Complete stability of cellular neural networks with unbounded time-varying delays

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1. Introduction

In the past decades, cellular neural networks (CNNs), proposed by Chua and Yang (1988a, 1988b), have been one of the most important research topics in many application fields, such as image processing (Chua & Yang, 1988b), associative memories (Liu & Michel, 1993), pattern recognition (Thiran, Crounse, Chua, & Halser, 1995), and so on. In a wide range of these applications, it is essential that the neural networks involved are multistable and completely stable, that is, they exhibit a large number of stable equilibrium points and every trajectory with a specified initial value converges to one of the equilibrium points. Thus, it is of great importance to study the complete stability, along with multistability, in both theory and applications.

There have been quite a few results on complete stability of neural networks reported in the literature. Note that, if the system just has one unique equilibrium point, then its complete stability is equivalent to the global stability of this equilibrium point, see Chen and Amari (2001), Chen and Wang (2007a, 2007b). While for the system with multiple equilibrium points, its complete stability is much more complicated, see Chen and Zheng (2010), Cheng and Shih (2009), Lu, Wang, and Chen (2011), Takahashi and Nishi (2006), Wang, Lu, and Chen (2010), Yi and Tan (2004), Zeng and Wang (2006), Zhang, Wei, and Xu (2005) and so on.

In a recent paper Takahashi and Nishi (2006), the cellular neural networks consisting of two cells were investigated, and under some assumptions, a set of necessary and sufficient conditions for such systems to be completely stable were presented. More generally, for n-neuron neural networks, the authors proposed a new approach to address the multistability in Lu et al. (2011) and Wang et al. (2010). And the complete stability of neural networks is a direct consequence of the identification of the attraction basin of each stable equilibrium point.

On the other hand, due to the inevitable existence of time delays in practical neural networks, considerable research attention has been paid to the delayed neural networks in recent years. In Zeng and Wang (2006), by using the contraction mapping principle and induction method, a set of sufficient conditions guaranteeing the complete stability of cellular neural networks with time-varying delays were proposed. For neural networks with constant delays, it also showed that the systems can have 3\textsuperscript{rd} equilibrium points and be completely stable under some conditions in Cheng and Shih (2009). And Chen and Zheng (2010) presented a new method and obtained some criteria unifying the delay-dependent and delay-independent cases on complete stability of delayed cellular neural networks. For more references on the dynamics of delayed neural networks, please refer to Kaslik and Sivasundaram (2011), Nie and Cao (2011), Wang, Lu, and Chen (2009) and so on.

It should be noted that most existing results are concerned with the neural networks with bounded time delays. As for systems with unbounded time-varying delays, it is hard to estimate the location of delayed items $u_i(t - \tau_i(t))$ and their effect on current states $u_i(t)$ \((i, j = 1, \ldots, n)\). And there are also fundamental difficulties...
in analyzing the multistability of neural networks and identifying the precise attraction basins of stable equilibrium points. To the best of our knowledge, there are only a few papers investigating the complete stability of multistable neural networks in the case that the time-varying delays are unbounded.

Motivated by these concerns, we will study the neural networks with unbounded time-varying delays in this paper.

Consider the following delayed cellular neural networks described as:

\[
\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + l_i, \quad i = 1, \ldots, n, \tag{1}
\]

where \(u_i(t)\) represents the state of the \(i\)-th unit at time \(t\); \(d_i > 0\) denotes the rate with which the \(i\)-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; \(\tau_j(t) \geq 0\) is the time delay; \(a_{ij}, b_{ij}\) correspond to the connection weights of the \(j\)-th unit on the \(i\)-th unit at time \(t\) and \(t - \tau_j(t)\), respectively; and \(l_i\) stands for the external input, \(i, j = 1, \ldots, n\). The activation function \(f_j(\cdot)\) is defined as

\[
f_j(\rho) = \frac{\lvert \rho + 1 \rvert - \lvert \rho - 1 \rvert}{2}, \quad j = 1, \ldots, n. \tag{2}
\]

The initial value is assumed to be

\[u_i(\theta) = \phi_i(\theta) \quad \text{for} \quad \theta \in (-\infty, 0], \tag{3}\]

where \(\phi_i(\theta) \in C((-\infty, 0], \mathbb{R}), i = 1, \ldots, n\).

