The proof of a conjecture on the lewin number of primitive non-powerful signed digraphs

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ABSTRACT

Suppose $S$ is a primitive non-powerful signed digraph. A pair of SSSD walks of $S$ are two directed walks of $S$, which have the same initial vertex, same terminal vertex and same length, but different signs. The lewin number of $S$, denoted $l(S)$, is the least positive integer $k$ such that there are both SSSD walks of lengths $k$ and $k + 1$ from some vertex $u$ to some vertex $v$ (possibly $u$ again) of $S$. This paper presents a proof of a conjecture on $l(S)$, which was put forward by You et al. [L.H. You, M.H. Liu, B.L. Liu, Bounds on the lewin number for primitive non-powerful signed digraphs, Acta Math. Appl. Sinica 35 (2012) 396–407].

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1. Introduction

Let $D = (V, A)$ be a digraph on $n$ vertices. We permit loops but no multiple arcs. A $u \rightarrow v$ walk in $D$ is a sequence of vertices $u, u_1, \ldots, u_k, v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_k, v)$. Furthermore, if $u \rightarrow v$ is a walk with distinct vertices, then $u \rightarrow v$ is called a path. A closed walk is a $u \rightarrow v$ walk, where $u = v$. A cycle is a closed $u \rightarrow v$ walk with distinct vertices except for $u = v$. 

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and a cycle on \( k \) vertices is denoted by \( C_k \). The length of a walk \( W \) is the number of arcs of \( W \), and is denoted by \( |W| \).

Suppose \( D \) is a digraph. If we assign a sign 1 or \(-1\) to each arc of \( D \), the resulting digraph \( S \) is called a signed digraph. If \( W \) is a walk of a signed digraph \( S \), the sign \( \text{sgn}(W) \) of \( W \) is defined as \( \text{sgn}(W) = \prod_{e \in E(W)} \text{sgn}(e) \). Two walks \( W_1 \) and \( W_2 \) in a signed digraph are called a pair of SSSD walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs. If the signed digraph \( S \) contains no pairs of SSSD walks, then \( S \) is called powerful. Otherwise, \( S \) is non-powerful.

It is well known that matrices and digraphs are closely related. Let \( A = (a_{ij}) \) be a square sign pattern matrix of order \( n \). The associated digraph \( D(A) \) (possibly with loops) is defined to be a digraph with vertex set \( V = \{1, 2, \ldots, n\} \) and arc set \( E = \{(i, j) | a_{ij} \neq 0\} \). The associated signed digraph \( S(A) \) of \( A \) is obtained from \( D(A) \) by assigning the sign of \( a_{ij} \) to each arc \((i, j)\) in \( D(A) \). For any positive integer \( k \), the entries \((A^k)_{ij} \neq 0 \) of the power \( A^k \) if and only if there is a walk of length \( k \) from vertex \( i \) to vertex \( j \) in \( D(A) \); the sign of the entries \((A^k)_{ij} \) is \( 1 \) (or \(-1\)) if and only if all walks of length \( k \) from vertex \( i \) to vertex \( j \) have the same sign \( 1 \) (or \(-1\)) in \( S(A) \); and the sign of the entries \((A^k)_{ij} \) is the ambiguous sign, say \# in [4], if and only if there is a pair of SSSD walks of length \( k \) from vertex \( i \) to vertex \( j \) in \( S(A) \). We call \( A \) powerful if each entry of \( A \) contains no \# entry, that is, \( A \) is powerful if and only if \( S(A) \) contains no pair of SSSD walks.

A nonnegative square matrix \( A \) is primitive if some power \( A^k > 0 \). A square sign pattern matrix \( A \) is called primitive if \( |A| \) is primitive, where \(|A|\) denotes the \((0,1)\)-matrix obtained from \( A \) by replacing each nonzero entry by 1.

The notation \( u \xrightarrow{k_1, \ldots, k_j} v \) is used to indicate that there is a \( u \rightarrow v \) walk of length \( k \). By \( u \xrightarrow{k_1} v \) and \( u \xrightarrow{k_2} v \), we mean that \( u \xrightarrow{k_1} v \) and \( u \xrightarrow{k_2} v \). We use the notation \( u \xrightarrow{k_{SSSD}} v \) to indicate that there is a pair of SSSD walks of length \( k \) from \( u \) to \( v \), and we use the notation \( u \xrightarrow{k_{SSSD}} v \) to indicate that \( u \xrightarrow{k_1} v \) and \( u \xrightarrow{k_2} v \). Now, from the above definitions, \( u \xrightarrow{k_{SSSD}} v \) and \( u \xrightarrow{k_{SSSD}} v \) imply that there is no walk, no pair of SSSD walks of length \( k \) from \( u \) to \( v \), respectively.

Let \( S \) be a signed digraph, and \( D \) be the corresponding underlying digraph of \( S \). Recall that \( D \) is primitive if and only if there exists some positive integer \( k \) such that there is a walk \( u \xrightarrow{k} v \) whenever \( u, v \in V(D) \). If \( D \) is primitive, then we call \( S \) primitive.

In 2007, You et al. [11] showed that a square sign pattern matrix \( A \) is primitive non-powerful if and only if there exists some positive integer \( k \) such that \( A^k = \#j \). The least such \( k \) is called the base of \( A \), and denoted by \( b(A) \). Namely, \( S \) is a primitive non-powerful signed digraph if and only if there exists some positive integer \( k \) such that \( u \xrightarrow{k_{SSSD}} v \) whenever \( u, v \in V(S) \). The least such \( k \) is called the base of \( S \), and denoted by \( b(S) \). Clearly, \( b(A) = b(S(A)) \).

Furthermore, the local base is defined in [1,3,6,9]. The research on the (local) bases of primitive non-powerful sign pattern matrices (signed digraphs) attracted much attention of the mathematics scholars [12].

In 1971, Lewin [2] presents a characterization of primitive matrices: A nonnegative irreducible square matrix is primitive if and only if for some positive integer \( k \) there is an ordered pair \((u, v)\) such that \((A^k)_{uv}(A^{k+1})_{uv} > 0 \). The result implies that a strongly connected digraph \( D \) is primitive if and only if there exists a positive integer \( k \) such that \( u \xrightarrow{k,k+1} v \) for some \( u, v \in V(D) \). Therefore, if \( D \) is primitive, then \( D \) must have a smallest \( k \) such that \( u \xrightarrow{k,k+1} v \) for some \( u, v \in V(D) \). Such smallest \( k \) is called the lewin number of \( D \), and denoted by \( l(D) \) in [8]. To be more precise, \( l(D) = \min\{t : u \xrightarrow{t,t+1} v \} \) for some \( u, v \in V(D) \).

