Exact traveling wave solutions for a generalized Hirota–Satsuma coupled KdV equation by Fan sub-equation method

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\textbf{A R T I C L E  I N F O}

Article history:
Received 28 January 2011
Received in revised form 28 March 2011
Accepted 22 April 2011
Available online 29 April 2011
Communicated by R. Wu

Keywords:
Fan sub-equation method
Generalized Hirota–Satsuma coupled KdV equation
Solitary wave solution
Periodic wave solution

\textbf{A B S T R A C T}

In this Letter, the Fan sub-equation method is used to construct exact solutions of a generalized Hirota–Satsuma coupled KdV equation. Many exact traveling wave solutions are successfully obtained, which contain more general solitary wave solutions and Jacobian elliptic function solutions with double periods. This method is straightforward and concise, and it can also be applied to other nonlinear evolution equations.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena in various scientific fields and especially in areas of physics such as plasma, fluid mechanics, biology, solid state physics, nonlinear optics and so on. Therefore the investigation of the analytical exact solutions to NLPDEs plays an important role in the study of nonlinear science, especially in the study of nonlinear physical science since the exact solutions can provide much physical information and thus lead to further applications. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The analytical exact solutions, if available, of those nonlinear equations facilitate the verification of the numerical solvers and aid in the stability analysis of solutions. In the past decades, exploring exact solutions for the nonlinear equations, in particular the solitary traveling wave solutions, has been a hot and very difficult topic in nonlinear science. Thus many researchers have paid more attention to these topics, and a variety of powerful methods for obtaining exact solutions to NLPDEs have been developed. Among these are bifurcation method of dynamical systems [1,2], tanh function method [3], extended tanh function method [4], F-expansion method [5,6], homogeneous balance method (HBM) [7], homotopy perturbation method (HPM) [8–11], homotopy analysis method (HAM) [12], differential transformation method (DTM) [13], variational iteration method (VIM) [14], exponential rational function method [15], fractional iteration method (FIM) [16] and so on.

Yomba [17] pointed out that the tanh function method, the extended tanh function method and the F-expansion method belonged to a class of methods called sub-equation method for which it appeared some basic relationships among the complicated NLPDEs in study and some simple and solvable nonlinear ordinary differential equations. Thus, these sub-equation methods consist of looking for the solutions of the NLPDEs in consideration as a polynomial in variable which satisfies an equation or equations (named sub-equation). For example, Riccati equation $\phi' = A + \phi^2$, auxiliary ordinary equation $\phi'^2 = B\phi^2 + C\phi^3 + D\phi^4$, first kind elliptic equation $\phi'^2 = A + B\phi^2 + D\phi^4$, the generalized Riccati equation $\phi' = r + p\phi + q\phi^2$ and so on.

Motivated by the existing sub-equation methods mentioned above, Fan [18] recently developed a new algebraic method, called Fan sub-equation method now, with computerized symbol computation, which greatly exceeded the applicability of those sub-equation methods in obtaining a series of analytical exact solutions of nonlinear equations. The idea of the Fan sub-equation method is to seek more new...
solutions of NLPDEs that can be expressed as polynomial in an elementary function which satisfies a more general sub-equation, called Fan sub-equation, than other sub-equations like Riccati equation, auxiliary ordinary equation, elliptic equation and generalized Riccati equation. As we know, the more general analytical exact solutions of the sub-equation are proposed, the more general corresponding exact solutions of NLPDEs will be obtained. Thus it is very important how to obtain more new solutions to the sub-equation. Fortunately, the Fan sub-equation method can just construct more general exact solutions to the sub-equation that can capture all the solutions of the Riccati equation, auxiliary ordinary equation, elliptic equation and generalized Riccati equation.

The important feature of the Fan method is to, without much extra effort and without considering the integrability of nonlinear equations, directly get a series of exact solutions in a uniform way, which readily covers all results of tanh function method, extended tanh function method, F-expansion method and some other sophisticated methods. The useful Fan sub-equation method is widely used by many researchers throughout the world such as in [19,20] and by the references therein. More recently, many authors [21–24] proposed the extended Fan sub-equation method by discussing different Fan sub-equations and obtained many excellent results by applying the extended method to a series of nonlinear equations.

In this Letter, a generalized Hirota–Satsuma coupled KdV equation will be studied by using the Fan sub-equation method mentioned above. As a result, we successfully find many new exact solutions which include solitary wave solutions and doubly periodic solutions in terms of the Jacobian elliptic function.

