Multiplicity results of periodic solutions for a class of first order delay differential equations

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Abstract

This paper is concerned with the periodic boundary value problem

\[ \begin{align*}
    u'(t) &= -\Lambda u(t + r) - f(t, u(t - r)), \\
    u(0) &= -u(2r), \\
    u(0) &= u(4r)
\end{align*} \]

where \( r > 0 \) is a given constant, \(-\frac{\pi}{2} \leq \Lambda < \frac{3\pi}{2} \) is a parameter, and \( f \in C(\mathbb{R}^1 \times \mathbb{R}^n, \mathbb{R}^n) \) satisfies \( f(t + r, z) = f(t, z) \) for all \( z \in \mathbb{R}^n \). The variational principle is given and some multiplicity results of periodic solutions of (1) are obtained via variational methods.

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1. Introduction and preliminaries

In this paper, we consider the multiplicity problems of periodic solutions for the following non-autonomous delay systems

\[ \begin{align*}
    u'(t) &= -\Lambda u(t + r) - f(t, u(t - r)), \\
    u(0) &= -u(2r), \\
    u(0) &= u(4r)
\end{align*} \]

via variational methods, where \( r > 0 \) is a given constant, \(-\frac{\pi}{2} \leq \Lambda < \frac{3\pi}{2} \) is a parameter, and \( f \in C(\mathbb{R}^1 \times \mathbb{R}^n, \mathbb{R}^n) \) satisfies \( f(t + r, z) = f(t, z) \) for all \( z \in \mathbb{R}^n \).

For autonomous delay differential equations dealing with scalar, the existence of the periodic solutions has been extensively studied in the past years via fixed point theory and some other techniques, for example, see [1–7]. It is not our purpose to give a survey in this paper. We only mention some related work here. In 2005, Zhiming Guo and Jianshe Yu [8] took the lead in using the variational approaches to study the existence of multiple periodic solutions for (1.1), and a multiplicity result was given by using a pseudo-index theory. However, up until now, there are a few existence and multiplicity results of periodic solutions for (1.1) dealing with variational approaches. In the present paper, our main purpose is to study the multiplicity of periodic orbits for the systems (1.1) via some recent critical point theorems for strongly indefinite functionals.

Now, we give some preliminaries. Let \( X \) and \( Y \) be Banach spaces with \( X \) being separable and reflexive, and set \( E = X \oplus Y \). Let \( S = X^* \) be a dense subset. For each \( s \in S \), there is a semi-norm on \( E \) defined by

\[ p_s : E \to \mathbb{R}^1, \quad p_s(u) = |s(x)| + \|y\| \quad \text{for } u = x + y \in X \oplus Y. \]
We denote by $\mathcal{T}_S$ the topology on $E$ induced by semi-norm family $\{p_i\}$, and let $w$ and $w^*$ denote the weak-topology and weak$^*$-topology, respectively. Clearly, the topology $\mathcal{T}_S$ is the product topology of the weak-topology on $X$ and the strong topology on $Y$.

For a functional $\Phi \in C^1(E, \mathbb{R}^1)$ we write $\Phi_0 = \{u \in E: \Phi(u) \geq 0\}$. Recall that $\Phi'$ is said to be weak sequentially continuous if for any $u_k \to u$ in $E$, one has $\lim_{k \to \infty} \Phi'(u_k)v \to \Phi'(u)v$ for each $v \in E$, i.e. $\Phi' : (E, w) \to (E^*, w^*)$ is sequentially continuous. For $c \in \mathbb{R}^1$ we say that $\Phi$ satisfies the $(C)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $1 + \|u_k\|\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence. Similarly, we say that $\Phi$ satisfies the $(PS)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

Suppose that

$\Phi_0$ for any $c \in \mathbb{R}^1$, $\Phi_0$ is $\mathcal{T}_S$-closed, and $\Phi' : (\Phi_0, \mathcal{T}_S) \to (E^*, w^*)$ is continuous,

$\Phi_1$ there exists a $\rho > 0$ such that $\kappa := \inf \Phi(\partial B_\rho \cap Y) > 0$, where $B_\rho = \{u \in E: \|u\| < \rho\}$,

$\Phi_2$ there exists a finite dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus S_0) < \inf \Phi(B_\rho \cap Y)$, where

$$E_0 := X \oplus Y_0, \quad \text{and} \quad S_0 = \{u \in E_0: \|u\| \leq R\},$$

$\Phi_3$ there exists an increasing sequence of finite dimensional subspaces $Y_n \subset Y$ and $R_n > \rho$ such that $\sup \Phi(E_n) < \infty$ and $\sup \Phi(E_n \setminus S_n) < \inf \Phi(B_\rho)$, where $E_n := X \oplus Y_n, S_n = \{u \in E_n: \|u\| \leq R_n\}$.

**Theorem 1.1.** (See [9].) Assume that $\Phi$ is even and $(\Phi_0)$--$(\Phi_2)$ are satisfied. Then $\Phi$ has at least $m = \dim Y_0$ pairs of critical points with critical values less than or equal to $\bar{c}$ provided $\Phi$ satisfies the $(C)_c$ condition for all $c \in [\kappa, \bar{c}]$.

**Theorem 1.2.** (See [10].) Assume that $\Phi$ is even and $(\Phi_0), (\Phi_1)$ and $(\Phi_3)$ are satisfied. Then $\Phi$ has an unbounded sequence of critical values provided $\Phi$ satisfies the $(PS)_c$ condition for every $c \in (0, \infty)$.

We need the following lemma which can be found in [10,11].

**Lemma 1.1.** Suppose $\Phi \in C^1(E, \mathbb{R}^1)$ be the form

$$\Phi(u) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(u) \quad \text{for} \quad u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R}^1)$ is bounded from below,

(ii) $\Psi : (E, w) \to \mathbb{R}^1$ is sequentially lower semicontinuous, that is, $u_k \to u$ in $(E, w)$ implies

$$\Psi(u) \leq \liminf_k \Psi(u_k),$$

(iii) $\Psi' : (E, w) \to (E^*, w^*)$ is sequentially continuous,

(iv) $v : E \to \mathbb{R}^1, \|v\|^2 = C_1$ and $v' : (E, w) \to (E^*, w^*)$ is sequentially continuous.