Recently, based on the Lewin’s result for primitive digraphs, You et al. [10] showed that a strongly connected signed digraph \( S \) is primitive non-powerful if and only if there exists a positive integer \( k \) such that \( u \xrightarrow{k,k+1_{SSSD}} v \) for some \( u, v \in V(S) \). Moreover, motivated by the concept of lewin number...
of a primitive digraph, You et al. [10] defined the lewin number \( l(S) \) of a primitive non-powerful signed digraph \( S \) as \( l(S) = \min \{ t : u \xrightarrow{t+1} v \text{ for some } u, v \in V(S) \} \).

By the relationship between digraphs and matrices, it is easy to know that the lewin number \( l(A) \) of a primitive non-powerful sign pattern matrix \( A \) is the smallest positive integer \( t \) such that the signs of \( (A^t)_{uv} \) and \( (A^{t+1})_{uv} \) are the ambiguous sign, \#, for some \( u, v \), where \( 1 \leq u, v \leq n \). Clearly, \( l(A) = l(S(A)) \).

In the coming discussion, we are only concerned with the primitive non-powerful signed digraphs, and to avoid repetition, \( S \) always indicates a primitive non-powerful signed digraph, and \( D \) denotes the corresponding underlying primitive digraph of \( S \).

Now, from the definitions of \( l(D) \), \( b(S) \), and \( l(S) \), it easily follows that \( l(D) \leq l(S) \leq b(S) \).

In the sequel, let \( q \) and \( s \) be two positive integers such that \( 2 \leq s < q \leq n \), the greatest common divisor of \( q \) and \( s \) is equal to \( 1 \), i.e., \( \gcd(q, s) = 1 \), and \( q + s \geq n + 1 \), and let \( D_{n,q,s} = (V, A) \) be the digraph, where \( V = \{v_1, v_2, \ldots, v_n\} \) and \( A = \{(v_i, v_{i+1}) : 1 \leq i \leq n - 1\} \cup \{(v_s, v_1), (v_n, v_{n-q+1})\} \) (see [8]). It is well known that \( D \) is primitive if and only if \( D \) is strongly connected and the greatest common divisor of the lengths of all the cycles of \( D \) is \( 1 \) (for instance, see [5]). Note that \( D_{n,q,s} \) has exactly two cycles, say \( C_q \) and \( C_s \), and \( \gcd(q, s) = 1 \). Thus, \( D_{n,q,s} \) is primitive. Throughout this paper, \( S_{n,q,s} \) denotes a non-powerful primitive signed digraph such that the corresponding underlying digraph of \( S_{n,q,s} \) is \( D_{n,q,s} \).

In [10], some bounds on the lewin number for primitive non-powerful signed digraphs were given, and the next conjecture was put forward.

**Conjecture 1.1** [10]. For any primitive non-powerful signed digraph \( S \) of order \( n \),

\[
l(S) \leq \begin{cases} \frac{3n^2 - 15n + 20}{2}, & \text{if } n \geq 12 \text{ is even}; \\ \frac{3n^2 - 9n + 6}{2}, & \text{if } n \geq 7 \text{ is odd}. \end{cases}
\]

Moreover, equality in the above two cases holds if and only if \( S \) is isomorphic to \( S_{n,n-1,n-3} \) or \( S_{n,n,n-2} \), respectively.

Unfortunately, when \( n = 14 \), the following theorem shows that there are exceptional signed digraphs for the equality of Conjecture 1.1. However, for the other cases, Conjecture 1.1 holds, since we have the next result, which will be shown in Section 3.

**Theorem 1.2.** For any primitive non-powerful signed digraph \( S \) of order \( n \),

\[
l(S) \leq \begin{cases} \frac{3n^2 - 15n + 20}{2}, & \text{if } n \text{ is even and } n \geq 12; \\ \frac{3n^2 - 9n + 6}{2}, & \text{if } n \text{ is odd and } n \geq 7. \end{cases}
\]

Moreover, equality in the former case holds if and only if \( S \cong S_{n,n-1,n-3} \) or \( S \cong S_{14,14,11} \), and equality in the later case holds if and only if \( S \cong S_{n,n,n-2} \).

### 2. The lewin number of \( S_{n,q,s} \)

Based on the characterization for powerful irreducible sign pattern matrices [4], You et al. [11] obtained an important characterization for primitive non-powerful signed digraphs in term of graph parameter.

**Proposition 2.1** [11]. Suppose \( S \) is a primitive signed digraph. Then, \( S \) is non-powerful if and only if \( S \) contains a pair of cycles \( C_{p_1} \) and \( C_{p_2} \) satisfying one of the following two conditions:

- \((A_1)\) \( p_1 \) is odd and \( p_2 \) is even and \( \text{sgn}(C_{p_2}) = -1 \).
- \((A_2)\) Both \( p_1 \) and \( p_2 \) are odd and \( \text{sgn}(C_{p_1}) = -\text{sgn}(C_{p_2}) \).
A pair of cycles $C_{p_1}$ and $C_{p_2}$ satisfying $(A_1)$ or $(A_2)$ is called a “distinguished cycle pair” in [11]. It is easy to check that if $C_{p_1}$ and $C_{p_2}$ are a distinguished cycle pair, then the (closed) walks $W_1 = p_2 C_{p_1}$ (walk around $C_{p_1}$ by $p_2$ times) and $W_2 = p_1 C_{p_2}$ have the same length $p_1 p_2$, but different signs:

$$\left(\text{sgn}(C_{p_1})\right)^{p_2} = -\left(\text{sgn}(C_{p_2})\right)^{p_1}. \quad (2.1)$$

The following lemma is well-known. For instance, see [7].

**Lemma 2.2.** Let $a$, $b$, $c$, $x_0$, $y_0$ be five integers. Suppose that $\gcd(a, b) = 1$, and $(x, y) = (x_0, y_0)$ is one solution of $ax + by = c$. Then, all the other integer solutions of $ax + by = c$ are of the form: $(x, y) = (x_0 + bt, y_0 - at)$, where $t$ is an arbitrary integer.

Let $W_1$ and $W_2$ be two walks of $D$. In the sequel, we always write $W_1 \cup W_2$ as $W_1 + W_2$.

