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Control of switched linear systems with input saturation

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The problem of control of switched linear systems with input saturation is considered. A synthesis method based on the minimum dwell time switching together with saturated feedbacks is presented. Sufficient conditions for stability are proposed in term of linear matrix inequalities. An illustrative example is presented to show the validity of the results.

Keywords: minimum dwell time; switched linear systems; input saturation; switching systems; stability; feedback control; continuous-time systems

1. Introduction

The past decade witnessed an enormous interest in the research of switched systems. Informally, a switched system is an indexed family of differential/difference equations and a rule that determines the switching between them. Many man-made systems and some real-life systems can be described by switched systems. Examples include car transmission systems (Johansson, Lygeros, and Sastry 2004), process control systems (Gao, Ye, Shi, Jiang, and Jia 2002), genetic regulatory networks (de Jong et al. 2004), etc. The rapidly developing area of control techniques based on switching between different controllers (Kulkarni and Ramadge 1996; Morse 1996; Narendra and Balakrishnan 1997) also motivate the study of switched systems. For this class of systems, results have been developed for stability analysis and stabilisation using the tools of common Lyapunov function (Shim, Noh, and Seo 1998; Dayawansa and Martin 1999; Liberon, Hespanha, and Morse 1999), multiple Lyapunov functions (Branicky 1994, 1998), dwell time or average dwell time (Hespanha and Morre 1999), dynamical dwell time approach (Xiang and Xiang 2009), LaSalle’s invariance principle (Yuan and Cheng 2008), linear matrix inequality (Montagner, Leite, Oliveira, and Peres 2006), Lie algebra (Liberon, Hespanha and Morse 1999; Agrachev and Liberon 2001), switched Lyapunov function (Daafouz, Riedinger, and Iung 2002), logic-based switching (Morse 1997), differential inclusion (Molchanov and Pyatnitsky 1989), Differential Petri Nets (Davrazosa and Koussoulas 2007) and optimal control (Teo 2002; Xu and Antsaklis 2004; Bengea and Decarlo 2005; Ho, Ling, Liu, Tam, and Teo 2008; Liu and Guo 2008). The reader is referred to Shorten, Wirth, Mason, Wulff, and King (2007) for recent developments in this area.

A very important issue which is always inherent to control systems is the presence of control saturations. From an engineering point of view, most physical actuators, sensors and interfacing devices are subject to saturation because of the existence of hard limitation. Considering these, we apply a saturation to the control. Due to the saturation nonlinearities, the study of stability and stabilisation becomes a challenging issue. Although there exist extensive literatures devoted to the control saturation problem for non-switched systems, the results on the stabilisation of switched systems with input saturations (SSIS) are relatively few. Therefore, the stabilisation of SSIS becomes a new field that is of great interest to some researchers. In Benzaouia, Saydy, and Akhrif (2004), the authors present sufficient conditions for stabilisation of discrete-time SSIS for arbitrary switching. Liu and Duan (2005) deal with the robust stabilisation of SSIS. The approach proposed in Song, Xiang, Chen, and Hu (2006) is to detect the existence of a common Lyapunov function to check the asymptotic stability of the SSIS. Stability analysis of switched systems using variational principles was proposed in Margaliot (2006). Optimal control of a class of linear hybrid systems with saturation was proposed in Schutter (2000).

Aforementioned works tackle the problem for an arbitrary switching case. That is, only the control inputs are utilised to stabilise the switched systems. It is worth noting that, besides the control inputs, the
switching law can also be taken as a control tool if it is
desirable. Usually the switching signals may be
generated by another logic-based process, such as
discrete-event system, etc. Particularly, the dwell time-
based switches and state-depending switches are
commonly used. The problem considered in this article
allows both the control inputs and dwell time-based
switching rules to be desirable to assure the expo-
ential stability of switched systems.

In the case of design of switching rule, we first
present a lemma which reveals the fact that a proper
minimal dwell time (MDT) (Hespanha and Morse 1999)
can assure the asymptotic stability provided so are all
subsystems. In this article we give an estimate for such
an MDT and obtain a large class of MDT-based
switching rules. Using the MDT-based switching rule to
stabilise switched systems has the advantage that each
subsystem with input saturation can be stabilised
independently. Then only an MDT needs to be deter-
mined to ensure the stability of the closed-loop switched
systems. To design control inputs, we use another
lemma proposed by Hu and Lin (2001) to place the
saturation nonlinearity into a convex hull of group of
linear feedbacks. Then, the study of a switched system
with saturated nonlinear subsystems is reduced to the
study of switched systems with linear subsystems.

