Consider the uncertain fuzzy time-delay system in (5). For given scalars \( d > 0 \) and \( \lambda > 0 \), the problem of robust stabilization with exponential estimates is solvable, if there exist matrices \( X \geq 0, Y \geq 0, Z \geq 0, \{G_i\}_{i \in \mathcal{R}}, \{U_i\}_{i \in \mathcal{R}}, \{V_i\}_{i \in \mathcal{R}}, \{A_i\}_{i \in \mathcal{R}}, \{\Omega_{ij}\}_{1 \leq i < j \leq r} \) and scalars \( \{\rho_{ij} > 0\}_{i,j \in \mathcal{R}} \) such that the following LMIs are satisfied:

\[
\begin{bmatrix}
A_1 & \Omega_{12} & \cdots & \Omega_{1r} \\
* & A_2 & \cdots & \Omega_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & A_r
\end{bmatrix} < 0, \\
\begin{bmatrix}
S_{ii} - A_i & \rho_{ij} \mathcal{H}_i & \mathcal{J}_{ij} \\
* & -\rho_{ii} I & 0 \\
* & * & -\rho_{ii} I
\end{bmatrix} < 0, \quad i \in \mathcal{R},
\]

\[
\begin{bmatrix}
S_{ij} + S_{ji} - \Omega_{ij} - \Omega_{ij}^T & \rho_{ij} \mathcal{H}_i & \mathcal{J}_{ij} \\
& -\rho_{ij} I & 0 \\
& * & -\rho_{ij} I & 0 \\
& * & * & 0 \\
& * & * & -\rho_{ij} I \\
& * & * & * & -\rho_{ij} I
\end{bmatrix} < 0, \quad 1 \leq i < j \leq r,
\]

where

\[
S_{ij} = \begin{bmatrix}
S^{(11)} & S^{(12)} & -dU_i & d\lambda X + dXA_i^T + dG_i^TB_i^T \\
* & S^{(22)} & -dV_i & dXA_i^T \exp(\lambda d) \\
* & * & dZ - 2dX & 0 \\
* & * & * & -dZ
\end{bmatrix},
\]

\[
\mathcal{H}_i = [C_i^T 0 0 dC_i^T]^T, \quad \mathcal{J}_{ij} = [D_{ij}X + D_{ij}G_j D_{ij}X \exp(\lambda d) 0 0]^T,
\]

\[
S^{(11)} = 2\lambda X + A_iX + XA_i^T + B_iG_j + G_j^TB_i^T + Y + U_i + U_i^T, \\
S^{(12)} = A_{di}X \exp(\lambda d) - U_i + V_i^T, \quad S^{(22)} = -Y - V_i - V_i^T.
\]

In this case, the controller gains of (7) are obtained as

\[
K_i = G_i X^{-1}, \quad i \in \mathcal{R}.
\]

Moreover, the corresponding exponential decay coefficient \( \sigma \) is given by

\[
\sigma = \max \left\{ \mu_4(\lambda) \sqrt{\frac{\mu_3(\lambda)}{\lambda_{\min}(X^{-1})}}, (\mu_3(\lambda)d + 1) \exp((\mu_3(\lambda) + \lambda)d) \right\},
\]

where

\[
\mu_3(\lambda) = \lambda + 1, \quad \mu_4(\lambda) = \lambda + \frac{1}{2}. 
\]
where
\[
\mu_1(\lambda) = \max_{i, j \in \mathbb{R}} \{\| A_i + B_i G_j X^{-1} + \lambda I \| + \| C_i \| \cdot \| D_{ii} + D_{3j} G_j X^{-1} \|\},
\]
\[
\mu_2(\lambda) = \max_{i \in \mathbb{R}} \{\| A_{di} \| + \| C_i \| \cdot \| D_{2i} \| \} \cdot \exp(\lambda d), \quad \mu_3(\lambda) = \mu_1(\lambda) + \mu_2(\lambda),
\]
\[
\mu_4(\lambda) = (\lambda_{\max}(X^{-1}) + d \cdot \lambda_{\max}(X^{-1}YX^{-1}) + \mu_1(\lambda)^2 d^2 \cdot \lambda_{\max}(Z^{-1})(\mu_3(\lambda)d + 1)^2 \exp(\mu_3(\lambda)d) + 2\mu_1(\lambda)\mu_2(\lambda)d^2 \cdot \lambda_{\max}(Z^{-1})(\mu_3(\lambda)d + 1)\exp(\mu_3(\lambda)d) + \mu_2(\lambda)^2 d^2 \cdot \lambda_{\max}(Z^{-1})).
\]