**Lemma 2.3.** Let $v_{t_1}$ and $v_{t_2}$ be two vertices of $S_{n, q, s}$, and $P_1$ and $P_2$ be two different paths from $v_{t_1}$ to $v_{t_2}$, where $|P_2| \geq |P_1|$. Suppose $v_{t_1} \xrightarrow{m \text{ SSD}} v_{t_2}$. Then,

1. either $m = |P_1| + (q - 1)s$ or $m \geq |P_1| + qs$;
2. or $m \geq |P_1| + qs$, then $v_{t_1} \xrightarrow{m - qs} v_{t_2}$.

**Proof.** By the definition of $D_{n, q, s}$, since there are two different paths from $v_{t_1}$ to $v_{t_2}$, we can conclude that $n - q + 1 \leq t_2 < t_1 \leq s$ and $|P_2| = |P_1| + q - s$. Furthermore, there exists no other path, except for $P_1$ and $P_2$, from $v_{t_1}$ to $v_{t_2}$. Suppose $W_1$ and $W_2$ are a pair of SSD walks of length $m$ from $v_{t_1}$ to $v_{t_2}$. Then, there exist nonnegative integers $a_i, b_i$, where $i = 1, 2$, such that one of the following three conditions holds:

- $(B_1)$ $W_i = P_i + a_i C_q + b_i C_s$, where $i = 1, 2$;
- $(B_2)$ $W_i = P_i + a_i C_q + b_i C_s$, where $i = 1, 2$;
- $(B_3)$ $W_i = P_i + a_i C_q + b_i C_s$, where $i = 1, 2$.

We divide the proof into the following two cases.

**Case 1.** $(B_1)$ holds.

Note that $|W_1| = |W_2|$. Thus, $m = |P_1| + a_1 q + b_1 s = |P_2| + a_2 q + b_2 s$, and hence $(a_1 - a_2) q + (b_1 - b_2) s = q - s$. Since $1 \times q + (-1) \times s = q - s$ and $\gcd(q, s) = 1$, by Lemma 2.2 there exists some integer $x$ such that

$$\begin{align*}
    a_1 &= a_2 + 1 + sx \\
    b_1 &= b_2 - 1 - qx
\end{align*} \quad (2.2)$$

If $x \geq 1$, then $a_1 \geq s + 1$ by Eq. (2.2). Thus, $m = |W_1| \geq |P_1| + a_1 q \geq |P_1| + (q + s + 1) > |P_1| + qs$. So, (1) holds. Moreover, $P_1 + (a_1 - s) C_q + b_1 C_s$ is a walk of length $m - qs$ from $v_{t_1}$ to $v_{t_2}$, and hence (2) holds.

If $x = 0$, then $a_1 = a_2 + 1$ and $b_2 = b_1 + 1$ by Eq. (2.2). Let $Q$ be the unique path from $v_{t_2}$ to $v_{t_1}$. Then, $W_1 + Q = a_1 C_q + (b_1 + 1) C_s = (a_2 + 1) C_q + b_2 C_s = W_2 + Q$, and hence $\text{sgn}(W_1) = \text{sgn}(W_2)$. This contradicts the fact that $W_1$ and $W_2$ are a pair of SSD walks.

If $x \leq -2$, then $b_1 \geq 2q - 1 > q$ by Eq. (2.2). Thus, $m = |W_1| \geq |P_1| + b_1 s \geq |P_1| + s(2q - 1) > |P_1| + qs$. So, (1) holds. Moreover, $P_1 + a_1 C_q + (b_1 - q) C_s$ is a walk of length $m - qs$ from $v_{t_1}$ to $v_{t_2}$, and hence (2) holds.

If $x = -1$, then $a_1 = a_2 + 1 - s$ and $b_1 = b_2 - 1 + q$ by Eq. (2.2). Thus, $a_2 = a_1 + s - 1 \geq s - 1$ and $b_1 \geq q - 1$.

If $a_2 = s - 1$ and $b_1 = q - 1$, then $a_1 = b_2 = 0$, and hence $m = |W_1| = |P_1| + (q - 1)s$. So, (1) holds. If $a_2 \geq s$, then $m = |W_2| \geq |P_2| + a_2 q > |P_1| + a_2 q \geq |P_1| + qs$. So, (1) holds. Moreover,
\[ P_2 + (a_2 - s)C_q + b_2 C_s \] is a walk of length \( m - qs \) from \( v_{t_1} \) to \( v_{t_2} \). So, (2) holds. (1) and (2) can be proved similarly when \( a_2 \geq s \).

**Case 2.** Either \((B_2)\) or \((B_3)\) holds.

We may suppose that \((B_2)\) holds, since the proof is similar when \((B_3)\) holds.

Note that \( |W_1| = |W_2| \). Thus, \( m = |P_1| + a_1 q + b_1 s = |P_1| + a_2 q + b_2 s \), and hence

\[
(a_1 - a_2)q = (b_2 - b_1)s. \tag{2.3}
\]

Since \( W_1 \) and \( W_2 \) are two SSSD walks, \( a_1 \neq a_2 \). By symmetry, we may suppose that \( a_1 > a_2 \). Note that \( \gcd(q, s) = 1 \). From Eq. (2.3), it follows that \( b_2 - b_1 = qx \), where \( x \) is a positive integer. Hence, \( b_2 \geq q \). Now, \( m = |W_2| = |P_1| + a_2 q + b_2 s \geq |P_1| + qs \). So, (1) holds. Furthermore, \( P_1 + a_2 C_q + (b_2 - q)C_s \) is a walk of length \( m - qs \) from \( v_{t_1} \) to \( v_{t_2} \). So, (2) holds. \(\square\)

**Lemma 2.4.** \( l(S_{n,q,s}) \leq l(D_{n,q,s}) + qs \).

**Proof.** Suppose \( u \) and \( v \) are two vertices of \( S_{n,q,s} \) such that \( u \xrightarrow{l(D_{n,q,s})} v \). Let \( Q_1 \) and \( Q_2 \) be two walks of lengths \( l(D_{n,q,s}) \) and \( l(D_{n,q,s}) + 1 \), respectively, from \( u \) to \( v \). Set \( V_1 = \{v_{n-q+1}, v_{n-q+2}, \ldots, v_s\} \). Since \( n - q + 1 \leq s \), \( V_1 \neq \emptyset \). It is easy to see that \( V(Q_1) \cap V_1 \neq \emptyset \) and \( V(Q_2) \cap V_1 \neq \emptyset \). Choose \( u_1 \in V(Q_1) \cap V_1 \) and \( u_2 \in V(Q_2) \cap V_1 \). Then, \( u_1 \in V(C_q) \cap V(C_s) \) and \( u_2 \in V(C_q) \cap V(C_s) \), where \( C_q \) and \( C_s \) are two cycles of \( D_{n,q,s} \).