The control synthesis presented in this article
combines the linear state feedbacks with MDT-based
switching strategy. We use linear state feedback control
to make the related estimation easily computable.
The verifiable conditions to ensure the stability of the closed-loop switched system are imposed. These
conditions are further expressed by LMIs, which are
numerically solvable. It is worth noting that the state-
dependent switched system is another interesting topic
( Geromel and Colaneri 2005). However, in this article
we consider only MDT-based switches.

For later reference, the following definitions are
given. $M^T$ is the transpose of the matrix $M$. $M > 0( < 0)$
means that $M$ is positive (negative) definite. For a
positive definite matrix $M$, $\lambda_{\text{min}}(M)$ and $\lambda_{\text{max}}(M)$ stand
for the minimal and maximal eigenvalues of $M$, re-
spectively. The set of all real and nature numbers are
denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. We use two
vector norms: $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ and $\|x\|_{\infty} = \max_{i=1}^n |x_i|$ for
$x \in \mathbb{R}^n$. The matrix norm used in this article is defined
as $\|M\| = \max_{x \neq 0} \frac{\|Mx\|_{\infty}}{\|x\|_{\infty}}$ for a square matrix $M$. sat$_c : \mathbb{R} \to \mathbb{R}$ is the scalar saturation function with upper
bound $c$, i.e. sat$_c(s) = \text{sign}(s) \min\{c, |s|\}$. Let $u_i = [u_i^1, u_i^2, \ldots, u_i^n]^T \in \mathbb{R}^m$, for the sake of simplicity, we also use
sat$_c(u_i) = [\text{sat}_c(u_i^1), \text{sat}_c(u_i^2), \ldots, \text{sat}_c(u_i^n)]^T$.

The rest of this article is organised as follows. In
Section 2, the problem formulation is given, a funda-
mental preliminary result is presented which provides
an estimate for MDT, and a useful lemma is recalled
which are utilised to deal with the saturation function.
Section 3 presents the main results which is the design
of saturated controllers and MDT-based switching
signals. An example illustrating the results is presented
in Section 4. Section 5 is a brief conclusion.

2. Problem formulation and preliminaries

Consider a switched linear system subject to input
saturation
\[ \dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}\text{sat}_c(u_{\sigma(t)}), \]
where $x \in \mathbb{R}^n$ is the system state, $\sigma : [0, +\infty) \to \mathcal{N} = \{1, 2, \ldots, N\}$ is a right-continuous mapping called
a switching signal, $A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}$ and $u_i \in \mathbb{R}^m$
denote the state matrix, input matrix and control input
of subsystem $i$, respectively; $N$ is the total number of
subsystems.

Let $S[\tau], \tau > 0$, be a set of switches, whose
successive switching instants $t_i, t_{i+1}$ satisfy $t_{i+1} - t_i \geq \tau$
for all $i \in \mathbb{N}$. The problem under consideration can be
stated as follows: design state feedbacks $u_i = K_i x,$
$K_i \in \mathbb{R}^{n \times m}, i \in \mathcal{N}$ and determine an MDT $\tau > 0$ such
that the equilibrium of switched system (1) is expon-
entially stable for all switching signals, $\sigma(t) \in S[\tau].$

When there is no saturation function in system (1),
under the conditions that $(A_i, B_i), i \in \mathcal{N}$ are control-
lable, the proposed problem can be effectively
addressed and the global exponential stability can be
achieved. However, with the saturation function, not
all the states of system (1) can be controlled to the
equilibrium. The problem of characterising the set of
all the states that can be steered to the origin by
appropriate choice of admissible control inputs and
switching signals is under extensive study and still
remains open. The objective of this article is to design
appropriate control inputs and switching signal which
can, if possible, steer every state in a pre-assigned
subset $W$ of $\mathbb{R}^n$ to the origin.

The following lemma provides an estimate of an MDT
to ensure the exponential stability.

**Lemma 2.1:** Let $\dot{x} = A_i x, A_i \in \mathbb{R}^{n \times n}, i \in \mathcal{N}$ be $N$ asy-
mmetrically stable linear systems, and $V(x) = x^T P x$, $i \in \mathcal{N}$ be the corresponding Lyapunov functions satisfying
\[ a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2, \quad i \in \mathcal{N} \]
and
\[ \frac{dV_i(x(t))}{dt} \mid_{A_i x = -c_i \|x\|^2}, \quad i \in \mathcal{N}. \]

Denote by $\mu_i = \frac{b_i}{a_i}, \lambda_i = \frac{c_i}{b_i}$ and set
\[ \tau > \max_{i \in \mathcal{N}} \left\{ \frac{\ln \mu_i}{\lambda_i} \right\}. \]
Then for any admissible switching signals $\sigma \in \mathcal{S}[\tau]$, the switched linear system $\dot{x} = A_\sigma(t)x$ is globally exponentially stable.