**Proof.** Set \( P = X^{-1}, Q = X^{-1}YX^{-1}, R = Z^{-1}, M_i = X^{-1}U_i X^{-1}, N_i = X^{-1}V_i X^{-1}, e_{ij} = \tilde{\rho}_{ij}, E = \text{diag}(X, X, X, Z), \Phi_i = E^{-1}A_i E^{-1} \) and \( \Psi_{ij} = E^{-1}\Omega_{ij} E^{-1} \). Then, pre-multiplying and post-multiplying (28) with \( \text{diag}(E^{-1}, E^{-1}, \ldots, E^{-1}) \) leads to (10). It is also easy to see [21,25]
\[
(X^{-1} - Z^{-1})Z(X^{-1} - Z^{-1}) \succeq 0,
\]
which implies
\[
-Z^{-1} \preceq X^{-1} Z X^{-1} - 2X^{-1}.
\]
By considering this and (32), pre-multiplying and post-multiplying (29) with
\[
\text{diag}(X^{-1}, X^{-1}, X^{-1}, Z^{-1}, \rho_{ii}^{-1} I, \rho_{ji}^{-1} I),
\]
we obtain (11). Similarly, by pre-multiplying and post-multiplying (30) with
\[
\text{diag}(X^{-1}, X^{-1}, X^{-1}, Z^{-1}, \rho_{ij}^{-1} I, \rho_{ji}^{-1} I, \rho_{ij}^{-1} I),
\]
we have (12). Therefore, by Theorem 1, the closed-loop system (8) is robustly exponentially stable with a decay rate \( \hat{\lambda} \) and the corresponding exponential decay coefficient is given by (33). \( \square \)

When there is no parameter uncertainty in (5), Theorem 2 reduces to the following one.

**Theorem 3.** Consider (5) with no parameter uncertainties; that is, \( \Delta A_i = 0, \Delta A_{di} = 0 \) and \( \Delta B_i = 0 \). For given scalars \( d > 0 \) and \( \hat{\lambda} > 0 \), there exists a state-feedback fuzzy controller in the form of (7) such that the resulting closed-loop system is exponentially stable with a decay rate \( \hat{\lambda} \), if there exist matrices \( X > 0, Y > 0, Z > 0, \{ G_i \}_{i \in \mathbb{R}}, \{ U_i \}_{i \in \mathbb{R}}, \{ V_i \}_{i \in \mathbb{R}}, \{ A_i \}_{i \in \mathbb{R}} \) and \( \{ \Omega_{ij} \}_{1 \leq i < j \leq r} \) such that (28) and the following LMIs are satisfied:
\[
S_{ii} - A_i < 0, \quad i \in \mathbb{R},
\]
\[
S_{ij} + S_{ji} - \Omega_{ij} - \Omega_{ij}^T < 0, \quad 1 \leq i < j \leq r,
\]
where \( S_{ij} \) is given by (31). In this case, the controller gains of (7) are obtained as
\[
K_i = G_i X^{-1}, \quad i \in \mathbb{R}.
\]
Moreover, the corresponding exponential decay coefficient \( \sigma \) is given by
\[
\sigma = \max \left\{ \sqrt{\frac{\mu_4(\lambda)}{\lambda_{\min}(X^{-1})}}, (\mu_3(\lambda)d + 1)\exp((\mu_3(\lambda) + \hat{\lambda})d) \right\},
\]
where
\[
\mu_1(\lambda) = \max_{i, j \in \mathbb{R}} \{\| A_i + B_i G_j X^{-1} + \lambda I \|\}, \quad \mu_2(\lambda) = \max_{i \in \mathbb{R}} \{\| A_{di} \| \} \cdot \exp(\lambda d), \quad \mu_3(\lambda) = \mu_1(\lambda) + \mu_2(\lambda),
\]
\[
\mu_4(\lambda) = (\lambda_{\max}(X^{-1}) + d \cdot \lambda_{\max}(X^{-1}YX^{-1}) + \mu_1(\lambda)^2 d^2 \cdot \lambda_{\max}(Z^{-1})(\mu_3(\lambda)d + 1)^2 \exp(\mu_3(\lambda)d) + 2\mu_1(\lambda)\mu_2(\lambda)d^2 \cdot \lambda_{\max}(Z^{-1})(\mu_3(\lambda)d + 1)\exp(\mu_3(\lambda)d) + \mu_2(\lambda)^2 d^2 \cdot \lambda_{\max}(Z^{-1}).
\]
3.2. Simplified LMI approach

In the previous subsection, some results are obtained based on the relaxed LMI approach. It should be pointed out that these results contain too many LMIs and decision variables, which are always difficult to check, especially when the number \( r \) is very large. Some may be more concerned with the computational burden than the design conservatism. Motivated by this, in this section we provide a simplified LMI approach for the study of the problem formulated in Section 2.