Note that \( C_q \) and \( C_s \) are all the cycles of \( D_{n,q,s} \) and \( S_{n,q,s} \) is non-powerful. Then, \( C_q \) and \( C_s \) must be a distinguished cycle pair by Proposition 2.1. By Eq. (2.1), \( qC_q \) and \( sC_s \) have different signs. Let \( W_1 = qC_q \) and \( W_2 = sC_s \). Then, \( W_1 \) and \( W_2 \) are a pair of SSSD walks from \( u_1 \) to \( u_1 \), and \( W_1 \) and \( W_2 \) are also a pair of SSSD walks from \( u_2 \) to \( u_2 \). Thus, \( Q_1 + W_1 \) and \( Q_1 + W_2 \) are a pair of SSSD walks (of length \( l(D_{n,q,s}) + qs \)) from \( u \) to \( v \), and \( Q_2 + W_1 \) and \( Q_2 + W_2 \) are a pair of SSSD walks (of length \( l(D_{n,q,s}) + qs + 1 \)) from \( u \) to \( v \). So,

\[
u \xrightarrow{l(D_{n,q,s}) + qs} u \xrightarrow{l(D_{n,q,s}) + qs + 1} \text{SSSD} \xrightarrow{v} v.
\]

Therefore, \( l(S_{n,q,s}) \leq l(D_{n,q,s}) + qs \). \(\square\)

**Theorem 2.5.** \( l(S_{n,q,s}) = l(D_{n,q,s}) + qs \).

**Proof.** Let \( u, v \) be two arbitrary vertices (possible \( u = v \)) of \( S_{n,q,s} \). In view of Lemma 2.4, it suffices to show that either \( u \xrightarrow{k \ ?} v \) or \( u \xrightarrow{k+1 \ ?} v \) for any \( k \in \{1, 2, \ldots, l(D_{n,q,s}) + qs - 1\} \).

Suppose there exists some integer \( k \in \{1, 2, \ldots, l(D_{n,q,s}) + qs - 1\} \) such that \( u \xrightarrow{k \ ?} v \) holds for some \( u, v \in V(S_{n,q,s}) \). As there are at most two different paths from \( u \) to \( v \), we divide the proof into the following two cases.

**Case 1.** There is a unique path from \( u \) to \( v \).

Let \( P \) be the unique path from \( u \) to \( v \), where \( |P| = 0 \) if and only if \( u = v \). Suppose \( W_1 \) and \( W_2 \) are a pair of SSSD walks of length \( k \), and \( W_3 \) and \( W_4 \) are a pair of SSSD walks of length \( k + 1 \) from \( u \) to \( v \). Then, there exist nonnegative integers \( a_i, b_i \) such that

\[
W_i = P + a_i C_q + b_i C_s, \quad i = 1, 2, 3, 4.
\]

Since \( W_1 \) and \( W_2 \) are a pair of SSSD walks, we have \( |W_1| = |W_2| = k \) and hence Eq. (2.3) holds. Without loss of generality, we may suppose that \( b_2 > b_1 \). By Eq. (2.3), it follows that \( b_2 = b_1 + qx \), where \( x \geq 1 \). Similarly, we may suppose that \( b_4 > b_3 \), and hence \( b_4 = b_3 + yq \), where \( y \geq 1 \). Thus, \( P + a_2 C_q + (b_2 - q)C_s \) and \( P + a_4 C_q + (b_4 - q)C_s \) are two walks of lengths \( k - qs \) and \( k + 1 - qs \), respectively, from \( u \) to \( v \). Therefore, \( l(D_{n,q,s}) \leq k - qs \leq (l(D_{n,q,s}) + qs - 1) - qs = l(D_{n,q,s}) - 1 \), a contradiction.
Case 2. There are two different paths from $u$ to $v$.

Let $P_1$ and $P_2$ be two different paths from $u$ to $v$, where $|P_2| \geq |P_1| \geq 1$. Since there are at most two different paths from $u$ to $v$, $k \geq |P_1| + q$ and $k + 1 \geq |P_1| + q$ by (1) of Lemma 2.3 and $s \geq 2$. Now, (2) of Lemma 2.3 implies that

$$u \stackrel{k - q, k + 1 - qs}{\rightarrow} v.$$

Thus, $l(D_{n,q,s}) \leq k - qs \leq (l(D_{n,q,s}) + qs - 1) - qs = l(D_{n,q,s}) - 1$, a contradiction. \□

By Theorem 2.5 and $l(D_{n,n,n-1}) = 1$, we have

Corollary 2.6 [10]. $l(S_{n,n,n-1}) = n(n - 1) + 1$.

Lemma 2.7 [8]. For any primitive digraph $D$ of order $n \geq 7$,

$$l(D) \leq \begin{cases} \frac{n^2 - 7n + 14}{2}, & \text{if } n \text{ is even;} \\ \frac{n^2 - 5n + 6}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, equality in the above two cases holds if and only if $D$ is isomorphic to $D_{n,n-1,n-3}$ or $D_{n,n,n-2}$, respectively.

From Theorem 2.5 and Lemma 2.7, it immediately follows that

Corollary 2.8. Suppose $n \geq 7$. Then,

1. $l(S_{n,n,n-2}) = \frac{3n^2 - 9n + 6}{2}$ if $n$ is odd;
2. $l(S_{n,n-1,n-3}) = \frac{3n^2 - 15n + 20}{2}$ if $n$ is even.

Corollary 2.9. $l(S_{14,14,11}) = 199$.

Proof. In view of Theorem 1.2, it suffices to prove that $l(D_{14,14,11}) = 45$. Let $v_1$ and $v_2$ be two arbitrary vertices of $S_{14,14,11}$. We only need to show that either $v_1 \xrightarrow{k} v_2$ or $v_2 \xrightarrow{k+1} v_1$ for any $k \in \{1, 2, \ldots, 44\}$. Assume to the contrary that there exists some positive integer $k \in \{1, 2, \ldots, 44\}$ such that $W_1$ and $W_2$ are two walks of lengths $k$ and $k + 1$, respectively, from $v_1$ to $v_2$. Because there are at most two different paths from $v_1$ to $v_2$, we consider the following two cases.

Case 1. There is a unique path from $v_1$ to $v_2$.