In the following, we will investigate the complete stability of neural networks (1). Due to the geometrical configuration of activation functions, the phase space \(\mathbb{R}^n\) can be divided into \(3^n\) subset regions. However, for an arbitrary solution \(u(t)\), it is impractical to limit its whole initial state (3) staying in the same subset region. Instead, we can follow the subset region where \(u(0)\) is located. And by rigorous analysis, we can get that \(u(t)\) would converge to some equilibrium point as \(t\) tends to \(+\infty\). Two examples are presented to illustrate the effectiveness of our results.

2. Preliminaries

First of all, we present some assumptions and definitions that will be used in the following sections.

Suppose that all delays satisfy \(\tau_j(t) \leq \tau_j\), \(i, j = 1, \ldots, n\), and \(\tau_j(t)\) is continuous, \(t - \tau_j(t)\) is monotonically increasing and tends to \(+\infty\) when \(t \to +\infty\).

Definition 1. The neural network (1) is said to be completely stable if for any initial value (3), the corresponding solution trajectory \(u(t)\) converges to a certain equilibrium point.

By the geometrical configuration of activation function (2), we denote an index \(\chi \in \{-1, 0, 1\}\) and use it to define the interval of \(\mathbb{R}\) as follows:

\[
I_\chi = \begin{cases} 
(\infty, -1), & \chi = -1 \\
[-1, 1], & \chi = 0 \\
(1, +\infty), & \chi = 1.
\end{cases}
\]

Therefore, the whole space \(\mathbb{R}^n\) can be divided into \(3^n\) subset regions as:

\[
\Phi_\xi = \prod_{k=1}^{n} I_{\xi_k}
\]

with index vectors \(\xi = [\xi_1, \ldots, \xi_n] = \{-1, 0, 1\}^n\), and \(\prod\) denotes the Cartesian product from left to right.

For each subset region \(\Phi_\xi\), we also define the corresponding index subsets as follows:

\[
\begin{align*}
N_1^\xi &= \{ i : \xi_i = -1 \}, \\
N_2^\xi &= \{ i : \xi_i = 0 \}, \\
N_3^\xi &= \{ i : \xi_i = 1 \}.
\end{align*}
\tag{4}
\]

Denote the number of zeros in the vector \(\xi\) as \(\delta(\xi) = \lvert N_2^\xi \rvert\). And let \(u^\xi = [u_1^\xi, \ldots, u_n^\xi]^T\) be the equilibrium point of system (1) located in \(\Phi_\xi\) if it exists.

3. Main results

In this section, we first consider the coexistence of multiple equilibrium points in the delayed cellular neural networks (1). Note that the equilibrium points of system (1) are the same as the equilibrium points of the following system without delays:

\[
\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} (a_{ij} + b_{ij}) f_j(u_j(t)) + l_i, \quad i = 1, \ldots, n. \tag{5}
\]

Directly applying the results in Wang et al. (2010), we have

Lemma 1. Suppose that

\[
\lvert l_i \rvert < -d_i + (a_{ii} + b_{ii}) - \sum_{j=1, j \neq i}^{n} \lvert a_{ij} + b_{ij} \rvert \tag{6}
\]

holds for \(i = 1, \ldots, n\). Then each subset region \(\Phi_\xi\) has an equilibrium point, and the system (1) has \(3^n\) equilibrium points in all.

For the detailed proof, readers can refer to Wang et al. (2010). Next, for those neural networks (1) with multistable dynamics, we are to address their complete stability by proving the following theorem.

Theorem 1. Suppose that

\[
\lvert l_i \rvert < -d_i + a_{ii} - \sum_{j=1, j \neq i}^{n} \lvert a_{ij} \rvert - \sum_{j=1, j \neq i}^{n} \lvert b_{ij} \rvert \tag{7}
\]

holds for \(i = 1, \ldots, n\). Then the system (1) has \(3^n\) equilibrium points, and it is completely stable.

As a direct consequence of Lemma 1, it is clear that under the condition (7), there are \(3^n\) equilibrium points. Before the proof of complete stability, we prove the following two lemmas.

Lemma 2. Suppose that (7) holds for \(i = 1, \ldots, n\). Let \(u(t)\) be an arbitrary solution of system (1) with \(u(0) \in \Phi_\xi\). If \(\delta(\xi) = 0\), then \(u(t)\) would stay in \(\Phi_\xi\) for all \(t \geq 0\), and converge to the equilibrium point \(u^\xi\) finally.