Then $\Phi$ satisfies $\Phi_0$.

2. The variational set

First of all, one can easily find that (1.1) can be transformed to the equations

$$\begin{cases}
  u'(t) = -\lambda \Delta u \left(t + \frac{\pi}{2}\right) - \lambda f \left(\lambda t, u\left(t - \frac{\pi}{2}\right)\right), \\
  u(0) = -u(\pi), \quad u(0) = u(2\pi)
\end{cases}$$

(2.1)

by making the change of variable $t \mapsto \frac{\pi}{2\lambda}t = \lambda^{-1}t$. This implies that a $4\pi$-periodic solution of (1.1) corresponds to a $2\pi$-periodic solution of (2.1). Hence we will only seek for the $2\pi$-periodic orbits of (2.1) in the sequel.

Throughout this paper, we always assume that

$(f_1)$ $f(t, z)$ is odd in $z$, i.e. $f(t, -z) = -f(t, z)$ for all $t \in [0, \pi]$,

$(f_2)$ there exists a continuously differentiable function $F(t, z) \in C^1(\mathbb{R}^1 \times \mathbb{R}^n, \mathbb{R}^1)$ such that $\nabla_z F(t, z) = f(t, z)$ for all $(t, z) \in \mathbb{R}^1 \times \mathbb{R}^n$.

$(f_3)$ one of the following conditions holds:
functions with dimension
in the space
Lu
grable 2
for all
v
for any
u
For any
We define an operator
By Proposition 6.6 in [12] we know that
Similar to the treatment in [8], we introduce the following variational set. Let
For any
u
such that
can also be viewed as a function belonging to
ζ
(t, z) =
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v
where
au
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0
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such that
lim
|ζ(t, z)| = 0 uniformly for all t ∈ [0, r].
There exist constants
> 0 and
p > 2 such that

for all
(t, z) ∈ [0, r] × \mathbb{R}^n.

L²(S¹, \mathbb{R}^n)
be the space of square integrable 2π-periodic vector-valued functions with dimension \(n\). Let \(C^{∞}(S¹, \mathbb{R}^n)\) be the space of 2π-periodic \(C^{∞}\) vector-valued functions with dimension \(n\). For any \(u \in C^{∞}(S¹, \mathbb{R}^n)\), it has the following Fourier expansion in the sense that it is convergent in the space \(L²(S¹, \mathbb{R}^n)\)

\[u(t) = \frac{a_0^u}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{+∞} (a_j^u \cos jt + b_j^u \sin jt),\]

where \(a_j^u, b_j^u \in \mathbb{R}^n\). Let \(H = H^{1/2}(S¹, \mathbb{R}^n)\) be the closure of \(C^{∞}(S¹, \mathbb{R}^n)\) with respect to the Hilbert norm

\[\|u\|_{H^{1/2}} \equiv \left[\frac{1}{2\pi} \int_0^{2\pi} \left(\|a_0^u\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right]^{1/2}.\]

More specifically, \(H^{1/2}(S¹, \mathbb{R}^n) = \{u \in L²(S¹, \mathbb{R}^n): \|u\|_{H^{1/2}} < +∞\}\) with the inner product

\[\langle u, v \rangle_H = \langle u, v \rangle_{L²} + \sum_{j=1}^{+∞} (1 + j)[(a_j^u, a_j^v) + (b_j^u, b_j^v)]\]

for any \(u, v \in H^{1/2}(S¹, \mathbb{R}^n)\), where \((\cdot, \cdot)\) denotes the usual inner product in \(\mathbb{R}^n\). The norm on \(H\) is defined by

\[\|u\|_H \equiv \left[\frac{1}{2\pi} \int_0^{2\pi} \left(\|\dot{u}\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right]^{1/2}.\]

By Proposition 6.6 in [12] we know that \(H\) is compactly embedded in \(L²(S¹, \mathbb{R}^n)\), where \(s \in [1, \infty)\). Now consider a functional \(I\) defined on \(H\), given by

\[I(u) = \frac{2π}{2} \int_0^{2π} \left[\frac{1}{2} \left(\|\dot{u}\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right] \left(\frac{1}{2}, u(t) + \frac{1}{2} \lambda \Lambda(u(t + \pi), u(t)) + \lambda F(\lambda t, u(t))\right) dt \tag{2.2}\]

for any \(u \in \mathcal{H}\), where \(\dot{u}(t)\) denotes the weak derivative of \(u\).

We define an operator \(L: H \to H^*\) as follows: for any \(u \in \mathcal{H}\), which is given by

\[Lu(v) = \frac{2π}{2} \int_0^{2π} \left[\left(\|\dot{u}\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right] \left(\frac{1}{2}, u(t) + \frac{1}{2} \lambda \Lambda(u(t + \pi), v(t))\right) dt\]

for all \(v \in \mathcal{H}\), where \(H^*\) denotes the dual space of \(H\). By the Riesz representation theorem, we can identify \(H^*\) with \(H\). Thus, \(L\) can also be viewed as a function belonging to \(H\) such that \(\langle Lu, v \rangle_H = Lu(v)\) for any \(u, v \in \mathcal{H}\).

For any \(u \in \mathcal{H}\), define a bounded linear operator \(\xi: H \to H\) as follows: \(\xi u(\cdot) = u(\cdot + \frac{π}{2})\). Set \(E = \{u \in \mathcal{H}: \xi^2 u = -u\}\), then \(E\) is a closed subspace of \(H\).

For any \(u, v \in \mathcal{H}\), we have

\[\langle \xi^2 Lu, v \rangle_H = \langle \xi^2 Lu, \xi^2 v \rangle_H = \langle Lu, \xi^2 v \rangle_H\]

\[= \int_0^{2π} \left[\left(\|\dot{u}\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right] \left(\frac{1}{2}, u(t + \pi) + \lambda \Lambda(u(t + \pi), v(t - \pi))\right) dt\]

\[= \int_0^{2π} \left[\left(\|\dot{u}\|^2 + \sum_{j=1}^{+∞} (1 + j)(|a_j^u|^2 + |b_j^u|^2)\right) dt\right] \left(\frac{1}{2}, u(t + \pi + \frac{3π}{2}) + \lambda \Lambda(u(t), u(t))\right) dt\]
\[
\begin{align*}
\int_0^{2\pi} \left[ -\dot{u} \left( t + \frac{\pi}{2} \right), v(t) \right] dt \\
= \langle -Lu, v \rangle_H
\end{align*}
\]

which implies that \( E \) is an invariant subspace of \( H \) with respect to \( L \). It is easy to check that \( L \) is a bounded linear operator on \( H \). Moreover, \( L \) is self-adjoint.