**Theorem 4.** For given scalars \( \lambda > 0 \) and \( d > 0 \), system (8) is robustly exponentially stable with a decay rate \( \lambda \), if there exist matrices \( P > 0, Q > 0, R > 0, P_1, P_2, \{ M_1 \}_{i \in \mathbb{N}}, \{ N_1 \}_{i \in \mathbb{N}}, \) and scalars \( \{ \varepsilon_i > 0 \}_{i \in \mathbb{N}} \) such that the following conditions hold for \( i \in \mathbb{N} \):

\[
\begin{bmatrix}
\Xi_1 & \Xi_2 & -dH^TM_i & \tilde{A}_i^T & \tilde{C}_i & \varepsilon_i \tilde{D}_{1i}^T \\
* & \Xi_3 & -dN_i & \tilde{A}_i^T & 0 & \varepsilon_i \tilde{D}_{2i}^T \\
* & * & -dR & 0 & 0 & 0 \\
* & * & * & -d^{-1}R^{-1}H\tilde{C}_i & 0 & 0 \\
* & * & * & * & -\varepsilon_i I & 0 \\
* & * & * & * & * & -\varepsilon_i I \\
\end{bmatrix} < 0, 
\]

where

\[
\Xi_1 = \tilde{P}^T\tilde{A}_i + \tilde{A}_i^T\tilde{P} + H^TQM_iH + H^T M_iH + H^T M_i^TH, \\
\Xi_2 = \tilde{P}^T\tilde{A}_i, \\
\Xi_3 = -Q - N_i - N_i^T, \\
\tilde{P} = \begin{bmatrix} P & 0 \\ P_1 & P_2 \end{bmatrix}, \\
\tilde{A}_i = \begin{bmatrix} A_i + \lambda I & B_i \\ K_i & -I \end{bmatrix}, \\
\tilde{A}_i = \begin{bmatrix} A_i \exp(\lambda d) & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{C}_i = \begin{bmatrix} C_i \\ 0 \end{bmatrix}, \\
\tilde{D}_{1i} = \begin{bmatrix} D_{1i}^T \\ D_{2i}^T \end{bmatrix}, \\
\tilde{D}_{2i} = D_{2i} \exp(\lambda d), \\
H = [I \ 0].
\]

Moreover, the corresponding exponential decay coefficient \( \sigma \) is given by (13).

**Proof.** Set

\[
\tilde{A}(h) = \sum_{i=1}^{r} h_i(A_i + \Delta A_i(t)), \\
\tilde{A}_d(h) = \sum_{i=1}^{r} h_i(A_{di} + \Delta A_{di}(t)), \\
\tilde{B}(h) = \sum_{i=1}^{r} h_i(B_i + \Delta B_i(t)), \\
K(h) = \sum_{i=1}^{r} h K_i.
\]

Then, \( \tilde{A}(h) \) and \( \tilde{A}_d(h) \) given in (17) and (18) are rewritten as \( \tilde{A}(h) = \tilde{A}(h) + \tilde{B}(h)K(h) + \lambda I \) and \( \tilde{A}_d(h) = \tilde{A}_d(h) \exp(\lambda d) \), respectively. Let \( \tilde{u}(t) = u(t) \exp(\lambda t) \). Then, \( \tilde{u}(t) = K(h)\tilde{y}(t) \). It is easy to obtain that

\[
2y(t)^T P_1 \tilde{y}(t) = 2y(t)^T P_1(\tilde{A}(h)\tilde{y}(t) + \tilde{B}(h)\tilde{u}(t) + \tilde{A}_d(h) \exp(\lambda d) y(t - d)) \\
+ 2[y(t)^T P_1 + \tilde{u}(t)^T P_2][K(h)\tilde{y}(t) - \tilde{u}(t)] \\
= z(t)^T \{ \tilde{P}^T \tilde{A}(h) + \tilde{A}(h)^T \tilde{P} \} z(t) + 2z(t)^T \tilde{P}^T \tilde{A}_d(h)Hz(t - d),
\]

where

\[
z(t) = \begin{bmatrix} y(t) \\ \tilde{u}(t) \end{bmatrix}, \\
\tilde{A}(h) = \begin{bmatrix} \tilde{A}(h) + \lambda I & \tilde{B}(h) \\ K(h) & -I \end{bmatrix}, \\
\tilde{A}_d(h) = \begin{bmatrix} \tilde{A}_d(h) \exp(\lambda d) \\ 0 \end{bmatrix}.
\]
Now, we consider a Lyapunov functional the same as that in (21). By considering the above, we have