Proof. By condition (7), we can find a positive constant \(\epsilon\) small enough such that

\[
-d_i(1 + \epsilon) + a_{ii} - \sum_{j=1, j \neq i}^{n} \lvert a_{ij} \rvert - \sum_{j=1, j \neq i}^{n} \lvert b_{ij} \rvert + l_i > 0, \tag{8}
\]

\[
-d_i(-1 - \epsilon) - a_{ii} + \sum_{j=1, j \neq i}^{n} \lvert a_{ij} \rvert + \sum_{j=1, j \neq i}^{n} \lvert b_{ij} \rvert + l_i < 0, \tag{9}
\]

hold for \(i = 1, \ldots, n\).
On the other hand, it is easy to get that \( \mathbb{N}^{2}_{x} = \emptyset \), since \( \delta(\xi) = 0 \).
Thus, for \( i \in \mathbb{N}^{1}_{x} \), if there exists some time point \( t_{0} \geq 0 \) such that \(-1 - \epsilon \leq u_{i}(t_{0}) \leq -1 \) while \( u(t) \in \Phi_{\xi} \) for \( 0 \leq t < t_{0} \), then,
\[
\frac{du_{i}(t)}{dt} \bigg|_{t=t_{0}} = -d_{i}u_{i}(t_{0}) + a_{ij}(u_{i}(t_{0})) + \sum_{j \neq i}a_{ij}(u_{j}(t_{0}))
\]
\[
\leq -d_{i}(-1 - \epsilon) - a_{ii} + \sum_{j \neq i}|a_{ij}| + \sum_{j \neq i}|b_{ij}| + I_{i}
\]
\[
\leq 0.
\]

Similarly, for \( i \in \mathbb{N}^{1}_{e} \), if there exists some time point \( t_{1} \geq 0 \) such that \( 1 < u_{i}(t_{1}) \leq 1 + \epsilon \) while \( u(t) \in \Phi_{\xi} \) for \( 0 \leq t < t_{1} \), then,
\[
\frac{du_{i}(t)}{dt} \bigg|_{t=t_{1}} = -d_{i}u_{i}(t_{1}) + a_{ij}(u_{i}(t_{1})) + \sum_{j \neq i}a_{ij}(u_{j}(t_{1}))
\]
\[
\leq -d_{i}(1 + \epsilon) + a_{ii} - \sum_{j \neq i}|a_{ij}| - \sum_{j \neq i}|b_{ij}| + I_{i}
\]
\[
> 0.
\]

By these two inequalities above, we know that \( u(t) \) would not leave the subset region \( \Phi_{\xi} \), but stay in it for all \( t \geq T_{0} \).

Note that \( t - \tau(t) \rightarrow +\infty \) when \( t \rightarrow +\infty \). Hence, we can find a time point \( T(\geq 0) \) big enough such that \( t - \tau(t) \geq T \) for all \( t \geq T_{0} \).

Lemma 3. Suppose that (7) holds for all \( i = 1, \ldots, n \). Let \( u(t) \) be an arbitrary solution of system (1) with \( u(0) \in \Phi_{\xi} \). If \( \delta(\xi) > 0 \), then \( u(t) \) would converge to the equilibrium point \( u^{e} \), or leave \( \Phi_{\xi} \) in finite time.

Proof. Without loss of generality, we suppose that \( u(t) \in \Phi_{\xi} \) for all \( t \geq 0 \) and denote the time point \( T(\geq 0) \) such that \( t - \tau(t) \geq 0 \) holds for all \( t \geq T_{0} \).

Then, the dynamics of \( u(t) \) \( (t \geq T) \) can be described by
\[
\frac{du_{i}(t)}{dt} = -d_{i}u_{i}(t) - (a_{ii} + b_{ii}) - \sum_{j \neq i}(a_{ij} + b_{ij})
\]
\[
+ \sum_{j \neq i}(a_{ij} + b_{ij}) + I_{i}, \quad i \in \mathbb{N}^{1}_{e},
\]
\[
\frac{du_{i}(t)}{dt} = -d_{i}u_{i}(t) + (a_{ii} + b_{ii}) - \sum_{j \neq i}(a_{ij} + b_{ij})
\]
\[
+ \sum_{j \neq i}(a_{ij} + b_{ij}) + I_{i}, \quad i \in \mathbb{N}^{1}_{e}.
\]

Define \( x_{i}(t) = u_{i}(t) - u^{e}_{i}, \ t \geq T, \ i = 1, \ldots, n \), then we have
\[
\frac{dx_{i}(t)}{dt} = -d_{i}x_{i}(t), \quad i = 1, \ldots, n.
\]

It is clear that \( x_{i}(t) \) would converge to 0 when \( t \) tends to +\( \infty \). That is, \( u(t) \) would converge to the equilibrium point \( u^{e} \). It completes the proof.