Let \( e_1, e_2, \ldots, e_n \) denote the usual normal orthogonal bases in \( \mathbb{R}^n \). Define the subspaces \( E^+ \) and \( E^- \) of \( E \) as follows

\[
E^+ = \text{span} \{ e_k \cos(2j - 1)t, e_k \sin(2j - 1)t: j \in \mathbb{Z}^+, j \text{ is even}, k = 1, 2, \ldots, n \},
\]

\[
E^- = \text{span} \{ e_k \cos(2j - 1)t, e_k \sin(2j - 1)t: j \in \mathbb{Z}^+, j \text{ is odd}, k = 1, 2, \ldots, n \},
\]

where \( \mathbb{Z}^+ \) is the set of all positive integers. By using the definition of \( E \) and a Fourier series argument, we see that \( E = E^+ \oplus E^- \). Moreover, for any \( u \in E^+ \), it has a Fourier expansion as follows

\[
u(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[ \alpha_{4j-1}^u \cos(4j - 1)t + \beta_{4j-1}^u \sin(4j - 1)t \right].
\]

Thus,

\[
\langle Lu, u \rangle_H = \int_0^{2\pi} \left[ -\dot{u} \left( t + \frac{\pi}{2} \right), u(t) \right] dt + \lambda \langle u(t + \pi), u(t) \rangle dt
\]

\[
= \int_0^{2\pi} \left[ -\dot{u} \left( t + \frac{\pi}{2} \right), u(t) \right] dt - \lambda \| u \|^2_{L^2}
\]

\[
= \sum_{j=1}^{+\infty} (4j - 1 - \lambda \Lambda) \left( |\alpha_{4j-1}^u|^2 + |\beta_{4j-1}^u|^2 \right)
\]

\[
\geq \left( 1 - \frac{\lambda A + 1}{4} \right) \sum_{j=1}^{+\infty} 4j |\alpha_{4j-1}^u|^2 + |\beta_{4j-1}^u|^2 \right) = \left( 1 - \frac{\lambda A + 1}{4} \right) \| u \|^2_H.
\]

Similarly, \( \langle Lu, u \rangle_H \leq -\frac{\lambda A + 1}{4} \| u \|^2_H \) for any \( u \in E^- \). Since \(-1 < \lambda \Lambda < 3\), we can define an equivalent norm \( \| \cdot \| \) on \( E \) given by

\[
\| u \|^2 = \| Lu^+, u^+ \|^2_H - \langle Lu^-, u^- \rangle_H
\]

for \( u = u^+ + u^- \in E^+ \oplus E^- \). Denote by \( \langle \cdot, \cdot \rangle \) the inner product corresponding to \( \| \cdot \| \) on \( E \). Clearly, the spaces \( E^+ \) and \( E^- \) are mutually orthogonal with respect to the inner products \( \langle \cdot, \cdot \rangle_H \) and \( \langle \cdot, \cdot \rangle_{L^2} \) by the orthogonality of trigonometric functions, where \( \langle \cdot, \cdot \rangle_{L^2} \) denotes the usual inner product on \( L^2(S^1, \mathbb{R}^n) \).

Let

\[
G(u) = \int_0^{2\pi} \lambda F(\lambda t, u(t)) dt
\]

for any \( u \in H \). Then \( I(u) \) can be rewritten as

\[
I(u) = \frac{1}{2} \left( \| u^+ \|^2 - \| u^- \|^2 \right) + G(u)
\]

for \( u = u^+ + u^- \in E \).

**Lemma 2.1.** \( G \) is weakly sequentially continuous on \( H \) under the assumption \((f_3)\).

**Proof.** Since \( f(t, z) \) is \( r \)-periodic in \( t \), by \((f_3)\), there are constants \( c_1, c_2 > 0 \) such that

\[
|f(t, z)| \leq c_1 + c_2 |z|^{p-1}
\]

for all \((t, z) \in \mathbb{R}^1 \times \mathbb{R}^n\). Let \( \{u_k\} \) be any sequence converging to some \( u \) weakly in \( H \). By the compactness of embedding, one has \( u_k \to u \) in \( L^p(S^1, \mathbb{R}^n) \). By Hölder inequality we have
\[ |G(u_k) - G(u)| = \left| \lambda \int_0^{2\pi} F(\lambda t, u_k) - F(\lambda t, u) \, dt \right| \]

\[ \leq \lambda \int_0^{2\pi} \left( c_1 + c_2 \left| u + s(u_k - u) \right|^p \right) |u_k - u| \, ds \, dt \]

where \( c_3 > 0 \) is a constant. This implies \( G(u_k) \to G(u) \). The proof is completed. \( \square \)

By Proposition B.37 in [12], we have the following lemma.

**Lemma 2.2.** Assume that \( f \) satisfies (f2) and (f3). Then the functional \( I \) is continuously differentiable on \( H \) and \( I'(u) \) is defined by

\[ I'(u) v = \int_0^{2\pi} \left[ \frac{1}{2} \left( \dot{u} \left( t + \frac{\pi}{2} \right) - \dot{u} \left( t - \frac{\pi}{2} \right), v(t) \right) + \lambda A(u(t + \pi), v(t)) + \lambda \left( f(\lambda t, u(t)), v(t) \right) \right] \, dt \]

for all \( v \in H \). In particular,

\[ I'(u) v = \int_0^{2\pi} \left[ \left( \dot{u} \left( t + \frac{\pi}{2} \right), v(t) \right) - \lambda A(u(t), v(t)) + \lambda \left( f(\lambda t, u(t)), v(t) \right) \right] \, dt \]

for all \( u, v \in E \).