Let $P$ be the unique path from $v_1$ to $v_2$, where $|P| = 0$ if and only if $t_1 = t_2$. Then, there exist nonnegative integers $a_i, b_i$ such that

$$W_i = P + a_i C_{14} + b_i C_{11}, \quad i = 1, 2.$$

Since $|W_2| = |W_1| + 1$, we get $14(a_2 - a_1) + 11(b_2 - b_1) = 1$. Since $14 \times 4 + 11 \times (-5) = 1$ and $\gcd(14, 11) = 1$, by Lemma 2.2 there exists some integer $x$ such that

$$\begin{cases} a_2 = a_1 + 4 + 11x \\ b_2 = b_1 - 5 - 14x \end{cases}$$

This contradicts $|W_1| < |W_2| \leq 45$.

Case 2. There are two different paths from $v_1$ to $v_2$.

Let $P_1$ and $P_2$ be two different paths from $v_1$ to $v_2$, where $|P_2| \geq |P_1| \geq 1$. It is easy to see that $1 \leq t_2 < t_1 \leq 11$ and $|P_2| = |P_1| + 3$. By virtue of Case 1, we may assume that there are nonnegative integers $a_i, b_i$, where $i = 1, 2$, such that one of the following two cases holds:
(i) \( W_1 = P_1 + a_1C_{14} + b_1C_{11} \), \( W_2 = P_2 + a_2C_{14} + b_2C_{11} \);
(ii) \( W_1 = P_2 + a_1C_{14} + b_1C_{11} \), \( W_2 = P_1 + a_2C_{14} + b_2C_{11} \).

If (i) holds, since \(|W_2| = |W_1| + 1\), we get \(14(a_1 - a_2) + 11(b_1 - b_2) = 2\). Since \(14 \times (-3) + 11 \times 4 = 2\) and \(\gcd(14, 11) = 1\), by Lemma 3.2 there exists some integer \(x\) such that
\[
\begin{align*}
a_1 &= a_2 - 3 + 11x \\
b_1 &= b_2 + 4 - 14x
\end{align*}
\]
This contradicts \(|W_1| < |W_2| \leq 45\), \(|P_2| \geq 4\) and \(|P_1| \geq 1\).

If (ii) holds, then \(14(a_2 - a_1) + 11(b_2 - b_1) = 4\) by \(|W_2| = |W_1| + 1\). Since \(14 \times 5 + 11 \times (-6) = 4\) and \(\gcd(14, 11) = 1\), by Lemma 3.2 there exists some integer \(x\) such that
\[
\begin{align*}
a_2 &= a_1 + 5 + 11x \\
b_2 &= b_1 - 6 - 14x
\end{align*}
\]
This contradicts \(|W_1| < |W_2| \leq 45\).

By combining the above arguments, we can conclude that \(l(S_{14, 14, 11}) = 199\).

3. The proof of Theorem 1.2.

**Lemma 3.1.** Let \(C_{p_1}\) and \(C_{p_2}\) be two cycles of a primitive digraph \(D\). If \(p_1 + p_2 \geq n + 1\) and \(p_2 = p_1 + 1\), then \(l(D) \leq p_1\).

**Proof.** Since \(p_1 + p_2 \geq n + 1\), \(C_{p_1}\) and \(C_{p_2}\) have a common vertex, say \(v\). Thus, \(C_{p_1}\) and \(C_{p_2}\) are two walks of lengths \(p_1\) and \(p_2\) from \(v\) to \(v\), respectively. So, \(l(D) \leq p_1\).

**Lemma 3.2.** Let \(S\) be a signed digraph on \(n\) vertices, \(C_{p_1}\) and \(C_{p_2}\) be a distinguished cycle pair of \(S\). Then, \(l(S) \leq l(D) + p_1p_2 + 2n - p_1 - p_2\).

**Proof.** Suppose \(W_1\) and \(W_2\) are two walks of lengths \(l(D)\) and \(l(D) + 1\), respectively, from \(u\) to \(v\). Let \(P_1\) be a shortest path from \(V(C_{p_1})\) to \(u\) and \(P_2\) be a shortest path from \(v\) to \(V(C_{p_2})\).

By Eq. (2.1), \(P_1 + W_1 + P_2 + p_2C_{p_1}\) and \(P_1 + W_1 + P_2 + p_1C_{p_2}\) are a pair of SSSD walks of length \(l(D) + |P_1| + |P_2| + p_1p_2\) from some vertex \(x \in V(C_{p_1})\) to some vertex \(y \in V(C_{p_2})\). Similarly, \(P_1 + W_2 + P_2 + p_2C_{p_1}\) and \(P_1 + W_2 + P_2 + p_1C_{p_2}\) are a pair of SSSD walks of length \(l(D) + 1 + |P_1| + |P_2| + p_1p_2\) from \(x\) to \(y\). Since \(|P_1| \leq n - p_1\) and \(|P_2| \leq n - p_2\), \(l(S) \leq l(D) + |P_1| + |P_2| + p_1p_2 \leq l(D) + 2n - p_1 - p_2 + p_1p_2\).

**Lemma 3.3.** Let \(S\) be a signed digraph on \(n\) vertices, \(C_{p_1}\) and \(C_{p_2}\) be a distinguished cycle pair of \(S\). If \(p_1 = p_2\), then \(l(S) \leq l(D) + 2n - p_1\).

**Proof.** Since \(C_{p_1}\) and \(C_{p_2}\) are a distinguished cycle pair of \(S\) and \(p_1 = p_2\), by Proposition 2.1 we get \(\text{sgn}(C_{p_1}) = -\text{sgn}(C_{p_2})\). It can be proved similarly with Lemma 3.2.

**Lemma 3.4.** Let \(C_{p_1}\) and \(C_{p_2}\) be a distinguished cycle pair of \(S\), and let \(W_1\) and \(W_2\) be two walks of lengths \(k\) and \(k + 1\), respectively, from \(u\) to \(v\) in \(S\). If \(V(W_1) \cap V(C_{p_i}) \neq \emptyset\) for \(1 \leq i \leq j \leq 2\), then \(l(S) \leq k + p_1p_2\).

**Proof.** It is easy to see that \(W_1 + p_2C_{p_1}\) and \(W_1 + p_1C_{p_2}\) are a pair of SSSD walks of length \(k + p_1p_2\) from \(u\) to \(v\) by Eq. (2.1). Similarly, \(W_2 + p_2C_{p_1}\) and \(W_2 + p_1C_{p_2}\) are a pair of SSSD walks of length \(k + 1 + p_1p_2\) from \(u\) to \(v\). Thus, \(l(S) \leq k + p_1p_2\).