Remark 1. From the proof of Lemma 2, we can see that, for any solution \( u(t) \) of system (1), if there exists some time point \( t_{0} \geq 0 \) such that \( u(t_{0}) \) enters a subset region \( \Phi_{\xi} \) with \( \delta(\xi) = 0 \), then under condition (7), \( u(t) \) would stay in \( \Phi_{\xi} \) for all \( t \geq t_{0} \) and converge to \( u^{e} \) finally.

Remark 2. We can also conclude that the equilibrium point \( u^{e} \) with \( \delta(\xi) = 0 \) is locally stable in \( \Phi_{\xi} \) from the proof of Lemma 2. Therefore, the system (1) has \( 2^{n} \) stable equilibrium points.
which implies that \( \lim_{t \to +\infty} x_k(t) = 0 \) clearly. In other words, \( u_i(t) \) would converge to \( u_i^* \) for all \( i = 1, \ldots, n \).

Case II: \( \exists t' \in \mathbb{N}_2^k \) such that \( \lim_{t \to +\infty} x_k(t') \neq 0 \\

Define
\[
E(t) = \sup_{t-t' \leq s \leq t} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right), \quad t \geq T.
\]

It is easy to see \( \max_{i \in \mathbb{N}_2^k} |x_i(t)| \leq E(t) \) for all \( t \geq T \) and we claim that there must exist some \( T_1 \geq T \), such that
\[
\max_{i \in \mathbb{N}_2^k} |x_i(T_1)| = E(T_1).
\]

Otherwise, denote \( \alpha_1 = T \), due to the continuity and monotonicity of \( t - \tau(t) \), we can find a time point \( \alpha_2 \) such that \( \alpha_2 > \alpha_1 \) and \( \alpha_2 - \tau(\alpha_2) = \alpha_1 \); for \( \alpha_2 \), we can also find a \( \alpha_3 \) such that \( \alpha_3 > \alpha_2 \) and \( \alpha_3 - \tau(\alpha_3) = \alpha_2 \); consequently, we can find a \( \alpha_4, \alpha_5, \ldots, \alpha_4 \) such that \( \alpha_k > \alpha_{k-1} \) and \( \alpha_k - \tau(\alpha_k) = \alpha_{k-1} \). It is clearly that the time sequence \( \{\alpha_k\}_{k=1}^{\infty} \) is increasing and \( \lim_{k \to +\infty} \alpha_k = +\infty \).

Then, for \( t = \alpha_1 \), there must exist some \( t_1 : 0 \leq t_1 < \alpha_1 \) such that
\[
\max_{i \in \mathbb{N}_2^k} |x_i(t_1)| = E(\alpha_1) = \sup_{\alpha_1 - \tau(\alpha_1) \leq s \leq \alpha_1} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right);
\]
for \( t = \alpha_2 \), there must exist some \( t_2 : \alpha_1 = t_2 < \alpha_2 \) such that
\[
\max_{i \in \mathbb{N}_2^k} |x_i(t_2)| = E(\alpha_2) = \sup_{\alpha_2 - \tau(\alpha_2) \leq s \leq \alpha_2} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right);
\]
\[
\cdots,
\]
for \( t = \alpha_k \), there must exist some \( t_k : \alpha_{k-1} = t_k < \alpha_k \) such that
\[
\max_{i \in \mathbb{N}_2^k} |x_i(t_k)| = E(\alpha_k) = \sup_{\alpha_k - \tau(\alpha_k) \leq s \leq \alpha_k} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right);
\]
\[
\cdots.
\]

Consequently, we can get another time sequence \( \{t_k\}_{k=1}^{\infty} \), which is nondecreasing and \( \lim_{k \to +\infty} t_k = +\infty \). While \( x_i(t), i \in \mathbb{N}_2^k \), is bounded, by the Bolzano–Weierstrass Theorem, we can select a subsequence of \( \{t_k\} \) (for convenience, we still denote it by \( \{t_k\} \)) such that \( x_i(t_k) \) converges to a constant \( c_i \) as \( k \to +\infty \), \( i \in \mathbb{N}_2^k \).