2. Exact solutions to generalized Hirota–Satsuma coupled KdV equation

In 1999, Wu et al. [25] introduced a 4 × 4 matrix spectral problem with three potentials and proposed a corresponding hierarchy of nonlinear equations. One of the typical equations is the following generalized Hirota–Satsuma coupled KdV equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\
\frac{\partial v}{\partial t} &= -v_{xxx} + 3uv_x, \\
\frac{\partial w}{\partial t} &= -w_{xxx} + 3uw_x.
\end{align*}
\]

Eq. (1) is reduced to a new complex coupled KdV equation [25] and the Hirota–Satsuma equation [26] with \( w = v^* \) and \( w = v \), respectively. More recently, several soliton solutions and periodic wave solutions for this equation were constructed by Fan [27,28] using an extended tanh function method, Jacobian elliptic functions expansion method and symbolic computation. Ganji and Rafei [29] discussed the soliton solutions with initial conditions by making use of the homotopy perturbation method. Moreover the discussed generalized Hirota–Satsuma coupled KdV equation has been studied by many authors via different approaches, for example, the improved F-expansion method by Zhang et al. [30], the modified extended tanh function method by Ali [31], the homotopy analysis method by Abbasbandy [32], direct algebraic method by Zhang [33] and Zayed et al. [34], Adomian’s decomposition method by Kaya [35] and the extended homogeneous balance method by Rady and Osman [36].

The aim of this Letter is to construct more types of analytical exact solutions for Eq. (1) by the Fan sub-equation method and symbolic computation. The key idea of this method is to take full advantages of the Fan sub-equation involving parameters and use its solutions to obtain the corresponding exact solutions of Eq. (1).

To solve Eq. (1) by using the Fan sub-equation method, we make the traveling wave transformation

\[
u(x, t) = u(\xi), \quad v(x, t) = v(\xi), \quad w(x, t) = w(\xi), \quad \xi = x - ct,
\]

where \( \xi \) is a wave variable and \( c \) is a wave speed.

Substituting Eq. (2) into Eq. (1) yields the ordinary differential equations for \( u(\xi), v(\xi), w(\xi) \)

\[
\begin{align*}
-cu' &= \frac{1}{2}u'' - 3uu' + 3(vw)', \\
-cv' &= -v'' + 3uv', \\
cw' &= -w'' + 3uw'.
\end{align*}
\]

where a prime denotes the derivative with respect to the variable \( \xi \).

We now seek the solutions of Eq. (3) in the form

\[
\begin{align*}
u &= \sum_{i=0}^{m} \alpha_i \phi^i, \\
v &= \sum_{i=0}^{n} \beta_i \phi^i, \\
w &= \sum_{i=0}^{p} \gamma_i \phi^i,
\end{align*}
\]

where \( \alpha_i, \beta_i, \gamma_i \) are constants to be determined later and the new variable \( \phi \) satisfies the Fan sub-equation

\[
\phi''(\xi) = \epsilon \sum_{j=0}^{4} \alpha_j \phi^j,
\]

In this Letter, a generalized Hirota–Satsuma coupled KdV equation will be studied by using the Fan sub-equation method mentioned above. As a result, we successfully find many new exact solutions which include solitary wave solutions and doubly periodic solutions in terms of the Jacobian elliptic function.
where $\epsilon = \pm 1$ and $\omega_j$ are constants. Thus the derivatives with respect to the variable $\xi$ become the derivatives with respect to the variable $\phi$ as follows

$$
\frac{d^2u}{d\xi^2} = \epsilon \sum_{j=0}^{\phi} \omega_j \phi^j \frac{d^2u}{d\phi^2},
$$

(6)

$$
\frac{d^2u}{d\xi^2} = \frac{1}{2} \sum_{j=1}^{6} \omega_j \phi^{j-1} \frac{d^2u}{d\phi^2} + \frac{1}{2} \omega_0 \phi \frac{d^2u}{d\phi^2}, \quad \cdots
$$

(7)

Substituting Eqs. (4)-(7) into Eq. (3), it follows from Eqs. (6) and (7) that each term in Eq. (3) takes the form like $\phi^k \sqrt{\sum_{j=0}^{6} \omega_j \phi^j}$ $(k = 0, 1, 2, \ldots)$. Balancing the term $v''$ with the term $u''$ and also $w''$ with $uw'$ in Eq. (3) gives

$$
n + 1 = m + n - 1, \quad p + 1 = m + p - 1,
$$

from which we have $m = 2$. Then after balancing $(vw)'$ with $u''$ (or with $u'$) in the first equation of Eq. (3), we can find

$$
n + p - 1 = m + 1 = 3 \quad \text{or} \quad n + p - 1 = m - 1 = 1,
$$

which imply $n + p = 4$ or $n + p = 2$. So we can choose $m = n = p = 2$ or $m = 2$, $n = p = 1$ and have the following two ansätze from Eq. (4)

$$
\begin{align*}
\alpha &= \alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2, \\
\beta &= \beta_0 + \beta_1 \phi + \beta_2 \phi^2, \\
\gamma &= \gamma_0 + \gamma_1 \phi + \gamma_2 \phi^2,
\end{align*}
$$

(8)

and

$$
\begin{align*}
\alpha &= \alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2, \\
\beta &= \beta_0 + \beta_1 \phi, \\
\gamma &= \gamma_0 + \gamma_1 \phi,
\end{align*}
$$

(9)

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1$ and $\gamma_2$ are constants to be determined and $\phi$ satisfies Eq. (5).

