Lifting of Outer Actions of Groups II*

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Abstract. In this paper we continue to study the lifting of outer actions of groups and introduce the concept of equivalence of two lifting homomorphisms, maximal lifting homomorphisms and maximal split extensions. Some criteria are obtained.

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This paper is a continuation of [2]. Fixing a group homomorphism \( \chi : G \to \text{Out} N \), where \( G \) and \( N \) are arbitrary groups (not necessarily finite), we call \( \chi \) an outer action of \( G \) on \( N \). Let \( \rho : \text{Aut} N \to \text{Out} N \) be the natural homomorphism. If there exists a group homomorphism \( \sigma : G \to \text{Aut} N \) such that \( \chi = \sigma \rho \), then \( \sigma \) is called a lifting homomorphism of \( \chi \). In [2], we obtained some necessary and sufficient conditions for an outer action to be lifted, and the number of its conjugacy classes.

In this paper we give another natural classification of all lifting homomorphisms for a given outer action, that is, define the equivalence of two lifting homomorphisms. We mainly consider the maximality of lifting homomorphisms and the corresponding split extensions, and obtain some criteria to guarantee the existence of such lifting homomorphisms. We keep the notation used in [1].

1 Equivalence of Lifting Homomorphisms

Let \( G \) and \( N \) be arbitrary groups. Suppose that \( \sigma : G \to \text{Aut} N \) is a group homomorphism, that is, \( \sigma \) is a group action of \( G \) on \( N \); then there is a semidirect product extension \( \mathcal{E}_\sigma : 1 \to N \to G \ltimes N \to G \to 1 \), where \( G \ltimes N = \{(g, a) \mid g \in G, a \in N\} \) is a group under the operation defined by

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\[(g,a)(h,b) = (gh,a^{h\gamma}b) \quad \text{for } g, h \in G \text{ and } a, b \in N. \quad (1)\]

It is well known that every split extension of \(N\) by \(G\) is equivalent to a semidirect product extension as above (see [1]).

Given two group actions \(\sigma_1, \sigma_2 : G \to \text{Aut} N\), if the corresponding semidirect product extensions are equivalent, then \(\sigma_1\) and \(\sigma_2\) are called equivalent, written as \(\sigma_1 \sim \sigma_2\). Since equivalent group extensions \(E_{\sigma_1}\) and \(E_{\sigma_2}\) have the same outer action \(\chi : G \to \text{Out} N\), both \(\sigma_1\) and \(\sigma_2\) are lifting homomorphisms of \(\chi\). In this way, we can define an equivalence relation on the set of all lifting homomorphisms of \(\chi\).

In addition, for any mapping \(\mu : G \to N\), if \((gh)^\mu = (g^{\mu})^{h^{\gamma}}h^\mu\) for all \(g, h \in G\), then \(\mu\) is said to be a \(\sigma\)-derivation from \(G\) to \(N\). The set of all \(\sigma\)-derivations from \(G\) to \(N\) is written as \(\text{Der}_\sigma(G, N)\) (see [1]). Notice that the action \(\sigma\) of \(G\) on \(N\) can naturally induce a group action of \(G\) on \(\text{Inn} N\), that is, each \(g \in G\) can act via conjugation on \(\text{Inn} N\) by \(g^\sigma \in \text{Aut} N\). Thus, we have the set of all \(\sigma\)-derivations from \(G\) to \(\text{Inn} N\), written as \(\text{Der}_\sigma(G, \text{Inn} N)\).

In other words, a mapping \(\tau : G \to \text{Inn} N\) is a \(\sigma\)-derivation if and only if \((gh)^\tau = (g^{\tau})^{h^{\gamma}}h^\tau = (h^{\gamma})^{-1}g^{\tau}h^{\gamma}h^\tau\) for all \(g, h \in G\).

**Proposition 1.** Let \(\sigma : G \to \text{Aut} N\) be a lifting homomorphism of the outer action \(\chi\).

(i) A mapping \(\sigma' : G \to \text{Aut} N\) is a lifting homomorphism of \(\chi\) if and only if \(g^{\sigma'} = g^{\sigma}g^\gamma\) for all \(g \in G\), where \(\tau \in \text{Der}_\sigma(G, \text{Inn} N)\). In particular, there is a bijection between the set of all lifting homomorphisms of \(\chi\) and the set \(\text{Der}_\sigma(G, \text{Inn} N)\).