Denote the index \( i_0 \in \mathbb{N}_2^k \) such that \( |c_{i_0}| = \max_{i \in \mathbb{N}_2^k} |c_i| \). Then, for any \( \epsilon \) with
\[
0 < \epsilon < \frac{-d_i + a_i}{\sum_{j \in \mathbb{N}_2^k} |a_{ij}| - \sum_{j \in \mathbb{N}_2^k} |b_{ij}|} |c_{i_0}|,
\]
there exists some \( K > 0 \), such that
\[
|c_i| - \epsilon < |x_i(t_k)| < |c_i| + \epsilon
\]
holds for all \( k \geq K \) and \( i \in \mathbb{N}_2^k \).

On the other hand, due to the monotonicity of \( t - \tau(t) \), it is easy to get that \( t_k - \tau(t_k) \in [\alpha_{k-2}, \alpha_{k-1}] \) (\( k > 2 \)), which implies that
\[
E(\alpha_k) = \max_{i \in \mathbb{N}_2^k} |x_i(t_k)| < \sup_{t_k - \tau(t_k) \leq s \leq \alpha_k} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right) = \sup_{t_k - \tau(t_k) \leq s \leq t_k} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right)
\]
\[
\leq \sup_{\alpha_{k-2} \leq s \leq \alpha_{k-1}} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right) = E(\alpha_{k-1}).
\]

Therefore, \( \{E(\alpha_k)\} \) is decreasing with respect to \( k \). And for \( k > K \), we can get that
\[
|x_i(t_k - \tau_i(t_k))| \leq \sup_{i \neq i_0 \mid \tau_i(t_k) \leq \alpha_k} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right)
\]
\[
\leq E(\alpha_k) < |c_{i_0}| + \epsilon, \quad i, j \in \mathbb{N}_2^k, \quad (18)
\]
and
\[
\frac{d|x_i(t_k)|}{dt} \bigg|_{t=t_k} = (-d_{i_0} + a_{i_0})|x_{i_0}(t_k)| + \text{sign}(x_{i_0}(t_k))
\]
\[
\times \left( \sum_{j \in \mathbb{N}_2^k} a_{ij}x_j(t_k) + \sum_{j \in \mathbb{N}_2^k} b_{ij}x_j(t_k - \tau_{ij}(t_k)) \right)
\]
\[
\geq (-d_{i_0} + a_{i_0})(|c_{i_0}| - \epsilon) - \sum_{j \in \mathbb{N}_2^k} |a_{ij}||(|c_j| + \epsilon) - \sum_{j \in \mathbb{N}_2^k} |b_{ij}||c_j|
\]
\[
> 0.
\]

It implies that we can find a \( \sigma > 0 \) such that \( |x_{i_0}(t)| > |x_{i_0}(t_k)|, t \in (t_k, t_k + \sigma) \), which contradicts the definition of \( t_k \), i.e.,
\[
|x_{i_0}(t_k)| = E(\alpha_k) = \sup_{\alpha_k - \tau(\alpha_k) \leq s \leq \alpha_k} \left( \max_{i \in \mathbb{N}_2^k} |x_i(s)| \right).
\]

Thus, it is natural to conclude that there exists some \( T_1 \geq T \), such that
\[
\max_{i \in \mathbb{N}_2^k} |x_i(T_1)| = E(T_1).
\]

Denote the index \( i_1 \in \mathbb{N}_2^k \) such that \( |x_{i_1}(T_1)| = \max_{i \in \mathbb{N}_2^k} |x_i(T_1)| \).