Substituting Eqs. (5) and (8) into Eq. (3), eliminating $\epsilon \sqrt{\sum_{j=0}^{6} \omega_j \phi^j}$ from system (3), and setting all the coefficients of like powers of $\phi^i$ $(i = 0, 1, 2, 3)$ to zero yields a system of algebraic equations with respect to the parameters $\alpha_i, \beta_i, \gamma_i$ $(i = 0, 1, 2)$ and $\epsilon$, namely

$$
\begin{align*}
6\alpha_1^2 - 12\alpha_2 \omega_4 - 12\beta_2 \gamma_2 &= 0, \\
9\alpha_1 \alpha_2 - 3\alpha_1 \omega_4 - 9\beta_2 \gamma_1 - 9\beta_1 \gamma_2 - \frac{15}{2} \alpha_2 \omega_3 &= 0, \\
-2\epsilon c_\alpha 2 + 6\alpha_0 \omega_4 + 3\alpha_1^2 - 6\beta_2 \gamma_0 - \frac{3}{2} \alpha_1 \omega_3 - 6\beta_0 \gamma_2 - 4\epsilon \alpha_2 \omega_2 - 6\beta_1 \gamma_1 &= 0, \\
-\epsilon c_\alpha 1 - \frac{3}{2} \alpha_2 \omega_1 - \frac{1}{2} \alpha_1 \omega_2 + 3\alpha_0 \alpha_1 - 3\beta_1 \gamma_0 - 3\beta_0 \gamma_1 &= 0, \\
24\beta_2 \omega_4 - 6\alpha_2 \beta_2 &= 0, \\
6\beta_1 \omega_4 + 15\beta_2 \omega_3 - 6\alpha_1 \beta_2 - 3\epsilon \alpha_2 \beta_1 &= 0, \\
8\beta_2 \omega_2 - 2\epsilon c_\beta 2 - 6\alpha_0 \beta_2 + 3\beta_1 \omega_3 - 3\epsilon \alpha_1 \beta_1 &= 0, \\
-\epsilon c_\beta 1 - 3\alpha_0 \beta_1 + 3\epsilon \beta_2 \omega_1 + \epsilon \beta_1 \omega_2 &= 0, \\
24\gamma_2 \omega_4 - 6\alpha_2 \gamma_2 &= 0, \\
6\gamma_1 \omega_4 + 15\gamma_2 \omega_3 - 6\epsilon \alpha_1 \gamma_2 - 3\epsilon \alpha_2 \gamma_1 &= 0, \\
8\gamma_2 \omega_2 - 2\epsilon c_\gamma 2 - 6\alpha_0 \gamma_2 + 3\epsilon \gamma_1 \omega_3 - 3\epsilon \gamma_1 \gamma_1 &= 0, \\
-\epsilon c_\gamma 1 - 3\alpha_0 \gamma_1 + 3\epsilon \gamma_2 \omega_1 + \epsilon \gamma_1 \omega_2 &= 0,
\end{align*}
$$

(10)

for which, with the aid of Maple, we get the following solutions

$$
\begin{align*}
\alpha_0 &= \frac{16\omega_2 \omega_4 - 4\epsilon c_\alpha \omega_4 - 3\epsilon \alpha_2^2}{12\omega_4}, \quad \alpha_1 = 2\omega_3, \quad \alpha_2 = 4\epsilon \omega_4, \\
\beta_0 &= \frac{-\epsilon \beta_2 (3\epsilon \alpha_2^2 - 16\omega_2 \omega_4 + 16\epsilon \omega_4)}{24\epsilon \omega_4}, \quad \beta_1 = \frac{\beta_2 \omega_3}{2\epsilon \omega_4}, \\
\gamma_1 &= \frac{2\epsilon \omega_3 \omega_4}{\beta_2}, \quad \gamma_2 = \frac{4\epsilon \alpha_2^2}{\beta_2}, \quad \gamma_1 = \frac{2\epsilon \omega_2 (4\epsilon \omega_2 \omega_4 - \epsilon \alpha_2^2)}{8\epsilon \omega_4^2}, \quad \omega_3 = \frac{\epsilon \omega_3 (4\epsilon \omega_2 \omega_4 - \epsilon \alpha_2^2)}{8\epsilon \omega_4^2}.
\end{align*}
$$