Moreover, \( G' : H \to H^* \) is a compact mapping and

\[ G'(u) v = \int_0^{2\pi} \lambda \left( f(\lambda t, u(t)), v(t) \right) \, dt, \]

for any \( v \in H \).

By the Riesz theorem, we can view \( G'(u) \) as an element of \( H \) for any \( u \in H \). In addition, one can easily prove that \( E \) is invariant with respect to \( G' \) under condition (f1) (see Zhiming Guo and Jianshe Yu [8]). As usual, we identify \( u \in H \) with its continuous representant.

Since \( E \) is invariant with respect to \( L \) and \( G' \), an argument as in [8] yields

**Lemma 2.3.** Assume that \( f \) satisfies (f1), (f2) and (f3). Then a critical point of functional \( I \) restricted to \( E \) is a \( 2\pi \)-periodic solution of system (2.1).

**Remark 2.1.** It is pointed in [8] that a critical point \( u \) of \( I \) in \( H \) is a weak solution of (2.1). However, a simple regularity argument shows that \( u \in C^1(S^1, \mathbb{R}^n) \) (see the proof of Theorem 6.10 in [12]).

**Remark 2.2.** As usual, we should deal with (2.2) in the space \( H \). But, according to Lemma 2.3, we only need to treat the functional \( I \) in the subspace \( E \) of \( H \). From now on we will view \( I \) as \( I|_E \).
3. Main results

In this section we denote by $Z^+$ the set of all positive integers; $c_i$ stand for different positive constants for $i \in Z^+$. The following hypotheses will be used in our main results.

(i)\( \lim_{|z| \to 0} \frac{|F(t, z)|}{|z|^2} = 0 \) uniformly for $t \in [0, r]$.

(ii)\( (Bz, z) > (A + \frac{1}{\lambda})|z|^2 \) for all $z \in \mathbb{R}^n \setminus \{0\}$, where $\lambda = \frac{2r}{\pi}$ appears in (2.1).

(iii)\( \text{for any positive integer } j, (-1)^{j+1} \lambda^{-1}(2j - 1) + A \notin \sigma(B), \) where $\sigma(B)$ is the set of all eigenvalues of $B$; $B$ is the $n \times n$ symmetric matrix appearing in (f3)(1).

Define $m = \max\{j \in Z^+: (A + 4j - 3)\lambda|z|^2 < (Bz, z) \text{ for } z \neq 0\}$. (f3) implies that $m$ is well defined. We have the following main result.

**Theorem 3.1**. Assume that $f$ satisfies (f1), (f3), (f3)(l) and (f3)–(f6). Then (1.1) possesses at least $2mn$ pairs of $4r$-periodic classical solutions.

**Proof.** We will show that $\Phi(u) = -I(u)$ satisfies all hypotheses of Theorem 1.1. The proof of this theorem will be divided into several parts.

**Step 1.** We prove that $\Phi$ satisfies (\( \Phi_0 \)). Let $X = E^+$, $Y = E^-$ and $\Psi(u) = G(u)$. Then

\[
\Phi(u) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(u) \quad \text{for } u = x + y \in X \oplus Y,
\]

and $\Psi(u) \in C^1(E, \mathbb{R})$ satisfies (ii) of Lemma 1.1 by Lemma 2.1.

Let \{\( u_k \)\} be any sequence converging to $u$ weakly in $E$. For $1 \leq r < \infty$, since the injection of $E$ into $L^r(S^1, \mathbb{R}^n)$ is continuous, the sequence \{\( u_k \)\} converges to $u$ weakly in $L^r(S^1, \mathbb{R}^n)$. Hence, in $L^r(S^1, \mathbb{R}^n)$, any convergent subsequence of \{\( u_k \)\} converges to $u$, and hence

\[
\int_0^{2\pi} |u|^r dt = \liminf_{k \to \infty} \int_0^{2\pi} |u_k|^r dt \leq \limsup_{k \to \infty} \int_0^{2\pi} |u_k|^r dt = \int_0^{2\pi} |u|^r dt.
\]

It follows that $u_k \to u$ in $L^r(S^1, \mathbb{R}^n)$ and $u_k \to u$ a.e. on $[0, 2\pi]$. Thus, for every $v \in E$ we get that $(f(\lambda t, u_k(t)), v(t)) \to (f(\lambda t, u(t)), v(t))$ a.e. for $t \in [0, 2\pi]$. Moreover, by (2.4), one has

\[
\left| \int_0^{2\pi} (f(\lambda t, u_k(t)), v(t)) dt \right| \leq \int_0^{2\pi} \left[ c_1 |v| + c_2 |u_k|^{p-1} |v| \right] dt
\]

\[
\leq c_1 \|v\|_{L^1} + c_2 \|u_k\|_{L^p}^{p-1} \|v\|_{L^p}
\]

\[
\to c_1 \|v\|_{L^1} + c_2 \|u\|_{L^p}^{p-1} \|v\|_{L^p}.
\]

Thus, the Vitali theorem is applicable.

\[
\int_0^{2\pi} (f(\lambda t, u_k(t)), v(t)) dt \to \int_0^{2\pi} (f(\lambda t, u(t)), v(t)) dt,
\]

that is, $\Psi'(u_k)v \to \Psi'(u)v$ for any $v \in E$. Hence $\Psi$ satisfies (iii) of Lemma 1.1. Moreover, note that $E$ is a Hilbert space. (iv) of Lemma 1.1 holds, obviously.

It remains to prove that $\Psi$ is bounded from below. Notice that $f \in C(\mathbb{R}^1 \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, z)$ is $r$-periodic in $t$. Hence (f3)(l) implies that there exists a constant $c > 0$ such that

\[
|f(t, z) - Bz| \leq \frac{1}{2} \left( A + \frac{1}{\lambda} \right) (|z| + c)
\]

for all $(t, z) \in \mathbb{R}^1 \times \mathbb{R}^n$, where $\lambda = \frac{2r}{\pi}$. Clearly, it can be deduced from (f4) that $F(t, 0) = 0$. Consequently, by (f5) and the above inequality, one has
\[
\Psi(u) = \int_0^{2\pi} \lambda F(\lambda t, u) dt = \int_0^{2\pi} \frac{1}{2} \int \lambda(f(\lambda t, su), u) ds dt
\]

\[
= \frac{1}{2} \int \lambda(Bu, u) dt + \frac{1}{2} \int \lambda(f(\lambda t, su) - sBu, u) ds dt
\]

\[
\geq \frac{\lambda A + 1}{2} \int |u|^2 dt - \frac{\lambda A + 1}{2} \int \int (s|u| + c)|u| ds dt
\]

\[
\geq \frac{\lambda A + 1}{4} \|u\|_2^2 - c_3 \|u\|_2^2.
\]

which implies that \(\Psi\) is bounded from below. By virtue of Lemma 1.1, \(\Phi\) satisfies (\(\Phi_0\)).