**Proof:** Since $\tau > \frac{\ln \mu_i}{\lambda_i - \ln \mu_i}$, $\lambda_i - \frac{\ln \mu_i}{\tau} > 0$. Let $\lambda_0 \in (0, \lambda_i - \frac{\ln \mu_i}{\tau})$ be a positive number. Then $\lambda_i - \lambda_0 > \frac{\ln \mu_i}{\tau}$. When $t \geq \tau$, we have

$$(\lambda_i - \lambda_0)t > \frac{\ln \mu_i}{\tau} t \geq \ln \mu_i,$$

which is equivalent to

$$e^{t \ln \mu_i - \lambda_0 t} \leq e^{t \lambda_0}, \quad t \geq \tau.$$  \hfill (5)

Note that the asymptotic stability of linear system $\dot{x} = A_kx$ is equivalent to its exponential stability. Its trajectory satisfies

$$\|x(t)\| \leq k_1\|x(0)\|e^{-k_1 t}, \quad t \geq 0$$

for some positive numbers $k_1$ and $r_1$. In fact, from (2) and (3) we have

$$\dot{V}_i(x(t)) \leq -\lambda_i V_i(x(t)), \quad t \geq 0,$$

which implies

$$V_i(x(t)) \leq e^{-\lambda_i (t-s)} V_i(x(s)), \quad t \geq s \geq 0.$$

Then

$$\|x(t)\| \leq \left[ \frac{V_i(x(0))}{a_i} \right]^{1 \frac{1}{2}} \leq \left[ \frac{V_i(x(0))e^{-\lambda_i t}}{a_i} \right] \leq \left( \mu_i \right)^{\frac{1}{2}} \|x(0)\| e^{-\lambda_i \frac{1}{2} t} = e^{t \ln \mu_i - \lambda_0 t} \|x(0)\|, \quad t \geq 0.$$

Thus, by the definition of matrix norm, the transition matrix $e^{A_i t}$ of $\dot{x} = A_i x$ satisfies

$$\|e^{A_i t}\| = \sup_{x_0 \neq 0} \frac{\|e^{A_i t}x_0\|}{\|x_0\|} = \sup_{x_0 \neq 0} \frac{\|x(t)\|}{\|x_0\|} \leq e^{t \ln \mu_i - \lambda_0 t}, \quad t \geq 0.$$  \hfill (6)

Let $\mu = \max_{i \in N} \mu_i$. Then

$$\|e^{A_i t}\| \leq e^{t \ln \mu_i - \lambda_0 t}, \quad t \geq 0.$$  \hfill (6)

For any switching signals $\sigma \in \mathcal{S}[\tau]$, let $0 = t_0, t_1, t_2, \ldots$ be the corresponding switching time series. Denote by $p(t_0)p(t_1)N$ the subsystem which is active during $[t_i, t_{i+1})$. Since $t_{i+1} - t_i \geq \tau$, (5) and (6) imply

$$\|e^{A_{p(t_i+1)-t_0}}\| \leq e^{-\lambda_0 (t_{i+1} - t_0)}.$$  

Denoting $\lambda_0 = \min_{i \in N} \{\lambda_0\}$, we have $\lambda_0 > 0$. Then the above inequality becomes

$$\|e^{A_{\sigma(t_i)-t_0}}\| \leq e^{-\lambda_0 (t_{i+1} - t_0)}.$$  

Let $\phi(t, t_0)$ be the transition matrix of the switched system $\dot{x} = A_{\sigma(t)}x$. For any $t, t_0 \in [0, +\infty)$, $t > t_0$, let $k, l$ be the integers such that $t_0 \in [t_k, t_{k+1})$ and $t \in [t_l, t_{l+1})$. Then

$$\phi(t, t_0) = \phi(t, t_1)\phi(t, t_2)\cdots \phi(t, t_{k-1})\phi(t, t_k)\phi(t, t_{k+1}, t_0)$$

$$= e^{A_{\sigma(t_1)-t_0}}e^{A_{\sigma(t_2)-t_1}}\cdots e^{A_{\sigma(t_k+1)-t_k}}e^{A_{\sigma(t_{k+1})-t_{k+1}}}$$

$$\leq e^{t \ln \mu_i - \lambda_0 t} e^{-\lambda_0 (t_l - t_0)} \cdots e^{-\lambda_0 (t_{k+1} - t_k)} e^{t \ln \mu_i - \lambda_0 t_{k+1} - t_{k+1}}$$