(11)

with $\beta_2 \neq 0$ and $\gamma_0$ being arbitrary constants.
Similarly, substituting Eqs. (5) and (9) into Eq. (3), eliminating \( \epsilon \sqrt{\sum_{j=0}^{3} \omega_j \phi_j} \) from system (3), and setting all the coefficients of like powers of \( \phi_i \) \((i = 0, 1, 2)\) to zero gives a system of algebraic equations with respect to the parameters \( \alpha_i, \beta_i, \gamma_i \) \((i = 0, 1, 2)\) and \( c \), namely

\[
\begin{aligned}
6\alpha_0^2 - 12\alpha_2\omega_4 &= 0, \\
\frac{15}{2}\alpha_2\omega_3 - 3\alpha_1\omega_4 + 9\alpha_1\alpha_2 &= 0, \\
6\alpha_0\alpha_2 - \frac{3}{2}\alpha_1\omega_3 - 4\alpha_2\omega_2 - 2\alpha_2 + 3\alpha_1^2 - 6\beta_1\gamma_1 &= 0, \\
-3\alpha_1 - 3\beta_1\gamma_0 - \frac{3}{2}\alpha_2\omega_1 - \frac{1}{2}\alpha_1\omega_2 - 3\gamma_1\beta_0 + 3\alpha_0\alpha_1 &= 0, \\
6\omega_4 - 3\alpha_2 &= 0, \\
3\alpha_3 - 3\alpha_1 &= 0, \\
-c + \omega_2 - 3\alpha_0 &= 0.
\end{aligned}
\] (12)

Solving the above algebraic equations, we have the following results

\[
\begin{aligned}
\alpha_0 &= \frac{\omega_2 - c}{3}, \quad \alpha_1 = \omega_3, \quad \alpha_2 = 2\omega_4, \\
\beta_0 &= \frac{(2\omega_2\omega_3 - 8\omega_3 - 12\omega_1\omega_4)\gamma_1 + (16\omega_4 + 8\omega_2\omega_4 - 3\alpha_0^2)\gamma_0}{12\gamma_1^2}, \\
\beta_1 &= -\frac{16\omega_4 + 8\omega_2\omega_4 - 3\alpha_0^2}{12\gamma_1},
\end{aligned}
\] (13)

with \( \gamma_1 \neq 0 \) and \( \gamma_0 \) being arbitrary constants.

Now we successfully establish two transformations (8) and (9) between Eqs. (3) and (5) under the conditions (11) and (13), respectively. Eqs. (8) and (9) are like a bridge connecting the more complex system (3) with the simpler system (5). If we know a solution of (5), we can easily obtain the corresponding solution of (3) (i.e., (1)) by using (8) and (9), respectively. Therefore we first discuss the exact solutions of the Fan sub-equation (5) and then study the corresponding solution of (3) according to the transformations (8) and (9), respectively.

**Case 1.** When \( \omega_0 = \omega_1 = \omega_3 = 0, \omega_2 > 0, \omega_4 < 0 \), Eq. (5) admits a hyperbolic function solution

\[
\phi = \sqrt{\frac{-\omega_2}{\omega_4}} \operatorname{sech}(\sqrt{\omega_2} x).
\] (14)

Substituting (14) and (11) into (8) yields the following new solitary wave solution of Eq. (1) of bell-type for \( u, v \) and \( w \)

\[
\begin{aligned}
u_1(x, t) &= \frac{4\omega_2 - c}{3} - 4\omega_2 \operatorname{sech}^2(\sqrt{\omega_2}(x - ct)), \\
v_1(x, t) &= \frac{\beta_2(8\omega_2\omega_4 - 8\omega_4 - 3\beta_2\gamma_0)}{12\omega_4^2} - \frac{\beta_2\omega_2}{\omega_4} \operatorname{sech}^2(\sqrt{\omega_2}(x - ct)), \\
w_1(x, t) &= \gamma_0 - \frac{4\omega_2\omega_4}{\beta_2} \operatorname{sech}^2(\sqrt{\omega_2}(x - ct)),
\end{aligned}
\] (15)

where \( \omega_2 > 0, \omega_4 < 0, \beta_2 \neq 0, c \) and \( \gamma_0 \) are arbitrary constants.

Similarly, substituting (14) and (13) into (9), we obtain the following new traveling wave solutions of Eq. (1) of bell-type for \( u, v \) and \( w \)

\[
\begin{aligned}
u_2(x, t) &= \frac{\omega_2 - c}{3} - 2\omega_2 \operatorname{sech}^2(\sqrt{\omega_2}(x - ct)), \\
v_2(x, t) &= \frac{2(2c + \omega_2)}{3\gamma_1^2} \left[ \gamma_0 \omega_4 + \gamma_1 \sqrt{-\omega_2\omega_4} \operatorname{sech}(\sqrt{\omega_2}(x - ct)) \right], \\
w_2(x, t) &= \gamma_0 - \frac{\gamma_1}{\omega_4} \sqrt{-\omega_2\omega_4} \operatorname{sech}(\sqrt{\omega_2}(x - ct)),
\end{aligned}
\] (16)

where \( \omega_2 > 0, \omega_4 < 0, \gamma_1 \neq 0, c \) and \( \gamma_0 \) are arbitrary constants.