Differentiating \( |x_i(t)| \), and we have
\[
\frac{d|x_i(t)|}{dt} \bigg|_{t=T_1} = (-d_{i_1} + a_{i_1}+1)|x_{i_1}(T_1)| + \text{sign}(x_{i_1}(T_1))
\]
\[
\times \left( \sum_{j \in \mathbb{N}_2^k} a_{ij}x_j(T_1) + \sum_{j \in \mathbb{N}_2^k} b_{ij}x_j(T_1 - \tau_{ij}(T_1)) \right)
\]
\[
\geq (-d_{i_1} + a_{i_1}+1)|x_{i_1}(T_1)| - \sum_{j \in \mathbb{N}_2^k} |a_{ij}||x_j(T_1)| - \sum_{j \in \mathbb{N}_2^k} |b_{ij}||x_j(T_1)|
\]

Moreover, we can find an index $i_k \in \mathbb{N}_3^d$ and an increasing time sequence $\{t^{(i)}_k\}_{k=1}^\infty$, $t_k^{(i)} \geq T_1$, such that $\lim_{k \to \infty} t_k^{(i)} = +\infty$, and $|x_i(t_k^{(i)})| = E(t_k^{(i)}), \ k = 1, 2, \ldots$. In the same way as (19), we can also get that

$$\frac{d|x_i(t)|}{dt} \bigg|_{t=t_k^{(i)}} \geq -d_i + a_i \epsilon - \sum_{j \neq i} a_{ij} - \sum_{j \neq i} |b_{ij}| \geq 0,$$

which implies that

$$E(t) \geq E(T_1), \quad \text{for all } t \geq T_1. \quad (20)$$

Moreover, we can find an index $i_k \in \mathbb{N}_3^d$ and an increasing time sequence $\{t^{(i)}_k\}_{k=1}^\infty$, $t_k^{(i)} \geq T_1$, such that $\lim_{k \to \infty} t_k^{(i)} = +\infty$, and $|x_i(t_k^{(i)})| = E(t_k^{(i)}), \ k = 1, 2, \ldots$. In the same way as (19), we can also get that

$$\frac{d|x_i(t)|}{dt} \bigg|_{t=t_k^{(i)}} \geq -d_i + a_i \epsilon - \sum_{j \neq i} a_{ij} - \sum_{j \neq i} |b_{ij}| \geq 0,$$

Therefore, $|x_i(t_k^{(i)})|$ would tend to $+\infty$ when $k \to \infty$, which contradicts the assumption that $u_i(t), \ i \in \mathbb{N}_2^d$, is bounded in the interval $[-1, 1]$. In other words, if $x_i(t)$ does not converge to 0 for some $i \in \mathbb{N}_2^d$, then the corresponding solution $u(t)$ would leave $\Phi_\epsilon$ in finite time. It completes the proof. \quad \square

Remark 3. The difficulty of the proof of Theorem 1 lies in that the states $u_i(t)$ and $u_j(t - t_q(t))$, $i, j = 1, \ldots, n$, are in different regions.

Furthermore, the results obtained not only give the complete stability of systems (1), but also address the issue of multistability, including the number of equilibrium points, their locations, and their stability (by Remark 2), which improve and extend the results existing in the literature.

Remark 4. As applications of Theorem 1, for neural networks (1) with bounded time delays $t_q(t) \leq t$ or some unbounded time delays, such as $t_q(t) \leq \mu t$, $t_q(t) \leq t - \frac{1}{1 + \epsilon}$, $t_q(t) \leq t - \epsilon^2$ (here $0 < \mu < 1$ and $0 < \alpha < 1$), proposed in Chen and Wang (2007b), we can directly get the following corollaries.

Corollary 1. Suppose that $t_q(t) \leq \tau_q, \ i, j = 1, \ldots, n,$ and

$$|l_i| < -d_i + a_i \cdot \sum_{j=1}^n |a_{ij}| - \sum_{j=1}^n |b_{ij}|$$

holds for $i = 1, \ldots, n$. Then the system (1) has $3^n$ equilibrium points, and is completely stable.

Corollary 2. Suppose that $t_q(t) \leq \mu t$ ($0 < \mu < 1$), or $t_q(t) \leq t - \frac{1}{1 + \epsilon}$, or $t_q(t) \leq t - \epsilon^2$ ($0 < \alpha < 1$), $i, j = 1, \ldots, n,$ and

$$|l_i| < -d_i + a_i \cdot \sum_{j=1}^n |a_{ij}| - \sum_{j=1}^n |b_{ij}|$$

holds for $i = 1, \ldots, n$. Then the system (1) has $3^n$ equilibrium points, and is completely stable.

Remark 5. Note that the condition (7) is a little more restrictive than (6). It is due to the specificity of delayed neural networks. An example is presented, in Section 4, to illustrate that condition (6) is insufficient for a delayed system to reach complete stability.