$$\leq e^{t \ln \mu_i - \lambda_0 t_{k+1} - t_k}$$

$$= e^{t \ln \mu_i - \lambda_0 t_{k+1} - t_k}.$$  

This means the switched system is exponentially stable. \hfill \Box

The main obstacle in the synthesis of system (1) is the presence of saturation function which is nonlinear in essence. The following technique from Hu and Lin (2001) can be used to place saturation nonlinearity into the convex hull of a group of linear feedbacks. Recall that for a group of points, $p_1, p_2, \ldots, p_l$, the convex hull of these points is defined as,

$$\text{co}\{p_k | k = 1, 2, \ldots, l\} = \left\{ \sum_{k=1}^{l} a_k p_k \right\} \text{such that } a_k = 1, a_k \geq 0 \right\}.$$  

Let $D$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are $2^m$ elements in $D$. Suppose that all elements of $D$ are labelled as $D_k$, $k = 1, 2, \ldots, 2^m$. Denote $D_k = I - D_k$, where $I$ is the $m \times m$ identity matrix.

**Lemma 2.2** (Hu and Lin 2001): Let $u, v \in \mathbb{R}^m$, $u = (u_1, u_2, \ldots, u_m)^T$, $v = (v_1, v_2, \ldots, v_m)^T$. Suppose $\|v\|_\infty \leq c$, where $c$ is the upper bound of saturation function defined before. Then

$$\text{sat}_c(u) \in \text{co}(D_k u + D_k v) | 1 \leq k \leq 2^m \}.$$  

3. Main results

In this section, the state feedbacks $u_i = K_i x$, $K_i \in \mathbb{R}^{n \times n}$, $i \in N$, combined with MDT-based switching law $\sigma(t)$ will be designed to render the stability of switched system

$$\dot{x} = A_{\sigma(t)} x + B_{\sigma(t)} \text{sat}_c(K_{\sigma(t)} x), \quad i \in N.$$  \hfill (7)

The basic idea behind our approach is that the feedback matrices $K_i$, $i \in N$ and the MDT $\tau$, including other design variables, be designed such that the close-loop systems

$$\dot{x} = A_{\sigma(t)} x + B_i \text{sat}_c(K_i x), \quad i \in N.$$  \hfill (8)
satisfy the conditions of Lemma 2.1. To use Lemma 2.1, the positive definite matrices $P_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{N}$ or the Lyapunov functions $V_i(x) = x^T P_i x$, $i \in \mathcal{N}$ for (8), must be designed. If such matrices $P_i$, $i \in \mathcal{N}$ are obtained, the domain of attraction of each subsystem (8) can be estimated as $\Phi(P_i, \rho) = \{x \in \mathbb{R}^n | x^T P_i x \leq \rho\}$, $i \in \mathcal{N}$ for some positive number $\rho$. Obviously, the positive number $\rho$ can be absorbed into positive definite matrix $P_i$. We simply set $\rho = 1$. Then we have $\Phi(P_i, 1) \subseteq \{x \in \mathbb{R}^n | x^T P_i x \leq 1\}$, $i \in \mathcal{N}$.

What stability conditions should be imposed on these design variable? Since $P_i$, $i \in \mathcal{N}$ are positive definite, set $a_i = \lambda_{\min}(P_i)$, $b_i = \lambda_{\max}(P_i)$. In order to get an estimate (3), we first use Lemma 2.2 to tackle the saturation function $\text{sat}(K_ix)$ which can be placed into a convex hull. To this end, introduce auxiliary matrices $H_i \in \mathbb{R}^{m \times n}$, $i \in \mathcal{N}$ such that

$$\Phi(P_i, 1) \subseteq \mathcal{L}(H_i), \quad i \in \mathcal{N},$$

where $\mathcal{L}(H_i) = \{x \in \mathbb{R}^n | ||H_ix||_\infty \leq c\}$. It follows easily that $||H_ix||_\infty \leq c$, $\forall x \in \Phi(P_i, 1)$. Let $K_ix$ and $H_ix$ correspond to $u$ and $\nu$ of Lemma 2.2, respectively. Then, for all $i \in \mathcal{N}$,

$$\text{sat}_i(K_ix) \in \text{co}\{D_kK_ix + D_k^T H_ix | 1 \leq k \leq 2^m\},$$

$$\forall x \in \Phi(P_i, 1)$$

and

$$A_i x + B_i \text{sat}_i(K_ix) \in \text{co}\{A_i x + B_i (D_kK_i + D_k^T H_i)x | 1 \leq k \leq 2^m\}, \quad \forall x \in \Phi(P_i, 1).$$

(9)