**Case 2.** For \( \omega_0 = \omega_1 = \omega_4 = 0, \omega_2 > 0 \), we can gain the following solution to Eq. (5)

\[
\phi = -\frac{\omega_2}{\omega_3} \operatorname{sech}^2 \left( \frac{\sqrt{\omega_2}}{2} (x - ct) \right).
\] (17)

Substituting (17) and (13) into (9), we give the solitary traveling wave solutions of Eq. (1) of bell-type for \( u, v \) and \( w \)
Remark 1. Substituting Eqs. (19) and (11) into (8) yields one family of solitary traveling wave solutions of Eq. (1) of bell-type for \( \gamma \) shows peak form (or valley form) if \( |\omega| > 1 \) and \( \omega \neq 0 \). Fig. 1. The plots of \( u_2(x,t) \), \( v_2(x,t) \) and \( w_2(x,t) \) with \( \omega_2 = 0.25, \omega_4 = -1, \epsilon = \gamma_0 = 1 \), shown in (a), (b) and (c) for \( \gamma_1 = 0.5 \) and shown in (d), (e) and (f) for \( \gamma_1 = -0.5 \), respectively.

\[
\begin{align*}
  u_3(x,t) &= \frac{\omega_2 - c}{3} - \omega_2 \text{sech}^2 \left( \frac{\sqrt{\omega_2}}{2}(x - ct) \right), \\
  v_3(x,t) &= \frac{2\omega_2 \omega_3 \gamma_1 - 8c\omega_3 \gamma_1 - 3\gamma_0 \omega^2}{12\gamma_1^2} - \frac{\omega_2 \omega_3}{4\gamma_1^2} \text{sech}^2 \left( \frac{\sqrt{\omega_2}}{2}(x - ct) \right), \\
  w_3(x,t) &= \gamma_0 - \frac{\omega_2 \gamma_1}{\omega_3} \text{sech}^2 \left( \frac{\sqrt{\omega_2}}{2}(x - ct) \right),
\end{align*}
\]

where \( \omega_2 > 0, \omega_3 \neq 0, \gamma_1 \neq 0, c \) and \( \gamma_0 \) are arbitrary constants.

To demonstrate the physical insight of the new solitary traveling wave solutions, here we take \( (u_2, v_2, w_2) \) as an example. It is obvious from their expressions that \( \sqrt{\omega_2} \) is the wave number and \( c \) is the traveling wave velocity. Moreover, \( 2\omega_2, \frac{2|2c + \omega_2|}{3\gamma_1^2}, \sqrt{-\omega_2 \omega_4} \) and \( |\gamma_1| \sqrt{-\omega_2 \omega_4} \) are the wave amplitudes of \( u_2, v_2 \) and \( w_2 \), respectively. For all \( \omega_2 > 0, u_2(x,t) \) is a solitary wave of peak type. \( v_2(x,t) \) shows peak form (or valley form) if \( \gamma_1(2c + \omega_2) > 0 \) (or < 0), while \( w_2(x,t) \) is in peak form (or valley form) for \( \gamma_1 > 0 \) (or < 0). By choosing \( \omega_2 = 0.25, \omega_4 = -1 \) and \( c = \gamma_0 = 1 \), the wave profiles of the solution \( (u_2(x,t), v_2(x,t), w_2(x,t)) \) for two different values of \( \gamma_1, \gamma_1 = 0.5 \) and \( \gamma_1 = -0.5 \), are displayed in Fig. 1, respectively. Clearly, in both cases, the solutions describe the traveling of waves in the \( x \)-direction, but different values of \( \gamma_1 \) yield different wave shapes.

Case 3. When \( \omega_1 = \omega_3 = 0, \omega_2 = \frac{\omega_2^2}{\omega_4}, \omega_4 < 0, \omega_4 > 0 \), Eq. (5) admits two hyperbolic function solutions

\[
\phi = \mp \sqrt{-\frac{\omega_2}{2\omega_4}} \tanh \left( \sqrt{\frac{\omega_2}{2}} x \right).
\]

Substituting Eqs. (19) and (11) into (8) yields one family of solitary traveling wave solutions of Eq. (1) of bell-type for \( u, v \) and \( w \).
\[
\begin{align*}
\hat{u}_4(x, t) &= \frac{4\omega_2 - c}{3} - 2\omega_2 \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{v}_4(x, t) &= \frac{\beta_2 (8\omega_2\omega_4 - 8c\omega_4 - 3\beta_2 \gamma_0)}{12\omega_4^2} - \frac{\beta_2 \omega_2}{2\omega_4} \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{w}_4(x, t) &= \gamma_0 - \frac{2\omega_2\omega_4}{\beta_2} \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right),
\end{align*}
\]

(20)

where \(\omega_2 < 0, \omega_4 > 0, \beta_2 \neq 0, c\) and \(\gamma_0\) are arbitrary constants.