**Step 2.** \(\Phi\) satisfies (\(\Phi_1\)). Indeed, for any \(\varepsilon > 0\), by (\(f_4\)), there is an \(l > 0\) such that

\[
|F(t, z)| \leq \varepsilon|z|^2
\]

for all \(t \in \mathbb{R}^l\) and \(|z| \leq l\). By (2.4),

\[
|F(t, z)| = \int_0^1 (f(t, sz), z) ds \leq \int f(t, sz)|z| ds \leq c_1|z| + \frac{c_2}{2}|z|^2 \leq \frac{c_1}{p-1}|z|^p + \frac{c_2}{2}|z|^p
\]

for all \(t \in \mathbb{R}^l\) and \(|z| \geq l\). Thus, for any \(\varepsilon > 0\), there is a \(c = c(\varepsilon) > 0\) such that

\[
|F(t, z)| \leq \varepsilon|z|^2 + c|z|^p
\]

for all \((t, z) \in \mathbb{R}^l \times \mathbb{R}^n\). Hence, for \(u \in Y\) and small \(\varepsilon\), we have

\[
\Phi(u) = \frac{1}{2} \|u\|_2^2 - \int_0^{2\pi} \lambda F(\lambda t, u) dt \geq \frac{1}{2} \|u\|_2^2 - \lambda \varepsilon \|u\|_2^2 - \lambda c \|u\|_p^p \geq \frac{1}{4} \|u\|_2^2 - c_4 \|u\|_p^p.
\]

Since \(p > 2\), there is a small \(\rho > 0\) such that \(\frac{1}{8} \rho^2 \geq c_4 \rho^p\). Therefore,

\[
\kappa := \inf \Phi(\partial B_\rho \cap Y) \geq \frac{1}{8} \rho^2 > 0
\]

and hence (\(\Phi_1\)) holds.

**Step 3.** (\(\Phi_2\)) is satisfied under the hypotheses of Theorem 3.1.

Let

\[Y_0 = \text{span}\{e_k \cos(4j - 3)t, e_k \sin(4j - 3)t: \ j \in \mathbb{Z}^+, \ j \leq m, \ k = 1, 2, \ldots, n\}\]

Obviously, \(Y_0 \subset Y\) and \(\dim Y_0 = 2mn\). In order to obtain the desired conclusion, it is sufficient to prove that \(\Phi(u) \to -\infty\) as \(u \in E_0 := X \cap Y_0\) and \(\|u\| \to \infty\).

By the definition of \(m\), there exists a \(\delta > 0\) such that

\[
(4m + \lambda A - 3 + \delta)|z|^2 \leq \lambda(Bz, z)
\]

(3.2)

for all \(z \in \mathbb{R}^n\) \(\setminus \{0\}\). Notice that

\[
(4m + \lambda A - 3)\|y\|_2^2 \geq \|y\|^2
\]

(3.3)

for any \(y \in Y_0\). Let \(\bar{F}(t, z) = F(\lambda t, z) - \frac{1}{2}(Bz, z)\). Then for \(u = x + y \in E_0\), by (3.2) and (3.3), one has
\[\Phi(u) = \frac{1}{2} (\|y\|^2 - \|x\|^2) - \int_0^{2\pi} \lambda F(\lambda t, u) \, dt\]

\[\leq \frac{1}{2} (4m + \lambda \Lambda - 3) \|y\|^2 - \frac{1}{2} \|x\|^2 - \frac{1}{2} \int_0^{2\pi} \lambda(Bu, u) \, dt - \int_0^{2\pi} \lambda \tilde{F}(t, u) \, dt\]

\[\leq \frac{1}{2} (4m + \lambda \Lambda - 3) \|y\|^2 - \frac{1}{2} \|x\|^2 - \frac{1}{2} (4m + \lambda \Lambda - 3 + \delta) \|u\|^2 - \int_0^{2\pi} \lambda \tilde{F}(t, u) \, dt\]

\[\leq \frac{1}{2} (4m + \lambda \Lambda - 3) \|y\|^2 - \frac{1}{2} \|x\|^2 - \frac{1}{2} (4m + \lambda \Lambda - 3 + \delta) \|y\|^2 - \int_0^{2\pi} \lambda \tilde{F}(t, u) \, dt\]

\[\leq - \frac{\delta}{8m + 2\lambda \Lambda - 6} \|y\|^2 - \frac{1}{2} \|x\|^2 - \int_0^{2\pi} \lambda \tilde{F}(t, u) \, dt\]

\[\leq - \left( \min \left\{ \frac{\delta}{8m + 2\lambda \Lambda - 6}, \frac{1}{2} \right\} \right) \|u\|^2 - \int_0^{2\pi} \lambda \tilde{F}(t, u) \, dt.\]

It remains to show that

\[\frac{1}{\|u\|^2} \int_0^{2\pi} \tilde{F}(t, u) \, dt \to 0\]  \hfill (3.4)

as \(\|u\| \to \infty\). For any \(\varepsilon > 0\), by the continuity of \(f\), \((f_3)\)(i) and the periodicity of \(f(\cdot, z)\) we know that there exists a positive constant \(c = c(\varepsilon)\) such that

\[|f(t, z) - Bz| \leq \varepsilon |z| + c\]  \hfill (3.5)

for \((t, z) \in \mathbb{R}^1 \times \mathbb{R}^n\). Thus, for \(u \in E_0\) with \(\|u\| \neq 0\) we have

\[\left| \frac{1}{\|u\|^2} \int_0^{2\pi} \tilde{F}(t, u) \, dt \right| = \frac{1}{\|u\|^2} \int_0^{2\pi} \left| \int_0^{1} (f(\lambda t, su) - B(su), u) \, ds \right| dt\]

\[\leq \frac{1}{\|u\|^2} \int_0^{2\pi} \left| \int_0^{1} (\varepsilon |su| + c) |u| \, ds \right| dt\]

\[\leq \frac{1}{\|u\|^2} \left( \frac{1}{2} \varepsilon \|u\|_2 + c \|u\|_1 \right)\]

\[\leq c_5 \left( \varepsilon + \frac{c}{\|u\|} \right),\]

which implies that (3.4) is true by the arbitrariness of \(\varepsilon\). Hence (\(\Phi_2\)) holds.