\[
V(y) = z(t)^T \{ \bar{P}^T \bar{A}(h) + \bar{A}(h)^T \bar{P} \} z(t) + 2z(t)^T \bar{P}^T \bar{A}_d(h)y(t - d) + d[H \bar{A}(h)z(t) + H \bar{A}_d(h)y(t - d)]^T \bar{R}_1[H \bar{A}(h)z(t) + H \bar{A}_d(h)y(t - d)] + 2[z(t)^T H^T M(h) + y(t - d)^T N(h)]^T H(z(t) - y(t - d) - \int_{t-d}^t \dot{y}(z) \, dz
\]

\[
= \frac{1}{d} \int_{t-d}^t \dot{\vartheta}(t, z)^T \Theta(h) \vartheta(t, z) \, dz,
\]

where

\[
M(h) = \sum_{i=1}^r h_i M_i, \quad N(h) = \sum_{i=1}^r h_i N_i, \quad \vartheta(t, z) = [z(t)^T \, y(t - d)^T \, \dot{y}(z)^T]^T,
\]

\[
\Theta(h) = \begin{bmatrix}
\Theta_1(h) & \Theta_2(h) & -dH^T M(h) \\
* & \Theta_3(h) & -dN(h) \\
* & * & -dR
\end{bmatrix} + \begin{bmatrix}
\bar{A}(h)^T H^T \\
\bar{A}_d(h)^T H^T \\
0
\end{bmatrix} (dR) \begin{bmatrix}
\bar{A}(h)^T H^T \\
\bar{A}_d(h)^T H^T \\
0
\end{bmatrix}^T,
\]

\[
\Theta_1(h) = \bar{P}^T \bar{A}(h) + \bar{A}(h)^T \bar{P} + H^T Q H + H^T M(h) H + H^T M(h)^T H,
\]

\[
\Theta_2(h) = \bar{P}^T \bar{A}_d(h) - H^T M(h) + H^T N(h)^T, \quad \Theta_3(h) = -Q - N(h) - N(h)^T.
\]

Now, by (38), there always exists a scalar \( \delta > 0 \) such that, for \( i \in \mathbb{N} \),

\[
\begin{bmatrix}
\Xi_1 + \delta I & \Xi_2 - dH^T M_i & \bar{A}_d^T H^T \\
* & \Xi_3 - dN_i & \bar{A}_d^T H^T \\
* & * & -dR
\end{bmatrix} + \varepsilon_i^{-1} \begin{bmatrix}
\bar{P}^T \bar{C}_i \\
\bar{P}^T \bar{C}_i \\
H \bar{C}_i
\end{bmatrix} + \varepsilon_i \begin{bmatrix}
\bar{D}_1^T \\
\bar{D}_2^T \\
H \bar{C}_i
\end{bmatrix}^T < 0,
\]

which implies

\[
\begin{bmatrix}
\Xi_1 + \delta I & \Xi_2 - dH^T M_i & \bar{A}_d^T H^T \\
* & \Xi_3 - dN_i & \bar{A}_d^T H^T \\
* & * & -dR
\end{bmatrix} + \begin{bmatrix}
\bar{P}^T \bar{C}_i \\
0 \\
H \bar{C}_i
\end{bmatrix} F_i(t) + \begin{bmatrix}
\bar{D}_1^T \\
\bar{D}_2^T \\
H \bar{C}_i
\end{bmatrix} F_i^T(t) < 0.
\]

This guarantees that

\[
\Theta(h) + \delta I \begin{bmatrix}
\Theta_1(h) & \Theta_2(h) & -dH^T M(h) \\
* & \Theta_3(h) & -dN(h) \\
* & * & -dR
\end{bmatrix} < 0.
\]

Applying the Schur complement equivalence to (40) yields that

\[
\Theta(h) < -\delta \begin{bmatrix}
I_{n+m} & 0 \\
0 & 0
\end{bmatrix}.
\]

By this and (39), we have

\[
V(y) \leq \delta \|z(t)\|^2 \leq -\delta \|y(t)\|^2.
\]

Therefore, system (19) is asymptotically stable. Furthermore, by following a similar line as in the proof of Theorem 1, we can prove that system (19) is robustly exponentially stable with a decay rate \( \lambda \) and the corresponding exponential decay coefficient \( \sigma \) is given by (13). \( \square \)
By Theorem 4, the design results are formulated as follows.