The **girth** of \(D\), denoted \(g(D)\), is the length of a shortest cycle in \(D\).
Corollary 3.5. Let $S$ be a signed digraph on $n$ vertices, and $C_{p_1}$ and $C_{p_2}$ be a distinguished cycle pair of $S$. If $l(D) \geq n$ and $g(D) \geq \left\lceil \frac{n+1}{2} \right\rceil$, then $l(S) \leq l(D) + p_1p_2$.

Proof. Suppose $W_1$ and $W_2$ are two walks of lengths $l(D)$ and $l(D) + 1$, respectively, from $u$ to $v$ in $D$. Since $l(D) \geq n$, $W_1$ contains at least one cycle $C_t$. Note that $g(D) \geq \left\lceil \frac{n+1}{2} \right\rceil$. Then, $V(C_t) \cap C_t \neq \emptyset$ and $V(C_t) \cap (C_{p_1} \cup C_{p_2}) \neq \emptyset$. Similarly, $W_2$ contains at least one cycle $C_s$ such that $V(C_s) \cap (C_{p_1} \cup C_{p_2}) \neq \emptyset$. Now, the result follows from Lemma 3.4. □

When $g(D) = 1$, we easily get $l(D) = 1$. For $g(D) \geq 2$, Shen and Neufeld [8] had shown that

Lemma 3.6 [8]. Let $D$ be a primitive digraph with $n \geq 7$ vertices and $2 \leq g(D) \leq 3$. Then, $l(D) \leq n - 2$.

Lemma 3.7 [8]. Let $D$ be a primitive digraph with $n \geq 7$ vertices and girth $g \geq 4$. Then,

$$l(D) \leq \begin{cases} \frac{g(n-5)}{2} + 1, & \text{if } g \text{ is even;} \\ \frac{g(n-4)}{2} + 1, & \text{if } g \text{ is odd and } n \text{ is even;} \\ \frac{g(n-3)}{2}, & \text{if } g \text{ is odd and } n \text{ is odd.} \end{cases}$$

Lemma 3.8. Let $S$ be a signed digraph on 14 vertices, and $D_{14,14,11}$ be a subgraph of $D$. If $C_{11}$ and $C_{14}$ are a distinguished cycle pair of $S$, then $l(S) \leq 199$, with the equality if and only if $S \cong S_{14,14,11}$.

Proof. Since $C_{14}$ and $C_{11}$ are a distinguished cycle pair of $S$ and $D_{14,14,11}$ is a subgraph of $D$, we have $l(S) \leq l(S_{14,14,11}) = 199$ by Corollary 2.9.

If $S \cong S_{14,14,11}$, then $l(S) = 199$ by Corollary 2.9. Next we shall show that $l(S) = 199$ implies that $S \cong S_{14,14,11}$. Assume to the contrary that $l(S) = 199$, but $D \not\cong D_{14,14,11}$. Then, $D$ contains at least another cycle $C_p$, where $1 \leq p \leq 13$.

If $g(D) \leq 7$, by Lemmas 3.6–3.7 we get $l(D) \leq 36$. Now, Lemma 3.2 implies that $l(S) \leq 36 + 11 \times 14 + 3 = 193$, a contradiction. If $g(D) \geq 8$ and $l(D) \leq 13$, then $l(S) \leq 13 + 11 \times 14 + 3 = 170$ by Lemma 3.2, a contradiction. If $g(D) \geq 8$ and $14 \leq l(D) \leq 44$, then $l(S) \leq 44 + 11 \times 14 = 198$ by Corollary 3.5, a contradiction.

Thus, $g(D) \geq 8$ and $l(D) \geq 45$, and hence $8 \leq p \leq 13$.

If $p = 8$, since there exists some vertex $u \in V(C_8) \cap V(C_{11})$ such that $4C_8$ and $3C_{11}$ are two walks of lengths 32 and 33, respectively, from $u$ to $u$, we get $l(D) \leq 32$, a contradiction.

If $p = 9$, since there exists some vertex $v \in V(C_9) \cap V(C_{14})$ such that $3C_9$ and $2C_{14}$ are two walks of lengths 27 and 28, respectively, from $v$ to $v$, we get $l(D) \leq 27$, a contradiction.

If $p \in \{10, 12, 13\}$, by Lemma 3.1 we have $l(D) \leq 13$, a contradiction.

So, $p = 11$. Since $l(D) \geq 45$, $D_{14,14,11} \subset D \subseteq D_{14,14,11}$, where $D_{14,14,11}$ is the digraph obtained from $D_{14,14,11}$ by adding the arcs $(v_{12}, v_2), (v_{13}, v_3), (v_{14}, v_4), (v_8, v_{12}), (v_9, v_{13}), (v_{10}, v_{14})$.

If $(v_{12}, v_2) \in A(D)$, then there are two paths $P_1$ and $P_2$ of lengths 3 and 11, respectively, from $v_{12}$ to $v_1$. Note that $P_1 + 3C_{14}$ and $P_2 + 3C_{11}$ are two walks of lengths 45 and 44, respectively, from $v_{12}$ to $v_1$. Then, $l(D) \leq 44$, a contradiction. If $(v_{13}, v_3) \in A(D)$ or $(v_{14}, v_4) \in A(D)$, a similar contradiction will be yielded.

If $(v_8, v_{12}) \in A(D)$, then there are two paths $P_3$ and $P_4$ of lengths 1 and 9, respectively, from $v_{11}$ to $v_{12}$. Note that $P_3 + 3C_{14}$ and $P_4 + 3C_{11}$ are two walks of lengths 43 and 42, respectively, from $v_{11}$ to $v_{12}$. Then, $l(D) \leq 42$, a contradiction. If $(v_8, v_{13}) \in A(D)$ or $(v_{10}, v_{14}) \in A(D)$, we can deduce a contradiction similarly.

Thus, $l(S) = 199$ implies that $S \cong S_{14,14,11}$. This completes the proof of this result. □

Lemma 3.9. Let $n \geq 12$ be even and $S$ be a signed digraph on $n$ vertices. Suppose $C_{n-3}$ and $C_n$ are a distinguished cycle pair of $S$. Then, $l(S) \leq \frac{1}{2} \left(3n^2 - 15n + 20\right)$, where the equality holds if and only if $S \cong S_{14,14,11}$. 
Proof. Let $D_1 = (V, A)$ be the digraph, where $V = \{v_1, v_2, \ldots, v_n\}$ and $A = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_{n-3}, v_1), (v_n, v_1)\}$. It is easy to see that $D$ has $D_1$ as its subgraph, and there are two paths $P_1$ and $P_2$ of lengths 1 and 4, respectively, from $v_{n-3}$ to $v_1$.