**Step 4.** \(\Phi\) satisfies the \((C)\) condition for any \(c \in \mathbb{R}^1\).

Let \(\{u_k\} \subset E\) be any sequence such that

\[\Phi(u_k) \to c, \quad (1 + \|u_k\|) \Phi'(u_k) \to 0\]  \hfill (3.6)

as \(k \to \infty\). We claim that \(\{u_k\}\) is bounded in \(E\). Assume by contradiction that \(\|u_k\| \to \infty\) as \(k \to \infty\). Let \(\varphi_k = \frac{u_k}{\|u_k\|}\), then \(\|\varphi_k\| = 1\). Without loss of generality, we can assume that \(\varphi_k \to \varphi\) in \(E\) and \(\varphi_k \to \varphi\) in \(L^2(S^1, \mathbb{R}^n)\). Hence for each \(v \in E\), by (3.5) and Hölder inequality, one has
An easy computation shows that
\[
\frac{1}{\|u_k\|} \int_0^{2\pi} (\lambda t, u_k) \, dt = \int_0^{2\pi} (B\varphi, v) \, dt, \quad \lambda \in \mathbb{R}
\]
and
\[
I = \frac{1}{\|u_k\|} \int_0^{2\pi} \left( (\lambda t, u_k) - B u_k, v \right) \, dt + \int_0^{2\pi} \left( B \varphi_k - B \varphi, v \right) \, dt.
\]

Then, by (3.8) one can obtain
\[
\leq \frac{1}{\|u_k\|} \int_0^{2\pi} (\lambda t, u_k) \, dt + \left( \sum_{1 \leq i, j \leq n} |b_{ij}| \right) \int_0^{2\pi} \varphi_k - \varphi \parallel v \parallel dt
\]
\[
\leq \frac{1}{\|u_k\|} (\|u_k\| \|v\|_{L^2} + c \|v\|_{L^1}) + c_0 \|\varphi_k - \varphi\|_{L^2} \|v\|_{L^2}
\]
\[
\leq c_7 \left( \parallel \epsilon \|u_k\| + \|\varphi_k - \varphi\|_{L^2} \parallel v \parallel. \right)
\]

By \( \|u_k\| \to \infty, \|\varphi_k - \varphi\|_{L^2} \to 0 \) and the arbitrariness of \( \epsilon \), we get that
\[
\frac{1}{\|u_k\|} \int_0^{2\pi} (\lambda t, u_k) \, dt \to \int_0^{2\pi} (B\varphi, v) \, dt
\]
as \( k \to \infty \). This yields
\[
\frac{\Phi'(u_k)v}{\|u_k\|} = \langle \varphi_k^+, v^- \rangle - \langle \varphi_k^-, v^+ \rangle - \frac{1}{\|u_k\|} \int_0^{2\pi} \lambda (f(\lambda t, u_k), v) \, dt
\]
\[
= \langle \varphi^-, v^- \rangle - \langle \varphi^+, v^+ \rangle - \int_0^{2\pi} \lambda (B\varphi, v) \, dt + o(1)
\]
\[
= -\langle L\varphi, v \rangle_H - \int_0^{2\pi} \lambda (B\varphi, v) \, dt + o(1).
\]

It can be deduced from the above equality that
\[
\langle L\varphi, v \rangle_H + \int_0^{2\pi} \lambda (B\varphi, v) \, dt = 0, \quad \forall v \in E.
\]

Using the definition of \( E \) we can set
\[
\varphi(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[ a_{2j-1}^\varphi \cos(2j-1)t + b_{2j-1}^\varphi \sin(2j-1)t \right]
\]
and
\[
v(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[ a_{2j-1}^v \cos(2j-1)t + b_{2j-1}^v \sin(2j-1)t \right].
\]

Then, by (3.8) one can obtain
\[
\sum_{j=1}^{+\infty} \left[ ((\lambda B + ((-1)^j(2j-1) - \lambda A)) I) a_{2j-1}^\varphi, a_{2j-1}^\varphi \right) + \left( (\lambda B + ((-1)^j(2j-1) - \lambda A)) I) b_{2j-1}^\varphi, b_{2j-1}^\varphi \right] = 0,
\]
where \( I \) is the \( n \times n \) unit matrix. For any \( j \), take \( v(t) = \frac{1}{\sqrt{\pi}} \epsilon_t \cos(2j-1)t \) and \( v(t) = \frac{1}{\sqrt{\pi}} \epsilon_t \sin(2j-1)t \), where \( i = 1, 2, \ldots, n \).
An easy computation shows that
\[
(\lambda B + ((-1)^j(2j-1) - \lambda A))I) a_{2j-1}^\varphi = 0
\]
and
\[
(\lambda B + ((-1)^j(2j-1) - \lambda A))I) b_{2j-1}^\varphi = 0.
\]
Hence, by \((-1)^{j+1} \lambda^{-1}(2j-1) + A \neq \sigma(B)\) we know \(\varphi \equiv 0\) and
\[
\frac{1}{\|u_k\|} \int_0^{2\pi} (f(\lambda t, u_k), \varphi_k^-) dt = \frac{1}{\|u_k\|} \int_0^{2\pi} (f(\lambda t, u_k), \varphi_k^-) dt - \frac{2\pi}{\|B\varphi, \varphi_k^-\|} dt.
\]
This shows by replacing \(\nu\) with \(\varphi_k^-\) in the proof of (3.7) that
\[
\frac{1}{\|u_k\|} \int_0^{2\pi} (f(\lambda t, u_k), \varphi_k^-) dt \to 0.
\]
It follows from (3.9) that
\[
o(1) = \frac{\Phi'(u_k)\varphi_k^-}{\|u_k\|} = \langle \varphi_k^-, \varphi_k^- \rangle - \frac{1}{\|u_k\|} \int_0^{2\pi} \lambda(f(\lambda t, u_k), \varphi_k^-) dt = \|\varphi_k^-\|^2 + o(1),
\]
which implies \(\|\varphi_k^-\| \to 0\). Similarly, \(\|\varphi_k^+\| \to 0\). It is impossible since \(\|\varphi_k\| = 1\) for any \(k\). Consequently, \(\{u_k\}\) is bounded in \(E\). Moreover, by the compactness of \(\Psi\), going if necessary to a subsequence, we can assume that \(u_k \to u\) and \(\Psi'(u_k) \to \Psi'(u)\) in \(E\). Then
\[
\|u_k^+ - u^+\|^2 = \|u_k^- - u^-\|_2^2 = \langle \Phi'(u_k), u_k^- - u^- \rangle = (\Phi'(u_k) - \Phi'(u))(u_k^- - u^-) + \langle \Psi'(u_k) - \Psi'(u), u_k^- - u^- \rangle \to 0.
\]
Similarly, \(\|u_k^+ - u^+\|^2 \to 0\). Hence \(u_k \to u\) in \(E\) and the \((C)_c\) condition is satisfied.