Similarly, from Eqs. (19), (13) and (9), we can establish a family of solitary wave solutions of Eq. (1) of bell-type for \(u\) and kink-type for \(v\) and \(w\)

\[
\begin{align*}
\hat{u}_5(x, t) &= \frac{\omega_2 - c}{3} - \omega_2 \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{v}_5(x, t) &= \frac{2c + \omega_2}{3\gamma_1} \left[ 2\gamma_0\omega_4 + \gamma_1 \sqrt{-2\omega_2\omega_4} \tanh \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right) \right], \\
\hat{w}_5(x, t) &= \gamma_0 + \frac{\gamma_1}{2\omega_4} \sqrt{-2\omega_2\omega_4} \tanh \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right),
\end{align*}
\]

(21)

where \(\omega_2 < 0, \omega_4 > 0, \gamma_1 \neq 0, c\) and \(\gamma_0\) are arbitrary constants.

**Remark 2.** In fact, two families of solitary wave solutions reported in [28] and [33] are the special cases of \((\hat{u}_4, \hat{v}_4, \hat{w}_4)\) by setting

\[
\omega_2 = -2k^2, \quad c = -\beta, \quad \gamma_0 = c_0, \quad \beta_2 = \frac{4k^2\omega_4}{c_2}
\]

and by setting

\[
\omega_2 = -2k^2 q_2, \quad \beta_2 = 2\omega_4, \quad \gamma_0 = c_0 - 2k^2 q_2,
\]

respectively. However, all of the other solitary wave solutions are new.

**Remark 3.** By taking

\[
\omega_2 = -2k^2, \quad c = -\beta, \quad \gamma_0 = c_0, \quad \gamma_1 = -\frac{c_1}{k} \sqrt{\omega_4}
\]

and by taking

\[
\omega_2 = -2k^2 q_2, \quad \gamma_0 = c_0 - \frac{c_2 q_2}{q_4}, \quad \gamma_1 = -\frac{c_2}{kq_4} \sqrt{q_2 \omega_4},
\]

respectively, the solitary wave solution \((\hat{u}_5, \hat{v}_5, \hat{w}_5)\) is full in agreement with ones constructed in [28] and [33].

**Case 4.** For \(\omega_0 = \omega_1 = 0, \omega_2 = \pm 2\sqrt{\omega_2 \omega_4}, \omega_2 > 0, \omega_4 > 0, \) Eq. (5) has two kinds of exact solutions

\[
\phi = -\sqrt{\frac{\omega_2 \omega_4}{2\omega_4}} \text{sign}(\omega_0) \left[ 1 + \tanh \left( \frac{\sqrt{\omega_2}}{2} x \right) \right].
\]

(22)

Substituting Eqs. (22) and (11) into (8) yields one family of solitary traveling wave solutions of Eq. (1) of bell-type for \(u\), \(v\) and \(w\)

\[
\begin{align*}
\hat{u}_6(x, t) &= -\frac{c + 2\omega_2}{3} + \omega_2 \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{v}_6(x, t) &= -\frac{\beta_2 (8\omega_2 \omega_4 + 2\omega_2 + 3\beta_2 \gamma_0)}{12\omega_4^2} + \frac{\beta_2 \omega_2}{4\omega_4} \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{w}_6(x, t) &= \gamma_0 + \frac{\omega_2 \omega_4}{\beta_2} \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right),
\end{align*}
\]

(23)

where \(\omega_2 > 0, \omega_4 > 0, \beta_2 \neq 0, c\) and \(\gamma_0\) are arbitrary constants.

From Eqs. (22), (13) and (9), we can obtain a family of solitary wave solutions of Eq. (1) of bell-type for \(u\) and kink-type for \(v\) and \(w\)

\[
\begin{align*}
\hat{u}_7(x, t) &= -\frac{\omega_2 + 2c}{6} + \frac{\omega_2}{2} \tanh^2 \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right), \\
\hat{v}_7(x, t) &= \frac{4c - \omega_2}{6\gamma_1^2} \left[ 2\gamma_0\omega_4 \pm \gamma_1 \sqrt{\omega_2 \omega_4} \left( 1 + \tanh \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right) \right) \right], \\
\hat{w}_7(x, t) &= \gamma_0 \pm \frac{\gamma_1}{2\omega_4} \sqrt{\omega_2 \omega_4} \left[ 1 + \tanh \left( \sqrt{-\frac{\omega_2}{2}} (x - ct) \right) \right],
\end{align*}
\]

(24)

where \(\omega_2 > 0, \omega_4 > 0, \gamma_1 \neq 0\) and \(c\) are arbitrary constants.
Remark 4. To the best of our knowledge, solutions (23) and (24) obtained for Eq. (1) have not been reported in literature.