4. Simulations

Example 1. Consider the following neural networks described by

$$\begin{align*}
\frac{du_1(t)}{dt} &= -u_1(t) + 6f_1(u_1(t)) + 0.3f_2(u_2(t)) + 0.5f_1(u_1(t - t_1(t))) + 0.2f_2(u_2(t - t_2(t))) + 0.1 \\
\frac{du_2(t)}{dt} &= -1.2u_2(t) + 0.2f_1(u_1(t)) + 6.5f_2(u_2(t)) + 0.2f_1(u_1(t - t_1(t))) + 0.3f_2(u_2(t - t_2(t))) + 0.1
\end{align*}$$

While for $i \in \mathbb{N}_3^d$, once $1 < u_i(t_0) \leq 1 + \epsilon$ for some $t_0 \geq 0$, then

$$\frac{du_i(t)}{dt} \bigg|_{t=t_0} \geq -d_i(1 + \epsilon) + a_i \cdot \sum_{j=1}^n |a_{ij}| - \sum_{j=1}^n |b_{ij}|$$

$$− \sum_{j=1}^n |b_{ij}| + l_i < 0.$$
where \( f_1, f_2 \) are defined as (2), and \( \tau_{11}(t) = \tau_{12}(t) = 5 + t - \frac{1}{2} \), \( \tau_{21}(t) = \tau_{22}(t) = 6 + t - \frac{1}{2} \).

It is easy to see that the condition (7) holds for \( i = 1, 2 \) and the time-varying delays satisfy the assumption in Section 2. Thus, by Theorem 1, the system (28) is completely stable.

In fact, direct calculations indicate that there are 3² equilibrium points existing in all. We pick 12 initial states arbitrarily to track the solutions by simulations. The dynamics of \( u_1(t) \), \( u_2(t) \) are depicted in Figs. 1 and 2, which show that each solution trajectory converges to one of the equilibrium points, as confirmed by our results.

In Remark 5, it is pointed out that the condition (6) is not enough to conclude the complete stability of delayed cellular neural networks (1). Here, let's change over the coefficients \( a_{ii} \) and \( b_{ii} \) (\( i = 1, 2 \)) of system (28) to illustrate this point.

Example 2. Consider the following neural networks described by

\[
\begin{align*}
\frac{du_1(t)}{dt} &= -u_1(t) + 0.5f_1(u_1(t)) + 0.3f_2(u_2(t)) \\
&+ 0.2f_1(u_1(t) - \tau_{11}(t)) + 0.2f_2(u_2(t) - \tau_{12}(t)) + 0.1 \\
\frac{du_2(t)}{dt} &= -1.2u_2(t) + 0.2f_1(u_1(t)) + 0.3f_2(u_2(t)) \\
&+ 0.2f_1(u_1(t) - \tau_{21}(t)) + 6.5f_2(u_2(t) - \tau_{22}(t)) + 0.1
\end{align*}
\]

where \( f_1, f_2 \) are defined as (2), and \( \tau_{11}(t) = \tau_{12}(t) = 5 + 0.7t, \tau_{21}(t) = \tau_{22}(t) = 6 + 0.4t \).

Obviously, the condition (6) holds for \( i = 1, 2 \), while (7) does not. Pick an initial value such as

\[
u_1(\theta) = \cos(2\theta), \quad u_2(\theta) = 2 \sin \theta, \quad \text{for } \theta \in (\infty, 0].
\]

By simulations, the dynamics of \( u_1(t), u_2(t) \) can be tracked, which are depicted in Figs. 3 and 4. It shows that \( u(t) \) does not converge to any equilibrium point. Thus the system (29) does not reach the complete stability.

5. Conclusions

Complete stability is an important dynamical property in delayed cellular neural networks. In this paper, we are concerned with the case that the time-varying delays are unbounded. Note that it is impractical to limit the whole initial value (3) staying in the same subset region of \( \mathbb{R}^n \), so it is difficult to track the dynamics of \( u(t) (t > 0) \) by the usual methods. Instead, we present a new approach to address the dynamics of \( u(t) \) based on the location of \( u(0) \). It shows that under some conditions, \( u(t) \) would converge to one of the equilibrium points as \( t \) tends to \( +\infty \). The results obtained extend the existing ones, and they can be verified by two illustrative examples.
References