**Theorem 5.** Consider the uncertain fuzzy time-delay system in (5). For given scalars \( d > 0 \) and \( \lambda > 0 \), the problem of robust stabilization with exponential estimates is solvable, if there exist matrices \( X > 0 \), \( Y > 0 \), \( Z > 0 \), \( X_1 \), \( X_2 \), \( \{G_i\}_{i \in \mathbb{N}} \), \( \{U_i\}_{i \in \mathbb{N}} \), \( \{V_i\}_{i \in \mathbb{N}} \), and scalars \( \{\rho_i > 0\}_{i \in \mathbb{N}} \) such that the following LMIs are satisfied for \( i \in \mathbb{N} \):

\[
\begin{bmatrix}
I_1 & I_2 & I_3 & -dU_i & X(A_i + \lambda I)^T + X_1^T B_1^T & \rho_i C_i & X D_{1i}^T + X_1^T D_{3i}^T
\end{bmatrix}
\begin{bmatrix}
* & -X_2 - X_2^T & 0 & 0 & X_2^T B_1^T & 0 & X_2^T D_{3i}^T
* & * & -Y - V_i - V_i^T & -dV_i & X A_{di}^T \exp(\lambda d) & 0 & X D_{2i}^T \exp(\lambda d)
* & * & * & dZ - 2dX & 0 & 0 & < 0
* & * & * & * & -d^{-1}Z & \rho_i C_i & 0
* & * & * & * & * & -\rho_i I & 0
* & * & * & * & * & -\rho_i I &
\end{bmatrix}
\]

where

\[
I_1 = (A_i + \lambda I)X + X(A_i + \lambda I)^T + B_1X_1 + X_1^T B_1^T + Y + U_i + U_i^T,
\]

\[
I_2 = B_1X_2 + G_i^T - X_1^T,
\]

\[
I_3 = A_{di}X \exp(\lambda d) - U_i + V_i^T.
\]

In this case, the controller gains of (7) are obtained as

\[
K_i = G_iX^{-1}, \quad i \in \mathbb{N}.
\]

Moreover, the corresponding exponential decay coefficient \( \sigma \) is given by (33).

**Proof.** By considering \(-XZ^{-1}X \leq Z - 2X\), it follows from (41) that

\[
\begin{bmatrix}
I_1 & I_2 & I_3 & -dU_i & X(A_i + \lambda I)^T + X_1^T B_1^T & \rho_i C_i & X D_{1i}^T + X_1^T D_{3i}^T
\end{bmatrix}
\begin{bmatrix}
* & -X_2 - X_2^T & 0 & 0 & X_2^T B_1^T & 0 & X_2^T D_{3i}^T
* & * & -Y - V_i - V_i^T & -dV_i & X A_{di}^T \exp(\lambda d) & 0 & X D_{2i}^T \exp(\lambda d)
* & * & * & dZ - 2dX & 0 & 0 & < 0
* & * & * & * & -d^{-1}Z & \rho_i C_i & 0
* & * & * & * & * & -\rho_i I & 0
* & * & * & * & * & -\rho_i I &
\end{bmatrix}
\]

This implies \(X_2 + X_2^T > 0\), which yields that \(X_2\) is nonsingular. Set \( P = X^{-1} > 0, P_1 = -X_2^{-1}X_1X^{-1}, P_2 = X_2^{-1} \). Then, it can be verified that

\[
\tilde{P} = \begin{bmatrix} P & 0 \\ P_1 & P_2 \end{bmatrix} = \begin{bmatrix} X & 0 \\ X_1 & X_2 \end{bmatrix}^{-1}.
\]

Let \( Q = X^{-1}YX^{-1}, R = Z^{-1}, M_i = X^{-1}U_iX^{-1}, N_i = X^{-1}V_iX^{-1} \) and \( \epsilon_i = \rho_i^{-1} \). With these notations, (45) can be rewritten as

\[
\begin{bmatrix}
\Gamma_1 & \Gamma_2 & -d\tilde{P}^{-T}H^TM_iP^{-1} & \tilde{P}^{-T}A_{1i}^TH^T & \epsilon_i^{-1}\tilde{C}_i & \tilde{P}^{-T}D_{1i}^T
\end{bmatrix}
\begin{bmatrix}
* & * & -d^{-1}P^{-1}M_iP^{-1} & -d^{-1}P^{-1}N_iP^{-1} & -d^{-1}P^{-1} \tilde{P}^{-1} \tilde{D}_{1i}^T
* & * & * & -d^{-1}R^{-1}N_iP^{-1} & -d^{-1}R^{-1} \tilde{P}^{-1} \tilde{D}_{2i}^T
* & * & * & * & -\epsilon_i^{-1}I
* & * & * & * & -\epsilon_i^{-1}I
\end{bmatrix}
\]

< 0.