(ii) Suppose that \(\sigma' : G \to \text{Aut} N\) is another lifting homomorphism of \(\chi\). If \(g^{\sigma'} = g^{\sigma}g^\gamma\) for all \(g \in G\), where \(\tau \in \text{Der}_\sigma(G, \text{Inn} N)\), then \(\sigma\) and \(\sigma'\) are equivalent if and only if \(\tau\) can be lifted to a \(\sigma\)-derivation from \(G\) to \(N\), that is, there exists some \(\mu \in \text{Der}_\sigma(G, N)\) such that \(\tau = \mu \rho\), where \(\rho : N \to \text{Inn} N\) is a group homomorphism determined by conjugation. In particular, \(\sigma \sim \sigma'\) if and only if \(g^{\sigma'} = g^{\sigma}(g^\mu)^\gamma\) for all \(g \in G\), where \(\mu \in \text{Der}_\sigma(G, N)\).

**Proof.** (i) Let \(\sigma' : G \to \text{Aut} N\) be a lifting homomorphism of the outer action \(\chi\). Since \(\chi = \sigma \rho = \sigma' \rho\), we have \((g^{\rho})^{\sigma'} = (g^{\rho})^{\sigma'}\) for all \(g \in G\). Since \(\text{Ker} \rho = \text{Inn} N\), we may write \(g^{\sigma'} = g^{\sigma}g^\gamma\), where \(g^\gamma \in \text{Inn} N\). For any \(h \in G\), we have \((gh)^{\sigma'} = g^{\sigma}h^{\gamma}\) and hence \((gh)^{\sigma} = (h^{\gamma})^{-1}g^{\sigma}h^{\gamma} = (g^\tau)^{h^{\gamma}}h^\tau\). Since \(\sigma\) is a group action of \(G\) on \(N\) and \(\text{Inn} N \triangleleft \text{Aut} N\), it follows that \(h^{\gamma} \in \text{Aut} N\) can act on \(\text{Inn} N\) by conjugation. The above formula \((gh)^{\tau} = (g^{\tau})^{h^{\gamma}}h^\tau\) implies \(\tau \in \text{Der}_\rho(G, \text{Inn} N)\).

Conversely, for \(\tau \in \text{Der}_\rho(G, \text{Inn} N)\), if \(g^{\sigma'} = g^{\sigma}g^\gamma\), then it is easy to verify that \(\sigma' : G \to \text{Aut} N\) is a lifting homomorphism of \(\chi\), which shows that there exists a bijection between the set of all lifting homomorphisms of \(\chi\) and the set \(\text{Der}_\sigma(G, \text{Inn} N)\) for the fixed lifting homomorphism \(\sigma\).

(ii) Suppose that \(\sigma\) and \(\sigma'\) are equivalent. Then there exists a group homomorphism \(\gamma\) fitting into the following commutative diagram, that is, two corresponding semidirect product extensions are equivalent:

\[
\begin{array}{c}
E_{\sigma'} : & 1 & \longrightarrow & N & \longrightarrow & G \ltimes_{\sigma'} N & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & & & \uparrow \gamma & & & & \downarrow & \\
E_{\sigma} : & 1 & \longrightarrow & N & \longrightarrow & G \ltimes_{\sigma} N & \longrightarrow & G & \longrightarrow & 1 \\
\end{array}
\]
By the formula (1) with \( \sigma \) replaced by \( \sigma' \), we have \((g, a) = (g, 1)(1, a)\) for all \( g \in G \) and \( a \in N \). By the commutativity of the diagram (2), it follows that \((1, a)\gamma = (1, a)\) and \((g, 1)\gamma = (g, g^\mu)\), where \(g^\mu \in N\). Thus, \((g, a)\gamma = (g, g^\mu)\). Let \((h, b) \in G \rtimes_{\sigma'} N\). Then \((g, a)(h, b) = (gh, a^{\sigma'}b) \in G \rtimes_{\sigma'} N\), whose image under \( \gamma \) is \((gh, (gh)^{\sigma'}a^{\sigma'}b) \in G \rtimes_{\sigma'} N\). Calculating with the group operation on \( G \rtimes_{\sigma'} N \) again, we obtain \((g, g^\mu)(h, h^\mu b) = (gh, (gh)^{\sigma'}h^\mu b)\). Since \( \gamma \) is a homomorphism, we get 
\[(gh)^{\mu}a^{\sigma'} = (g^{\mu}a)^{h^\mu} = (g^{\mu})^{h^\mu}a^{h^\mu}.\] 
Obviously, \(1^\mu = 1\), and if we put \(g = 1\), then \(h^\mu a^{h^\sigma} = a^{h^\sigma}h^\mu\), namely \(a^{h^\sigma} = (h^\mu)^{-1}a^{h^\sigma}h^\mu = (a^{h^\sigma})^{h^\mu}\). A simple calculation yields \((gh)^\mu = (g^{\mu})^{h^\mu}h^\mu\), which shows that \( \mu \in \text{Der}_G(G, N) \). Clearly, \( \mu \) lifts \( \tau \).