To show the physical insight of these new solutions, we take \((u_2, v_2^\pm, w_2^\pm)\) as an example. The solutions \(u_2(x, t), v_2^\pm(x, t)\) and \(w_2^\pm(x, t)\) travel with the same wave velocity \(c\) and wave number \(\sqrt{\omega_2}\). However, for all \(\omega_2 > 0\), \(u_2(x, t)\) is a smooth solitary wave solution with valley form, while \(v_2^\pm(x, t)\) and \(w_2^\pm(x, t)\) are of kink form or anti-kink form determined by different values of \(\omega_2\), \(c\) and \(\gamma_1\). The wave profiles of the solutions \(u_2(x, t), v_2^\pm(x, t)\) and \(w_2^\pm(x, t)\) for \(\omega_2 = 0.25, \omega_4 = 1, c = 0.5, \gamma_0 = 1\) and \(\gamma_1 = -0.5\) are displayed in Fig. 2. The solutions describe the traveling of waves in the \(x\)-direction. Note that \(v_2^\pm(x, t)\) and \(v_2^\mp(x, t)\), also \(w_2^\pm(x, t)\) and \(w_2^\mp(x, t)\), show different shapes, i.e., one is of kink form and another is of anti-kink form, although they take the same parametric values of \(\omega_2\), \(\omega_4\), \(c\), \(\gamma_0\) and \(\gamma_1\).

Case 5. When \(\omega_1 = \omega_3 = 0\), Eq. (5) admits three Jacobian elliptic doubly periodic solutions

\[
\phi = \frac{-\omega_2 k^2}{\omega_4(2k^2 - 1)} \quad \text{cn}\left(\frac{\omega_2}{\sqrt{2k^2 - 1}}, k\right), \quad \text{for} \quad \omega_0 = \frac{\omega_2^2 k^2 (k^2 - 1)}{\omega_4(2k^2 - 1)^2}, \quad \omega_2 > 0, \quad \omega_4 < 0; \quad (25)
\]

\[
\phi = \frac{-\omega_2}{\omega_4(2 - k^2)} \quad \text{dn}\left(\frac{\omega_2}{2 - k^2}, k\right), \quad \text{for} \quad \omega_0 = \frac{\omega_2^2 (1 - k^2)}{\omega_4(k^2 - 2)^2}, \quad \omega_2 > 0, \quad \omega_4 < 0; \quad (26)
\]

\[
\phi = \pm \frac{-\omega_2 k^2}{\omega_4(k^2 + 1)} \quad \text{sn}\left(\frac{-\omega_2}{k^2 + 1}, k\right), \quad \text{for} \quad \omega_0 = \frac{\omega_2^2 k^2}{\omega_4(k^2 + 1)^2}, \quad \omega_2 < 0, \quad \omega_4 > 0. \quad (27)
\]

Substituting Eqs. (25), (11) and (13) into Eqs. (8) and (9), respectively, yields two families of Jacobian elliptic doubly periodic wave solutions

\[
\begin{align*}
\begin{cases}
u_s(x, t) = 4\omega_2 - c & \frac{4\omega_2^2 - c}{3 - \frac{4\omega_2^2}{2k^2 - 1}} \cdot \text{cn}\left(\sqrt{\frac{\omega_2}{2k^2 - 1}}(x - ct), k\right),\\
w_s(x, t) = 4\omega_2 \frac{3}{2k^2 - 1} & \frac{\beta_2(8\omega_2\omega_4 - 8c\omega_4 - 3\beta_2 \gamma_0)}{12\omega_4^2} \cdot \text{cn}\left(\sqrt{\frac{\omega_2}{2k^2 - 1}}(x - ct), k\right),
\end{cases}
\end{align*}
\]

\[
\begin{cases}
u_s(x, t) = \frac{2\beta_2}{(2k^2 - 1)} & \frac{\beta_2(8\omega_2\omega_4 - 8c\omega_4 - 3\beta_2 \gamma_0)}{12\omega_4^2} \cdot \text{cn}\left(\sqrt{\frac{\omega_2}{2k^2 - 1}}(x - ct), k\right),
\end{cases}
\]

\[
\begin{cases}
u_s(x, t) = \frac{\omega_2}{2k^2 - 1} & \frac{2\beta_2}{(2k^2 - 1)} \cdot \text{cn}\left(\sqrt{\frac{\omega_2}{2k^2 - 1}}(x - ct), k\right).
\end{cases}
\]
In the limit case when $\omega_2 > 0$, $\omega_4 < 0$, $\beta_2 \neq 0$, $\gamma_1 \neq 0$, $k \in (\sqrt{2}, 1)$, $c$ and $\gamma_0$ being arbitrary constants.

From Eqs. (26), (11) and (13) along with Eqs. (8) and (9), we can obtain two families of periodic wave solutions of Eq. (1)

\[
\begin{align*}
\begin{align*}
&\frac{\omega_2 - c}{3} - \frac{2k^2\omega_2}{2k^2 - 1} \text{cn}^2\left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right), \\
&\frac{2(2c + \omega_2)}{3\gamma_1^2} \left[\gamma_0\omega_4 + k\gamma_1 \sqrt{-\omega_2\omega_4} \text{cn} \left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right)\right], \\
&\frac{k\gamma_1}{\omega_4} \sqrt{-\omega_2\omega_4} \text{cn} \left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right)
\end{align*}
\end{align*}
\]

(29)

with $\omega_2 > 0$, $\omega_4 < 0$, $\beta_2 \neq 0$, $\gamma_1 \neq 0$, $k \in (0, 1)$, $c$ and $\gamma_0$ being arbitrary constants.