Finally, \(\Phi\) is even since \(f(t, z)\) is odd in \(z\) and \(F(t, 0) = 0\). Hence Theorem 3.1 follows from Theorem 1.1. The proof is completed. \(\square\)

**Theorem 3.2.** Assume that \(f\) satisfies (f_1), (f_2), (f_3), (f_4), (f_5) and the following condition
\[(f_7) \quad (Bz, z) < (A - \frac{2}{3})|z|^2 \quad \text{for all} \ z \in \mathbb{R}^n \setminus \{0\}.
\]
Then (1.1) possesses at least \(2mn\) pairs of \(4r\)-periodic classical solutions, where \(m = \max\{j \in \mathbb{Z}^+: (A - \frac{4j-1}{2})|z|^2 > (Bz, z) \text{ for } z \neq 0\}.

**Proof.** Let \(X = E^-, Y = E^+, \Phi(u) = I(u), \Psi(u) = -G(u)\) and
\[Y_0 = \text{span}\{e_k \cos(4j-1)t, e_k \sin(4j-1)t: j \in \mathbb{Z}^+, j \leq m, k = 1, 2, \ldots, n\}.
\]
Then the conclusion will be obtained by the same argument as Theorem 3.1. The proof is completed. \(\square\)

**Theorem 3.3.** Assume that \(f\) satisfies (f_1), (f_2), (f_3), (f_4), (f_5) and the following condition
\[(f_8) \quad \hat{F}(t, z) \geq 0 \quad \text{and} \quad \hat{F}(t, z) \to +\infty \quad \text{as} \ |z| \to \infty \quad \text{uniformly for} \ t \in [0, r], \text{where} \ \hat{F}(t, z) = \frac{1}{2} (f(t, z) - F(t, z).
\]

Let \(m\) be given by Theorem 3.1. Then (1.1) possesses at least \(2mn\) pairs of \(4r\)-periodic classical solutions.

**Proof.** Let \(\Phi\) and \(\Psi\) be that in Theorem 3.1. From the proof of Theorem 3.1 we see that the condition (f_5) was only used to prove the \((C)_c\) condition. Hence, it is sufficient to prove that \(\Phi\) satisfies the \((C)_c\) condition.

Let \(\{u_k\} \subset E\) be any sequence such that \(\Phi(u_k) \to c, (1 + \|u_k\|)\Phi'(u_k) \to 0\) as \(k \to \infty\). We claim that \(\{u_k\}\) is bounded in \(E\). Assume by contradiction that \(\|u_k\| \to \infty\) as \(k \to \infty\). Let \(\varphi_k = \frac{u_k}{\|u_k\|}\), then \(\|\varphi_k\| = 1\). Without loss of generality, we can assume that \(\varphi_k \to \varphi\) in \(E, \varphi_k \to \varphi \in L^2(S^1, \mathbb{R}^n)\) and \(\varphi_k(t) \to \varphi(t)\) for almost all \(t \in [0, 2\pi]\). If \(\varphi \equiv 0\), the argument of Theorem 3.1 shows \(\|\varphi_k\| \to 0\). This contradicts \(\|\varphi_k\| = 1\). Hence the case \(\varphi \equiv 0\) will not occur, and hence \(\varphi \neq 0\). Set \(\Omega = \{t \in [0, 2\pi]: \varphi(t) \neq 0\}\). Then \(\Omega\) has a positive measure and \(u_k(t) \to \infty\) for all \(t \in \Omega\). It follows from (f_8) that
\[
c = \lim_{k \to \infty} \left[ \Phi(u_k) - \frac{1}{2} \Phi'(u_k) u_k \right] = \lim_{k \to \infty} \int_0^{2\pi} \lambda \hat{F}(t, u_k) dt \geq \int_{\Omega} \liminf_{k \to \infty} \lambda \hat{F}(t, u_k) dt \to +\infty.
\]
This is a contradiction. Therefore, \(\{u_k\}\) is bounded. Moreover, by arguing as in Theorem 3.1 we know that \(\{u_k\}\) has a convergent subsequence. The proof is completed. \(\square\)
At the end of this paper, we discuss the infinitely many solutions for system (1.1).

**Theorem 3.4.** Assume that $f$ satisfies $(f_1)$, $(f_2)$, $(f_3)(II)$, $(f_4)$ and the following condition

(f$_9$) there exists an $\tilde{c} > 0$ such that

$$f(t, z, z) \geq pF(t, z) > 0$$

for $t \in [0, r]$ and $|z| \geq \tilde{c}$, where $p$ appears in $(f_3)(II)$.

Then (1.1) possesses an unbounded sequence of $4r$-periodic classical solutions.

**Proof.** Let $\Phi$, $\Psi$ and $X$, $Y$ be given by Theorem 3.1. The proof of this theorem will be completed with the aid of Theorem 1.2.