If $1 \leq g(D) \leq n - 6$, since $n \geq 12$ is even, Lemmas 3.6–3.7 imply that $l(D) \leq \frac{\lfloor \frac{n-1}{2} \rfloor (n-5) + 1}{2}$. By Lemma 3.2 and $n \geq 12$,

$$l(S) \leq l(D) + 3 + n(n-3) \leq \frac{(n-6)(n-5)}{2} + 4 + n(n-3) < \frac{1}{2} \left(3n^2 - 15n + 20\right).$$

If $l(D) \leq n - 1$, Lemma 3.2 implies that $l(S) \leq n - 1 + 3 + n(n-3) < \frac{1}{2}(3n^2 - 15n + 20)$.

Thus, we may suppose that $g(D) \geq n - 5 \geq \left\lceil \frac{n+1}{2} \right\rceil$ and $l(D) \geq n$ in the following. There are three cases to be considered.

Case 1. $n \equiv 1 \pmod{3}$.

Note that $P_1 + \frac{n-1}{2}C_{n-3}$ and $P_2 + \frac{n-7}{2}C_n$ are two walks of lengths $\frac{1}{3}(n-7)n+5$ and $\frac{1}{3}(n-7)n+4$, respectively, from $v_{n-3}$ to $v_1$. Using Corollary 3.5 and $n \geq 13$, we have

$$l(S) \leq l(D) + n(n-3) \leq \frac{1}{3}(n-7)n + 4 + n(n-3) < \frac{1}{2} \left(3n^2 - 15n + 20\right).$$

Case 2. $n \equiv -1 \pmod{3}$.

Note that $P_1 + \frac{n-2}{3}C_{n-3}$ and $P_2 + \frac{n-5}{3}C_n$ are two walks of lengths $\frac{1}{3}(n-5)n+3$ and $\frac{1}{3}(n-5)n+4$, respectively, from $v_{n-3}$ to $v_1$. By Corollary 3.5 and $n \geq 14$

$$l(S) \leq l(D) + n(n-3) \leq \frac{1}{3}(n-5)n + 3 + n(n-3) < \frac{1}{2} \left(3n^2 - 15n + 20\right).$$

If $l(S) = \frac{1}{2} \left(3n^2 - 15n + 20\right)$, then $n = 14$ and $l(S) = \frac{1}{2} \left(3 \times 14^2 - 15 \times 14 + 20\right) = 199$. By Lemma 3.8, we can conclude that $S \cong S_{14,14,11}$. Conversely, if $S \cong S_{14,14,11}$, then $l(S) = 199$ by Lemma 3.8.

Case 3. $n \equiv 0 \pmod{3}$.

Then, gcd$(n, n-3) \geq 3$. Recall that $D$ is primitive. The greatest common divisor of the lengths of all the cycles in $D$ is 1. Thus, there exists some cycle $C_p$ in $D$ such that $p \neq n - 3$ and $p \neq n$. Since $g(D) \geq n - 5$, it follows that $p \geq n - 5$.

If $p = n - 5$, then $g(D) = n - 5$. Since $n$ is even, $\text{sgn}(C_n) = -1$ by Proposition 2.1. Note that $n - 5$ is odd. Then, $C_{n-5}$ and $C_n$ are also a distinguished cycle pair of $S$ by Proposition 2.1. Now, Lemma 3.7 and Corollary 3.5 imply that

$$l(S) \leq l(D) + n(n-5) \leq \frac{(n-5)(n-4)}{2} + 1 + n(n-5) < \frac{1}{2} \left(3n^2 - 15n + 20\right).$$

If $n - 4 \leq p \leq n - 1$, since $p \notin \{n - 3, n\}$, we get $p \in \{n-4, n-2, n-1\}$. Thus, $l(D) \leq n - 1$ by Lemma 3.1. Using Corollary 3.5 and $n \geq 12$, we have

$$l(S) \leq l(D) + n(n-3) \leq n - 1 + n(n-3) < \frac{1}{2} \left(3n^2 - 15n + 20\right).$$

This completes the proof of this result. □

Lemma 3.10. Let $S$ be a signed digraph on $n \geq 12$ vertices. Suppose $C_{n-1}$ and $C_{n-3}$ are a distinguished cycle pair of $S$. Then, $l(S) \leq \frac{1}{2} \left(3n^2 - 15n + 20\right)$, where the equality holds if and only if $S \cong S_{n,n-1,n-3}$.

Proof. By Proposition 2.1, $n$ is even.

If $1 \leq g(D) \leq n - 4$, since $n \geq 12$ is even, by Lemmas 3.2, 3.6 and 3.7
The result already holds. Since Proof of Theorem 1.2.

Subcase 2.1.

Case 1.

Case 2.

can be proved similarly that the result holds.

If \( l(D) \leq n - 1 \), by Lemma 3.2 we get \( l(S) \leq n + 3 + (n - 1)(n - 3) < \frac{1}{2} \left( 3n^2 - 15n + 20 \right) \).

Thus, we suppose that \( g(D) = n - 3 > \left\lceil \frac{n+1}{2} \right\rceil \) and \( l(D) \geq n \) in the following. Now, Corollary 3.5 and Lemma 2.7 imply that

\[
l(S) \leq l(D) + (n - 1)(n - 3) \leq \frac{n^2 - 7n + 14}{2} + (n - 1)(n - 3) = \frac{3n^2 - 15n + 20}{2}. \tag{3.1}
\]

If \( l(S) = \frac{3n^2 - 15n + 20}{2} \), then \( l(D) = \frac{n^2 - 7n + 14}{2} \) and hence \( S \cong S_{n,n-1,n-3} \) by inequality (3.1) and Lemma 2.7. Conversely, if \( S \cong S_{n,n-1,n-3} \), then \( l(S) = \frac{3n^2 - 15n + 20}{2} \) by Corollary 2.8. \( \square \)

**Proof of Theorem 1.2.** Note that \( S \) is primitive non-powerful. By Proposition 2.1, \( S \) has a distinguished cycle pair \( C_{p_1} \) and \( C_{p_2} \), where \( p_2 \geq p_1 \). We consider the following two cases.