A sufficient condition ensures the stability of $\dot{x} = A_i x + B_i \text{sat}_i(K_ix)$ is that all the vertex matrices $A_i + B_i (D_kK_i + D_k^T H_i)$, $1 \leq k \leq 2^m$ in the above convex hull are stable. In view of the candidate Lyapunov function $V_i(x) = x^T P_i x$ introduced for subsystem $i$, it is quite natural to define

$$Q_i^k = -[A_i + B_i (D_kK_i + D_k^T H_i)]^T P_i$$

$$- P_i [A_i + B_i (D_kK_i + D_k^T H_i)], \quad i \in \mathcal{N}, \quad 1 \leq k \leq 2^m.$$  

(10)

Then under the conditions $Q_i^k > 0$ for all $i \in \mathcal{N}$, $1 \leq k \leq 2^m$, we can reach the estimate (3). Moreover, the positive number $c_i$ in (3) can be obtained as $c_i = \max_{1 \leq k \leq 2^m}(\lambda_{\min}(Q_i^k))$. Finally, in case of MDT $\tau$, defining $\mu_i = b_i/a_i$, $\lambda_i = c_i/b_i$ and applying Lemma 2.1 to systems (8), we need the estimate of MDT as $\tau > \max_{i \in \mathcal{N}}(\ln \mu_i/\lambda_i)$.

The remaining task is finding the estimate of the domain of attraction of system (7). It is worth noting that exact description of the set of all the states which can be steered to the origin by appropriate control inputs and an MDT-based switching signal seems to be very difficult. To avoid the difficulty, we use an Euclidean ball $W$ as the set of allowed initial conditions. Since the switching type we considered here is MDT based, the $W$ must locate inside every subsystem’s null controllable region (Lin 1999), especially, $W \subseteq \Phi(P_i, 1)$, $i \in \mathcal{N}$. Otherwise, there will be no solution to the proposed problem.

Summarising the above discussion, we have all the design requirements.

**Theorem 3.1:** Let $W$ be an Euclidean ball in $\mathbb{R}^n$ centred at the origin. If there exist matrices $K_i \in \mathbb{R}^{m \times n}$, $H_i \in \mathbb{R}^{m \times n}$ and positive definite matrices $P_i > 0$ such that

(a) $W \subseteq \Phi(P_i, 1)$, $i \in \mathcal{N}$,

(b) $\Phi(P_i, 1) \subseteq \mathcal{L}(H_i)$, $i \in \mathcal{N}$

(c) $Q_i^k > 0$, $i \in \mathcal{N}$, $1 \leq k \leq 2^m$,

then the switched system (7) with initial condition $x_0 \in W$ is exponentially stable for any $\sigma \in \mathcal{S}[\tau]$ with $\tau > \max_{i \in \mathcal{N}}(\ln(b_i/a_i))$, where $a_i = \lambda_{\min}(P_i)$, $b_i = \lambda_{\max}(P_i)$ and $c_i = \max_{1 \leq k \leq 2^m}(\lambda_{\min}(Q_i^k))$.

**Proof:** First, we claim that any trajectories $x(t)$ of system (7) with initial states in $W$ will stay in $W$ at any switching instants. For any switching signals $\sigma \in \mathcal{S}[\tau]$, let $t = t_0, t_1, t_2, \ldots$ be the corresponding switching instant series. Then for any $t > 0$, there exists an integer $i$ such that $t \in [t_i, t_{i+1})$. Also suppose that during $[t_i, t_{i+1})$, mode $p(i)$ is active, where $p(i) \in \mathcal{N}$. For the given positive definite matrices $P_i$, $i \in \mathcal{N}$, let $V_{p(i)}(x) = x^T P_{p(i)} x$ be a Lyapunov candidate function corresponding to mode $p(i)$. It is obvious that

$$a_{p(i)} ||x||^2 \leq V_{p(i)}(x) \leq b_{p(i)} ||x||^2,$$

(11)

where $a_{p(i)} = \lambda_{\min}(P_{p(i)})$, $b_{p(i)} = \lambda_{\max}(P_{p(i)})$. It follows from (8) to (10) that

$$\dot{V}_{p(i)}(x) = 2x^T P_{p(i)} [A_{p(i)} + B_{p(i)} \text{sat}(K_{p(i)} x)]$$

$$\leq \max_{1 \leq k \leq 2^m} \{x^T [A_{p(i)} + B_{p(i)} (D_kK_{p(i)} + D_k^T H_{p(i)})] P_{p(i)} x$$