First, since the assumption $(f_9)$ is the Ambrosetti–Rabinowitz condition, it is well known that there exist constants $c_1, c_2 > 0$ such that

$$F(t, z) \geq c_1|z|^p - c_2$$

for $(t, z) \in \mathbb{R}^1 \times \mathbb{R}^n$. This implies that $\Psi$ is bounded from below. Moreover, by the argument of Theorem 3.1 we see that $\Phi$ satisfies $(\Phi_0)$ and $(\Phi_1)$.

Next, we check that $\Phi$ satisfies $(\Phi_3)$. To do this, let $\tilde{Y}$ be any finite dimensional subspace of $Y$. It is sufficient to prove that $\Phi(u) \to -\infty$ as $u \in \tilde{E} := X \oplus \tilde{Y}$ and $\|u\| \to \infty$.

Since $\tilde{Y}$ is finite dimensional, there is a $\tilde{\delta} = \delta(\tilde{Y}) > 0$ such that

$$\|y\|_{12} \geq \delta\|y\|$$

for any $y \in \tilde{Y}$. Hence for $u = u^+ + u^- \in X \oplus \tilde{Y}$, by (3.10), (3.11) and $\frac{2}{p} < 1$, one has

$$\Phi(u) = \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - \int_0^{2\pi} \lambda F(\lambda t, u) dt \leq \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - \int_0^{2\pi} \lambda (c_1|u|^p - c_2) dt \leq \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - c_3\left(\int_0^{2\pi} \|u^+ + u^-\|^2 dt\right)^{\frac{p}{2}} + 2\pi \lambda c_2 \leq \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - c_3\|u^-\|^p_{L_2} + 2\pi \lambda c_2 \leq \frac{1}{2}\left(\|u^-\|^2 - \frac{1}{2}\|u^+\|^2 - c_3\tilde{\delta}\|u^-\|^p + 2\pi \lambda c_2 \leq -\frac{1}{2}\|u^+\|^2 - c_4 = -\frac{1}{2}\|u\|^2 + c_4.

This yields $\Phi(u) \to -\infty$ as $u \in \tilde{E}$ and $\|u\| \to \infty$. Hence $(\Phi_3)$ holds.

Finally, we prove that the $(PS)_c$ condition holds for any $c \in (0, \infty)$. Let $[u_k]$ be any sequence such that $\Phi(u_k) \to c > 0$ and $\Phi'(u_k) \to 0$ as $k \to \infty$. We can assume $\|\Phi'(u_k)\| \leq 1$. By $(f_9)$ and (3.10) we have

$$2c + \|u_k\| \geq 2\Phi(u_k) - \Phi'(u_k)u_k \geq \int_0^{2\pi} \lambda \left[\left(\int f(\lambda t, u_k, u_k) - 2F(\lambda t, u_k) dt\right] dt \geq \int_0^{2\pi} \lambda (p - 2)F(\lambda t, u_k) dt + \int_0^{2\pi} \lambda \left[\left(\int f(\lambda t, u_k, u_k) - pF(\lambda t, u_k)\right) dt\right]$$
\[\int_0^{2\pi} \lambda(p - 2)(c_1|u_k|^p - c_2) \, dt - c_5 \geq \int_0^{2\pi} \lambda c_6 \|u_k\|_{L^p}^p - c_7 \]

which implies

\[\|u_k\|_{L^p} \leq c_6 (1 + \|u_k\|^\frac{1}{p}) \quad \text{(3.12)}\]

Write \(u_k = u_k^++u_k^- \in X \oplus Y\). Then for large \(k\), by \((f_3)(\text{II})\) and Hölder inequality we get that

\[\|u_k^+\| \geq \|\Phi'(u_k)u_k^+\| \geq \|u_k^+\|^2 - \int_0^{2\pi} \lambda(f(\lambda t, u_k), u_k^-) \, dt \geq \|u_k^+\|^2 - \int_0^{2\pi} \lambda a(1 + |u_k|^p-1)|u_k^-| \, dt \geq \|u_k^-\|^2 - \lambda a\|u_k^-\|_{L^1} - \lambda a\|u_k\|_{L^p}^{p-1}\|u_k^-\|_{L^p} \geq \|u_k^-\|^2 - c_9\|u_k^-\| - c_{10}\|u_k\|_{L^p}^{p-1}\|u_k^-\|^\frac{1}{p} \quad \text{(3.13)}\]

This yields

\[\|u_k^-\| \leq c_{11} (1 + \|u_k\|_{L^p}^{p-1}) \quad \text{(3.14)}\]

Similarly, one can easily get that

\[\|u_k^+\| \leq c_{11} (1 + \|u_k\|_{L^p}^{p-1}) \quad \text{(3.14')}\]

The combination of (3.12)–(3.14) shows that

\[\|u_k\| \leq \|u_k^-\| + \|u_k^+\| \leq c_{12} (1 + \|u_k\|^\frac{p-1}{p}) \quad \text{(3.14'')}\]

It implies that \(\{u_k\}\) is bounded in \(E\). Moreover, \(\{u_k\}\) has a convergent subsequence according to the argument in Theorem 3.1. Hence \((PS)\) condition holds for any \(c \in (0, \infty)\).

We have pointed out the fact that \(\Phi\) is even in Theorem 3.1. By virtue of Theorem 1.2, \(\Phi\) has a sequence of critical points \(\{u_n\} \subset E\) such that \(|\Phi(u_n)| \to \infty\). If \(\{u_n\}\) is bounded in \(E\), then by the assumption \((f_3)\)(II) and the definition of \(\Phi\), one know that \(\{|\Phi(u_n)|\}\) is also bounded, a contradiction. Hence \(\{u_n\}\) is unbounded in \(E\). The proof is completed. 

**Remark 3.1.** Similar to the treatment of Theorem 3.2, we can get the same conclusion as Theorem 3.4 by replacing \((f_3)\) with the following condition

\((-f_3)\) there exists an \(\bar{r} > 0\) such that

\[-f(t, z, z) \geq p F(t, z) > 0 \quad \text{for} \ t \in [0, \bar{r}] \text{ and } |z| \geq \bar{r}\]

**References**