Similarly, substituting Eqs. (27), (11) and (13) into Eqs. (8) and (9), respectively, we can give two families of Jacobian elliptic doubly periodic wave solutions

\[
\begin{align*}
\begin{align*}
&\frac{\omega_2 - c}{3} - \frac{2\omega_2}{2k^2 - 1} \text{dn}^2\left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right), \\
&\frac{2(2c + \omega_2)}{3\gamma_1^2} \left[\gamma_0\omega_4 + \gamma_1 \sqrt{-\omega_2\omega_4} \text{dn} \left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right)\right], \\
&\frac{k\gamma_1}{\omega_4} \sqrt{-\omega_2\omega_4} \text{dn} \left(\sqrt{\frac{\omega_2}{2k^2 - 1}} (x - ct), k\right)
\end{align*}
\end{align*}
\]

(30)

with $\omega_2 > 0$, $\omega_4 < 0$, $\beta_2 \neq 0$, $\gamma_1 \neq 0$, $k \in (0, 1)$, $c$ and $\gamma_0$ being arbitrary constants.

Remark 5. In the limit case when $k \to 1$, $\text{sn}(\xi, k) \to \text{tanh} \xi$, $\text{cn}(\xi, k) \to \text{sech} \xi$ and $\text{dn}(\xi, k) \to \text{sech} \xi$, then both $(u_8, v_8, w_8)$ and $(u_{10}, v_{10}, w_{10})$ tend to $(u_1, v_1, w_1)$, and both $(u_6, v_6, w_6)$ and $(u_{11}, v_{11}, w_{11})$ tend to $(u_2, v_2, w_2)$. Obviously, $(u_{12}, v_{12}, w_{12}) \to (u_4, v_4, w_4)$ and $(u_{13}, v_{13}, w_{13}) \to (u_5, v_5, w_5)$ as $k \to 1$. This fact shows that some solitary wave solutions given above can be obtained from the limits of the Jacobian elliptic periodic wave solutions.

Remark 6. Two families of periodic wave solutions given in [27] and [30] are the special cases of $(u_{12}, v_{12}, w_{12})$ by taking

\[
\begin{align*}
\omega_2 = -(k^2 + 1)c^2, & \quad c = -\beta, & \quad \beta_2 = \frac{4\alpha^2k^2\omega_4}{c^2}, & \quad \gamma_0 = c_0
\end{align*}
\]

and by taking
Step 2: Expand the solution of (35) in the form 

\[ u(x, t) = \sum_{i=0}^{m} \alpha_i \phi^i(\xi), \]

where \( \phi(\xi) \) is a known function.

Step 3: Determine \( n \) by substituting (36) with (5) into (35) and balancing the linear term of the highest order with the nonlinear term in Eq. (35).

Step 4: Substituting (36) and (5) into Eq. (35) again and collecting all coefficients of \( \phi^k \sqrt{\sum_{j=0}^{l} \omega_j \phi^j} \) \( (l = 0, 1, 2, \ldots) \), then setting these coefficients to zero will give a set of algebraic equations with respect to \( \alpha_i \) \( (i = 0, 1, \ldots, n) \) and \( c \).

Step 5: Solve these algebraic equations to obtain \( \alpha_i \) and \( c \). Substituting these results into (36) yields the general form of traveling wave solutions.

Step 6: For each solution to Eq. (5) which depends on the special conditions chosen for the \( \omega_0, \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \), it follows from (36) obtained from the above steps that the corresponding exact solution of (35) can be constructed.

Remark 7. The periodic wave solution \( (u_{13}, v_{13}, w_{13}) \) is the same as one obtained in [27] by taking

\[ \omega_2 = -(k^2 + 1)\omega^2, \quad c = \beta, \quad \gamma_0 = c_0, \quad \gamma_1 = \frac{c_1}{k\alpha} \sqrt{\omega_4}. \]

Remark 8. All of the other Jacobian elliptic wave solutions with double period in Eqs. (28)-(33) are new.
From the above procedure, it is easy to find that the Fan sub-equation method is a powerful and effective mathematical tool to solve the generalized Hirota–Satsuma coupled KdV equation and a lot of new solutions can be obtained in the same time. It is also a promising method to solve many other nonlinear equations.

Acknowledgements

We would like to express our sincere thanks to the editors and the referees for their valuable and helpful suggestions and comments. This research is supported by National Natural Science Foundation of China (Nos. 11061010, 61004101), Guangxi Natural Science Foundation (Nos. 2011GXNSFA018136, 2011GXNSFB018059), China Postdoctoral Science Foundation (No. 20100480952) and Postdoctoral Science Foundation of Central South University.

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