(46)
where

\[
\begin{align*}
I_1 &= \bar{A}_i P_i^{-1} + \bar{F}^T A_i^T + \bar{F}^T H^T Q H \bar{F}^{-1} + \bar{F}^T H^T M_i H \bar{P}^{-1} + \bar{F}^T H^T M_i^T H \bar{F}^{-1}, \\
I_2 &= \bar{A}_{di} P_i^{-1} - \bar{F}^T H^T M_i P_i^{-1} + \bar{F}^T H^T N_i^T P_i^{-1}, \\
I_3 &= -\bar{P}^{-1} Q \bar{P}^{-1} - \bar{P}^{-1} N_i \bar{P}^{-1} - \bar{P}^{-1} N_i^T \bar{P}^{-1}.
\end{align*}
\]

By pre-multiplying diag(\(\bar{P}^T, P, P, I, \epsilon_I, \epsilon_I\)) and post-multiplying diag(\(\bar{P}, P, P, I, \epsilon_I, \epsilon_I\)) to (46), we obtain the conditions in (38). Therefore, by Theorem 4, the closed-loop system (8) is robustly exponentially stable with a decay rate \(\lambda\) and the corresponding exponential decay coefficient is given by (33).

**Theorem 6.** Consider (5) with no parameter uncertainties; that is, \(\Delta A_i = 0\), \(\Delta A_{di} = 0\) and \(\Delta B_i = 0\). For given scalars \(d > 0\) and \(\lambda > 0\), there exists a state-feedback fuzzy controller in the form of (7) such that the resulting closed-loop system is exponentially stable with a decay rate \(\lambda\), if there exist matrices \(X > 0\), \(Y > 0\), \(Z > 0\), \(X_1, X_2, \{G_i\}_{i \in \mathfrak{N}}, \{U_i\}_{i \in \mathfrak{N}}\) and \(\{V_i\}_{i \in \mathfrak{N}}\) such that the following LMIs are satisfied for \(i \in \mathfrak{N}\):

\[
\begin{bmatrix}
I_1 & I_2 & I_3 & -dU_i & X(A_i + \lambda I)^T + X_i^T B_i^T \\
* & -X_2 - X_2^T & 0 & 0 & X_2^T B_i^T \\
* & * & -Y - V_i - V_i^T & -dV_i & X A_{di}^T \exp(\lambda d) \\
* & * & * & dZ - 2dX & 0 \\
* & * & * & * & -d^{-1}Z
\end{bmatrix} < 0,
\]

where \(I_1, I_2\) and \(I_3\) are given in (42) and (43), respectively. In this case, the controller gains of (7) are obtained as

\[K_i = G_i X_i^{-1}, \quad i \in \mathfrak{N}.\]

Moreover, the corresponding exponential decay coefficient \(\sigma\) is given by (37).

**Remark 2.** In the proof of Theorem 4, by noting the fact that \(K(h)\gamma(t) - \tilde{u}(t) = 0\) and by introducing two slack variables \(P_1\) and \(P_2\), the product \(\tilde{F}(h) K(h)\) are successfully separated. By applying such an approach, several simplified decay-rate-dependent results are obtained in Theorems 5 and 6 for fuzzy controller design and exponential estimates. In comparison with the results obtained by the relaxed LMI approach, however, the results obtained by using the simplified LMI approach may be more conservative to some extent, because the slack variables \(P_1\) and \(P_2\) introduced in Theorem 4 (or \(X_1\) and \(X_2\) in Theorems 5 and 6) are fixed (independent of the fuzzy basis functions). This will be shown by examples given in the next section.

**Remark 3.** One way to assess the efficiency of a computational approach involving LMIs is to consider the number of decision variables involved. We note that the numbers of decision variables involved in the LMI-based conditions of Theorems 2, 3, 5 and 6 are \((8r^2 + 2r + 1.5)n^2 + (mr + 2r + 1.5)n + r^2, (8r^2 + 2r + 1.5)n^2 + (mr + 2r + 1.5)n, (2r + 1.5)^2 + (mr + m + 1.5)n + m^2 + r\) and \((2r + 1.5)n^2 + (mr + m + 1.5)n + m^2\), respectively, where \(n, m, r, m\) and \(r\) are the dimension of the state, the dimension of the input and the number of IF–THEN rules, respectively. Particularly, for \(m = 1\) and \(r = 3\), we provide some data in Table 3 by considering systems with different dimensions. It is seen from the table that the LMI-based conditions in Theorem 5 (Theorem 6) involve a lot less decision variables than those in Theorem 2 (Theorem 3). Therefore, the LMI-based design conditions obtained by the simplified LMI approach are much simpler and requires less computational burdens than those obtained by the relaxed LMI approach, and thus more applicable for T–S fuzzy systems with larger dimensions and more IF–THEN rules.