$$+ x^T P_{p(i)} [A_{p(i)} + B_{p(i)} (D_kK_{p(i)} + D_k^T H_{p(i)})] x]\}

$$= \max_{1 \leq k \leq 2^m} x^T Q_{p(i)}^k x$$

$$\leq -c_{p(i)} ||x||^2,$$

(12)

where $c_{p(i)} = \max_{1 \leq k \leq 2^m}(\lambda_{\min}(Q_{p(i)}^k))$. Using (11) and (12), a similar argument as in the proof of Lemma 2.1 yields

$$||x(t)||^2 \leq \mu_{p(i)} e^{-\lambda_{p(i)}(t-t_0)} ||x(t_0)||^2.$$  

By the continuity of $x(t)$, letting $t \to t_{i+1}$ , we have

$$||x(t_{i+1})||^2 \leq \mu_{p(i)} e^{-\lambda_{p(i)}(t_{i+1}-t_i)} ||x(t_i)||^2.$$
Using the above inequality repeatedly, we get
\[
\|x(t_{i+1})\|^2 \leq \mu_{p(0)} \mu_{p(1)} e^{-\lambda_{p(0)} t_{i+0}} \mu_{p(1)} e^{-\lambda_{p(1)} t_{i+1}} \cdots e^{-\lambda_{p(N)} t_{i+N-1}} \|x(t_0)\|^2
\]
\[
\leq \mu_{p(0)} \mu_{p(1)} e^{-\lambda_{p(0)} t_{i+0}} - \lambda_{p(1)} t_{i+1}]^2 \|x(t_0)\|^2.
\]
(13)
Since \( \tau > \max_{\in \mathbb{N}} \{\ln \mu_i / \lambda_i\} \) and
\[
\max_{\in \mathbb{N}} \left\{ \ln \mu_i / \lambda_i \right\} > \ln \mu_{p(0)} + \cdots + \ln \mu_{p(0)},
\]
it follows that
\[
\tau > \ln \mu_{p(0)} + \cdots + \ln \mu_{p(0)} / \lambda_{p(0)} + \cdots + \lambda_{p(0)}.
\]
Equivalently,
\[
\mu_{p(0)} \cdots \mu_{p(0)} e^{-\lambda_{p(0)} t_{i+0}} > 1.
\]
Thus inequality (13) is reduced to
\[
\|x(t_{i+1})\|^2 \leq \|x(t_0)\|^2.
\]
Since \( x_0 \in W \) and \( W \) is an Euclidian ball, it follows easily that at any switching instants \( t_{i+1} \), we have \( x(t_{i+1}) \in W, i \in \mathbb{N} \).
To proceed, let us note that \( W \subset \Phi(P_i, 1) \) and \( \Phi(P_i, 1) \) is a positively invariant set of subsystem (8) for all \( i \in \mathbb{N} \). Therefore, \( x_0 \in W \) leads directly to \( x(t) \subset \Phi(P_i, 1), \forall i \in \{t_i, t_{i+1}\} \). Using (11) and (12) and noting that \( \tau > \max_{\in \mathbb{N}} \{\ln (b_i/a_i) + \epsilon_i / h_i\} \), we can conclude from Lemma 2.1 that the switched system (7) is exponentially stable for all \( x_0 \in W \).

**Remark 1:** The average dwell time problem for switched linear systems has been widely discussed (Pola, Polderman, and Benedetto 2004; Shorten et al. 2007). Here our boundary for dwell time is not as tight as what they obtained, because the saturation may weaken the stabilisation capability.

Next, we express the conditions in the above theorem by means of LMIs. Since the domain of attraction \( W \) is assumed to be an Euclidian ball, estimating it converts to estimating the radius of the ball. Based on this consideration, let \( W = r \mathcal{B} = \{x|x \in \mathcal{B}\} \), where \( r \) is a positive number and \( \mathcal{B} \) is an Euclidian ball centred at the origin with radius one. Condition (a) is satisfied if and only if \( rw \in \Phi(P_i, 1), \forall w \in r \mathcal{B} \), where \( r \mathcal{B} \) stands for the boundary of \( \mathcal{B} \). Theorem 3.1 has the following corresponding form.