**Case 1.** Either \( p_1 + p_2 \leq n \) or \( l(D) \leq n - 1 \).

If \( p_1 + p_2 \leq n \), since \( g(D) \leq p_1 \leq \frac{n}{2} \) and \( n \geq 7 \), by Lemmas 3.2, 3.6 and 3.7

\[
l(S) \leq l(D) + 2n - p_1 - p_2 + p_1 p_2
\]

\[
\leq \frac{n(n - 3)}{4} + 2n - 1 + (p_1 - 1)(p_2 - 1)
\]

\[
\leq \frac{n^2 + 5n - 4}{4} + \frac{1}{4}(p_1 + p_2 - 2)^2
\]

\[
\leq \frac{2n^2 + n}{4}.
\]

Note that \( \frac{2n^2 + n}{4} < \frac{3n^2 - 15n + 20}{2} < \frac{3n^2 - 9n + 6}{2} \) by \( n \geq 7 \). Thus, the result follows. If \( l(D) \leq n - 1 \), it can be proved similarly that the result holds.

**Case 2.** \( p_1 + p_2 \geq n + 1 \) and \( l(D) \geq n \).

Then, \( C_{p_1} \) and \( C_{p_2} \) have some common vertex. If \( p_1 = p_2 \), by Lemmas 2.7 and 3.3

\[
l(S) < l(D) + 2n \leq \frac{n^2 - 5n + 6}{2} + 2n < \frac{3n^2 - 15n + 20}{2}.
\]

The result already holds. Since \( l(D) \geq n \), \( g(D) \geq 4 \) by Lemma 3.6. Thus, we may suppose that \( p_2 > p_1 \geq g(D) \geq 4 \) in the sequel. We divide the proof into the following five subcases.

**Subcase 2.1.** \( 4 \leq p_1 \leq n - 5 \).

Thus, \( 4 \leq g(D) \leq n - 5 \).

If \( n \) is odd, by Lemmas 3.2 and 3.7

\[
l(S) \leq l(D) + 2n - 9 + p_1 p_2 \leq \frac{(n - 5)(n - 3)}{2} + 2n - 9 + n(n - 5) < \frac{3n^2 - 9n + 6}{2}.
\]

If \( n \) is even, by Lemmas 3.2 and 3.7

\[
l(S) \leq l(D) + 2n - 9 + p_1 p_2 \leq \frac{(n - 5)(n - 4)}{2} + 2n - 8 + n(n - 5) < \frac{3n^2 - 15n + 20}{2}.
\]
Subcase 2.2. $p_1 = n - 4$.

Now, $p_2 > p_1$ implies that $n - 3 \leq p_2 \leq n$ and hence $4 \leq g(D) \leq n - 4$.

If $n - 3 \leq p_2 \leq n - 1$, since $n \geq 8$, using Lemmas 3.2 and 3.7, we get

$$l(S) \leq l(D) + 7 + p_1 p_2 \leq \frac{(n - 4)(n - 3)}{2} + 7 + (n - 4)(n - 1) < \frac{3n^2 - 15n + 20}{2}.$$ 

If $p_2 = n$, by Proposition 2.1 we can conclude that $n$ is odd because $C_{n-4}$ and $C_n$ are a distinguished cycle pair. From Lemmas 3.2 and 3.7, it follows that

$$l(S) \leq l(D) + 4 + p_1 p_2 \leq \frac{(n - 4)(n - 3)}{2} + 4 + n(n - 4) < \frac{3n^2 - 9n + 6}{2}. $$

Subcase 2.3. $p_1 = n - 3$.

Then, $n - 2 \leq p_2 \leq n$.

If $p_2 = n - 2$, it follows from Lemmas 3.1 and 3.2 that

$$l(S) \leq l(D) + 3 + (n - 2)(n - 3) \leq n - 3 + 5 + (n - 2)(n - 3) < \frac{3n^2 - 15n + 20}{2}. $$

If $p_2 = n - 1$, then $n$ is even by Proposition 2.1. By Lemma 3.10, the result holds.

If $p_2 = n$, in view of Lemma 3.9 we may suppose that $n$ is odd. Because $p_1 = n - 3$, by Lemmas 2.7 and 3.2

$$l(S) \leq l(D) + 3 + n(n - 3) \leq \frac{n^2 - 5n + 6}{2} + 3 + n(n - 3) < \frac{3n^2 - 9n + 6}{2}. $$

Subcase 2.4. $p_1 = n - 2$.

Then, $n - 1 \leq p_2 \leq n$.

If $p_2 = n - 1$, then Lemmas 3.1 and 3.2 imply that

$$l(S) \leq l(D) + 3 + (n - 1)(n - 2) \leq n - 2 + 3 + (n - 1)(n - 2) = n^2 - 2n + 3. \quad (3.2)$$

If $n \in \{7, 9, 11\}$, the result follows from inequality (3.2). If $n \geq 12$, inequality (3.2) implies that $l(S) \leq n^2 - 2n + 3 < \frac{3n^2 - 15n + 20}{2}$, the result also holds.

If $p_2 = n$, then $n$ is odd by Proposition 2.1. If $4 \leq g(D) \leq \lceil \frac{n+1}{2} \rceil - 1 \leq \frac{n}{2}$, it follows from Lemmas 3.2 and 3.7 that

$$l(S) \leq l(D) + 2 + n(n - 2) \leq \frac{n(n - 3)}{4} + 2 + n(n - 2) < \frac{3n^2 - 9n + 6}{2}. $$

Thus, we may suppose that $g(D) \geq \lceil \frac{n+1}{2} \rceil$ in the following. Note that $l(D) \geq n$. By Corollary 3.5 and Lemma 2.7,

$$l(S) \leq l(D) + n(n - 2) \leq \frac{n^2 - 5n + 6}{2} + n(n - 2) = \frac{3n^2 - 9n + 6}{2}. \quad (3.3)$$

If $l(S) = \frac{3n^2 - 9n + 6}{2}$, then $l(D) = \frac{n^2 - 5n + 6}{2}$ and hence $S \cong S_{n,n,n-2}$ by inequality (3.3) and Lemma 2.7. Conversely, if $S \cong S_{n,n,n-2}$, then $l(S) = \frac{3n^2 - 9n + 6}{2}$ by Corollary 2.8.

Subcase 2.5. $p_1 = n - 1$.

Then, $p_2 = n$, and hence $D$ contains $D_{n,n,n-1}$ as its subgraph. By Corollary 2.6,

$$l(S) \leq l(S_{n,n,n-1}) = n^2 - n + 1. \quad (3.4)$$

If $n \in \{7, 9, 11\}$, the result follows from inequality (3.4). If $n \geq 12$, inequality (3.4) implies that $l(S) \leq n^2 - n + 1 < \frac{3n^2 - 15n + 20}{2}$, the result also holds.
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