**Remark 4.** For T–S fuzzy time-delay systems with and without parameter uncertainties, sufficient conditions for the solvability of the problem of robust stabilization with exponential estimates have been presented in Theorems 2, 3, 5 and 6, respectively. Note that these conditions are explicitly dependent on the decay rate \(\lambda\). Thus, the presented results can be applied to design fuzzy controllers for the case where decay rates are specified. It is also worth noting that the decay coefficient \(\sigma\) can be exactly computed once the given LMIs are feasible. This enables us to further estimate the upper bound of the state trajectories of the closed-loop system.
Table 1
Numerical results for Example 1

<table>
<thead>
<tr>
<th>Cases</th>
<th>By Theorem 2</th>
<th>By Theorem 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 0.5 )</td>
<td>( \sigma = 18.8447 )</td>
<td>( \sigma = 40.4552 )</td>
</tr>
<tr>
<td>( \lambda = 1.0 )</td>
<td>( \sigma = 140.2475 )</td>
<td>( \sigma = 215.6266 )</td>
</tr>
<tr>
<td>( \lambda = 2.0 )</td>
<td>( \sigma = 4.5560 \times 10^3 )</td>
<td>( \sigma = 1.2344 \times 10^4 )</td>
</tr>
</tbody>
</table>

4. Numerical examples

In this section, we provide two examples to demonstrate the effectiveness of the proposed methods. The first one considers parameter uncertainties, while there is no uncertainty in the second example.

Example 1. Consider the following T–S fuzzy time-delay system with parameter uncertainties [28]:

\[
\dot{x}(t) = \sum_{i=1}^{2} h_i [(A_i + C_i F_i(t) D_{1i}) x(t) + (A_{di} + C_i F_i(t) D_{2i}) x(t - 0.2) + (B_i + C_i F_i(t) D_{3i}) u(t)],
\]

where

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_{d1} = A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.5 - 1.5\beta \end{bmatrix}.
\]

\[
B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} -0.03 \\ 0 \end{bmatrix}, \quad D_{11} = D_{12} = \begin{bmatrix} -0.15 \\ 0.2 \end{bmatrix}, \quad D_{21} = D_{22} = \begin{bmatrix} -0.05 \\ 0.08 \end{bmatrix}, \quad D_{31} = D_{32} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta = \frac{0.01}{\pi},
\]

\[
h_1 = \left(1 - \frac{1}{1 + \exp(-3(x_2/0.5 - \pi/2))}\right) \times \left(\frac{1}{1 + \exp(-3(x_2/0.5 + \pi/2))}\right),
\]

\[
h_2 = 1 - h_1, \quad F_1(t) = F_2(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}, \quad \cos t.
\]

For \( \lambda = 0.5, 1 \) and \( 2 \), respectively, it is found that the LMI-based conditions in Theorems 2 and 5 are feasible. Using the solutions of the LMIs we obtain the decay coefficient and controller gains that are shown in Table 1. From the table we can see that, for the same decay rate \( \lambda \), the decay coefficient \( \sigma \) obtained by Theorem 2 is less than that obtained by Theorem 5. Hence, the relaxed LMI approach may lead to less conservative results than the simplified LMI approach. On the other hand, for this example, the number of decision variables in LMIs (28)–(30) is 169, while that in LMIs (41) is only 34. Hence, the conditions obtained by the simplified LMI approach are much easier to be checked than those by the relaxed LMI approach.

Example 2. Consider the following truck-trailer model with a time-delay:

\[
\begin{align*}
\dot{x}_1(t) &= -a \frac{v_f}{L_0} x_1(t) - (1 - a) \frac{v}{L_0} x_1(t - d) + \frac{v}{l_0} u(t), \\
\dot{x}_2(t) &= a \frac{v}{L_0} x_1(t) + (1 - a) \frac{v}{L_0} x_1(t - d), \\
\dot{x}_3(t) &= \frac{v}{l_0} \sin \left[ x_2(t) + a \frac{v}{2L} x_1(t) + (1 - a) \frac{v}{2L} x_1(t - d) \right],
\end{align*}
\]
where \( l = 2.8, L = 5.5, \nu = -1.0, \bar{i} = 2.0, t_0 = 0.5, a = 0.7 \) and \( d = 0.1 \). This system can be represented by the following \( T\)-\( S \) fuzzy model [24]:

**Plant Rule 1:** If \( \theta(t) = x_2(t) + a(\bar{v}\bar{i}/2L)x_1(t) + (1 - a)(\bar{v}\bar{i}/2L)x_1(t - d) \) is about 0, THEN

\[
\dot{x}(t) = A_1x(t) + A_{d1}x(t - d) + B_1u(t).
\]  

**Plant Rule 2:** If the decay rate is specified, THEN

\[
\dot{x}(t) = A_2x(t) + A_{d2}x(t - d) + B_2u(t).
\]

In (49) and (50),

\[
A_1 = \begin{bmatrix}
-a\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
\frac{\bar{v}\bar{i}^2}{2L_0} & \frac{\bar{v}\bar{i}}{\bar{t}_0} & 0
\end{bmatrix},
A_{d1} = \begin{bmatrix}
-(1 - a)\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
(1 - a)\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
(1 - a)\frac{\bar{v}\bar{i}^2}{2L_0} & 0 & 0
\end{bmatrix},
B_1 = \begin{bmatrix}
\frac{\bar{v}\bar{i}}{\bar{t}_0} \\
0 \\
0
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
-a\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
\frac{\bar{v}\bar{i}^2}{2L_0} & \frac{\bar{v}\bar{i}}{\bar{t}_0} & 0
\end{bmatrix},
A_{d2} = \begin{bmatrix}
-(1 - a)\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
(1 - a)\frac{\bar{v}\bar{i}}{L_0} & 0 & 0 \\
(1 - a)\frac{\bar{v}\bar{i}^2}{2L_0} & 0 & 0
\end{bmatrix},
B_2 = \begin{bmatrix}
\frac{\bar{v}\bar{i}}{\bar{t}_0} \\
0 \\
0
\end{bmatrix}.
\]

Set \( b = 10t_0/\pi \) and the membership functions as

\[
h_1 = \left(1 - \frac{1}{1 + \exp(-3(\bar{t}(t) - 0.5\pi))}\right),
\]

\[
h_2 = 1 - h_1.
\]

Table 2 shows the numerical results obtained by Theorems 3 and 6, respectively. When the decay rate is specified, it is found that the decay coefficient obtained by Theorem 3 is less than that obtained by Theorem 6. This may show that the results in Theorem 3 are less conservative than those in Theorem 6. It is also worth noting that the number of decision variables in LMIs (28), (34) and (35) is 360, while that in LMIs (47) is only 64. Therefore, solving the LMIs in Theorem 6 requires less computational burdens than that in Theorem 3.

With the initial condition given by \( \theta(t) = [0.5\pi, 0.75\pi, -5]^T \) for \( t \in [-0.1, 0] \), Figs. 1 and 2 show the trajectories of \( |x(t)| \) by considering different decay rates of system (49)–(50) with a fuzzy controller obtained by Theorems 3 and 6, respectively. It is seen that the state trajectories converge more quickly if the decay rate is larger.

Moreover, define

\[
\lambda\alpha = \lim_{t \to \infty} \left(-\frac{\ln |x(t)|}{t}\right),
\]
Table 3
Number of decision variables ($m = 1, r = 3$)

<table>
<thead>
<tr>
<th>Cases</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>348</td>
<td>756</td>
<td>1323</td>
<td>2049</td>
<td>8064</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>45</td>
<td>88</td>
<td>146</td>
<td>219</td>
<td>809</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>339</td>
<td>747</td>
<td>1314</td>
<td>2040</td>
<td>8055</td>
</tr>
<tr>
<td>Theorem 6</td>
<td>42</td>
<td>85</td>
<td>143</td>
<td>216</td>
<td>806</td>
</tr>
</tbody>
</table>

Fig. 1. $|x(t)|$ of the closed-loop system (by Theorem 3).

Fig. 2. $|x(t)|$ of the closed-loop system (by Theorem 6).
which is called the achieved decay rate. It follows from (9) that

$$\frac{-\ln|\chi(t)|}{t} \geqslant \lambda - \frac{\ln(\sigma|\phi_d|)}{t}, \quad t > 0,$$

which implies that $\hat{\lambda}_d \geq \lambda$. For this example, we provide the approximate values of $\hat{\lambda}_d$ in Table 2 for different prescribed decay rates $\lambda$. It can be found that the prescribed decay rate is a good estimate of the achieved decay rate.

5. Conclusions

For T–S fuzzy systems with state delays, this paper has presented several sufficient conditions for the existence of state-feedback fuzzy controllers guaranteeing that the closed-loop system is exponentially stable with a specified decay rate. Different from the existing results in the literature, our results are explicitly dependent on the decay rate and the exponential estimates can be obtained. Two numerical examples have been provided to demonstrate the effectiveness of the proposed design approaches.

References