**Theorem 3.2:** Let \( \mathcal{B} \) be an Euclidian ball centred at the origin with radius one. If, for any \( w_1, w_2, \ldots, w_s \) on the boundary of \( \mathcal{B} \) with \( s \) large enough, there exist a positive number \( r \), positive definite matrices \( X_i \in \mathbb{R}^{n \times n} \), matrices \( Y_i \in \mathbb{R}^{n \times n} \) and matrices \( Z_i \in \mathbb{R}^{n \times n} \) such that the following LMI s hold
\[
(d') \left( \begin{array}{c} 1 \\
rw \\
X_i
\end{array} \right) \geq 0, \quad i \in \mathcal{N}, \quad 1 \leq i \leq s
\]
\[
(b') \left( \begin{array}{c} c^2 \\
z_i^T \\
X_i
\end{array} \right) \geq 0, \quad i \in \mathcal{N}, \quad 1 \leq j \leq m
\]
\[
(c') A_i X_i + X_i A_i^T + B_i (D_i Y_i + D_i^T Z_i)
+ (D_i Y_i + D_i^T Z_i) B_i^T < 0, \quad i \in \mathcal{N}, \quad 1 \leq k \leq 2^m,
\]
where \( z_i \) is the \( j \)-th row of matrix \( Z_i \), \( c \) is the upper bound of saturation function, then the switched system (7) with initial condition \( x_0 \in W = r \mathcal{B} \) is exponentially stable for any \( \sigma \in \mathcal{S}[\tau] \) with \( \tau > \max_{\in \mathbb{N}} \{\ln (b_i/a_i) + \epsilon_i / h_i\} \) for \( P_i = X_i^{-1}, K_i = Y_i X_i^{-1}, A_i = Z_i X_i^{-1} \).

**Proof:** We first show that the condition (a) is equivalent to condition (d'). Using the standard Schur complement (Zhang 2005), it follows from \( P_i = X_i^{-1} \) that condition (d') is equivalent to
\[
r^2 w^T P_i w_i \leq 1, \quad i \in \mathcal{N}, \quad 1 \leq i \leq s,
\]
which is also equivalent to
\[
rw_i \in \Phi(P_i, 1), \quad i \in \mathcal{N}, \quad 1 \leq i \leq s.
\]
Since \( rw_1, \ldots, rw_s \) are arbitrarily selected \( s \) points on the boundary of \( W \), the above relation is equivalent to
\[
W \subset \Phi(P_i, 1), \quad i \in \mathcal{N}.
\]
We proceed to show the equivalence of the constraints (b) and (b'). Note that \( \Phi(P_i, 1) \subset \mathcal{L}(H_i) \) if and only if all the hyperplane \( h_i = c, 1 \leq j \leq m \), lie completely outside of the ellipsoid \( \Phi(P_i, 1) \), i.e. at each point \( x \) on the hyperplane \( h_i = c, 1 \leq j \leq m \), we have \( x^T P_i x \geq 1 \). This means that constraint (b) is equivalent to
\[
\min_x \{x^T P_i x | h_i = c\} \geq 1, \quad i \in \mathcal{N}, \quad 1 \leq j \leq m.
\]
(14)
Using the Lagrangian multiplier method to a positive definite quadratic form with linear restriction yields
\[
\min_x \{x^T P_i x | h_i = c\} = [(h_i/c) P_i^{-1} (h_i/c)^T]^{-1}.
\]
(15)
According to (15), the minimum in (14) can be calculated as
\[
\min_x \{x^T P_i x | h_i = c\} = [(h_i/c) P_i^{-1} (h_i/c)^T]^{-1},
\]
\[
i \in \mathcal{N}, \quad 1 \leq j \leq m.
\]
Consequently, constraint (b) is equivalent to
\[
h_i P_i^{-1} (h_i)^T \leq c^2, \quad i \in \mathcal{N}, \quad 1 \leq j \leq m.
\]
Also by the Schur complement (Zhang 2005), the above inequality is equivalent to
\[
\begin{pmatrix}
    c^2 & h_i^T P_i^{-1} \\
    P_i^{-1}(h_i^T) & P_i^{-1}
\end{pmatrix} \geq 0, \quad 1 \leq j \leq m, \quad i \in \mathcal{N}.
\]

Note that \( P_i = X_i^{-1}, K_i = Y_i X_i^{-1}, H_i = Z_i X_i^{-1} \), hence the above constraint is exactly the constraint \((b')\).

We prove the last equivalence of the constraints \((c)\) and \((c')\). This is obvious by the fact that \( P_i = X_i^{-1}, K_i = Y_i X_i^{-1}, H_i = Z_i X_i^{-1} \).

From this theorem, the stabilising feedback matrices \( K_i, i \in \mathcal{N} \) can be obtained from the solutions \( X_i, Y_i, Z_i, i \in \mathcal{N} \) of the LMIs \((a'), (b')\) and \((c')\) by taking \( K_i = X_i^{-1}, i \in \mathcal{N} \).

**Remark 2:** In numerical computation, we are not able to check the conditions in Theorem 3.2 for all points on the boundary of \( \partial \mathcal{X} \). So we can only choose some sampling points on \( \partial \mathcal{X} \). A natural question here is: how many points do we need and how to choose them. It is obvious that the more the chosen points, the more reliable the result. Taking the computing complexity into consideration, only a reasonable number of points can be chosen. The number can be chosen by a standard way as in numerical iteration: double the number each time and as the iteration error is small enough we stop. Another problem is how to choose the points. We let the points be distributed as balanced as possible. It can be easily done through the spherical coordinates by dividing the angular parameters.

### 4. An illustrative example

Consider the switched linear control system \((7)\), where \( A_i \) and \( B_i \) are as follows:

\[
A_1 = \begin{pmatrix}
0 & 1.4915 \\
3.6583 & -0.2905
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
0 \\
5
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
-0.8743 & -0.5053 \\
-0.1474 & 0.2710
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
1 \\
1
\end{pmatrix}.
\]

Suppose the upper bound \( c \) of the saturation function sat is 1. We take eight points on the boundary of the unit circle. They are as follows:

\[
w_1 = (0, 1)^T, \quad w_2 = (1/2, \sqrt{3}/2)^T, \quad w_3 = (1, 0)^T,
\]

\[
w_4 = (1/2, -\sqrt{3}/2)^T, \quad w_5 = (0, -1)^T,
\]

\[
w_6 = (-1/2, -\sqrt{3}/2)^T, \quad w_7 = (-1, 0)^T,
\]

\[
w_8 = (-1/2, \sqrt{3}/2)^T.
\]

Then conditions of Theorem 3.2 are converted into 25 LMIs with seven variables \( r, X_1, X_2, Y_1, Y_2, Z_1 \) and \( Z_2 \). Using the LMI Toolbox in Matlab we obtain the solutions to these LMIs and the matrix \( P_i, K_i, H_i, i = 1, 2 \) and \( r \) can be obtained as follows:

\[
P_1 = \begin{pmatrix}
0.3625 & 0.1488 \\
0.1488 & 0.3028
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0.2938 & 0.0243 \\
0.0243 & 0.3520
\end{pmatrix},
\]

\[
K_1 = (-1.2765 - 0.3240), \quad K_2 = (0.4633 - 0.6774),
\]

\[
H_1 = (-0.8110 - 0.3733)^T, \quad H_2 = (0.1570 - 0.4115)^T
\]

and \( r = 1.4810 \). The two ellipsoid sets \( \Phi(P_i, 1), i = 1, 2 \) of invariance and the Euclidian ball \( W \) of attraction of the switched system are presented in Figure 1.

With these matrices, we can calculate \( a_1 = 0.3617, a_2 = 0.2850, b_1 = 0.9689, b_2 = 0.3608, c_1 = 0.3824, c_2 = 0.1425 \) by taking \( a_i = \lambda_{\min}(P_i), b_i = \lambda_{\max}(P_i) \) and \( c_i = \max_{k=1,2}(\lambda_{\min}(Q_k^i)) \) for \( i = 1, 2 \). Then, \( \mu_1 = b_1/a_1 = 2.6787, \mu_2 = b_2/a_2 = 1.2660, \lambda_1 = c_1/b_1 = 0.3947, \ lambda_2 = c_2/b_2 = 0.3948, \tau = \max(\ln a_1, \ln a_2) = 2.4965 \). Let the dwell time of the switched system to be 2.5 and the corresponding MDT-based switching signal be that subsystem 1 is active during \([0, 2.5) \cup [5, 7.5) \cup \cdots \) and subsystem 2 is active during \([2.5, 5) \cup [7.5, 10) \cup \cdots \). In simulation set the initial state \( x_0 = (-1.3, -0.1)^T \) which is locate inside the \( W \). The values of \( x_1(t) \) and \( x_2(t) \) versus time are shown in the left side of Figure 2. The simulation results show that the given feedbacks and the MDT-based switching signal ensure the stability of the switched system for \( x_0 \in W \). At the same time, we plot the values of \( \text{sat}(u_1(t)), \text{sat}(u_2(t)) \) versus time in right side of Figure 2, where one can see that the control signal \( u \) has been saturated. We simulate with another initial state \( x_0 = (0.53, 1.35)^T \) which also locate inside the \( W \). The time responses of \( x(t) \) and \( u(t) \) are
show in Figure 3, from which we see that the state trajectory convergence and the control $u$ has been saturated.

5. Conclusion

In this article, we consider the problem of stabilisation of switched linear systems with input saturation. We provide stability conditions for such systems. These conditions can be converted to LMIs, which are numerically solvable. A class of switching signals, which are MDT based, together with state feedbacks are designed to stabilise the switched systems. An illustrative example is presented to show the validity of the results.

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