Conversely, if \( \mu \in \text{Der}_G(G, N) \) is a lifting of \( \tau \), then \( \tau = \mu \sigma \) and hence \(a^{h^\sigma} = (a^{h^\sigma})^{h^\mu} = (a^{h^\sigma})^{h^\mu}\) for all \( h \in G \) and \( a \in N \). We let \((g, a)\gamma = (g, g^\mu a)\). Then it is easy to verify that \( \gamma \) is a group homomorphism and makes the diagram (2) commute. This shows that two semidirect product extensions \( \mathcal{E}_{\sigma'} \) and \( \mathcal{E}_\sigma \) are equivalent and hence \( \sigma \sim \sigma' \).

\[\square\]

### 2 Maximal Lifting Homomorphisms

We write \( G_0 \) for the image of \( \chi \). Let \( \chi = \chi_{0} : \chi_{0} : G \to G_{0} \) is an epimorphism and \( \iota : G_{0} \to \text{Out} N \) an inclusion mapping. Clearly, \( g^\chi = g^{\chi_{0}} \) for all \( g \in G \).

If \( \sigma : G \to \text{Aut} N \) is a lifting homomorphism of \( \chi \), then \( g^\chi = (g^\chi)^{\sigma} \) for all \( g \in G \), where \( \rho : \text{Aut} N \to \text{Out} N \) is the natural homomorphism. From \( g^\chi = 1 \) we have \( g^\chi = 1 \), so \( \text{Ker} \sigma \leq \text{Ker} \chi \). If \( \text{Ker} \sigma = \text{Ker} \chi \), then \( \sigma \) is called a \textbf{maximal lifting homomorphism} of \( \chi \); and the corresponding semidirect product extension \( \mathcal{E}_{\sigma} : 1 \to N \to G \rtimes_{\sigma} N \to G \to 1 \) is called a \textbf{maximal split extension}. More generally, if a group extension \( \mathcal{E} : 1 \to N \to E \to G \to 1 \) is equivalent to some maximal split extension \( \mathcal{E}_{\sigma} \), then \( \mathcal{E} \) is also called a maximal split extension.

When \( \chi : G \to \text{Out} N \) is monomorphic, every lifting homomorphism of \( \chi \) (if exists) is maximal. In general, any lifting homomorphism of \( \chi \) need not be maximal. For example, choose a group \( N \) such that \( \text{Aut} N \) is not split over \( \text{Inn} N \). Let \( G = \text{Aut} N \) and \( \chi : G \to \text{Out} N \) be the natural homomorphism. Then \( \chi \) can be lifted, but there is no maximal lifting of \( \chi \) by Theorem 1 below. In addition, even if there is a maximal lifting, not all lifting homomorphisms are maximal. For example, if \( \chi \) is trivial, then it can be lifted to the trivial homomorphism from \( G \) to \( \text{Inn} N \). In this case, we may take \( G \) to be a non-trivial \( p \)-group and \( N \) a non-abelian \( p \)-group for any prime \( p \), then there is a non-trivial homomorphism \( \sigma : G \to \text{Inn} N \). Clearly, \( \sigma \) is a lifting homomorphism of \( \chi \) and \( \text{Ker} \sigma < G = \text{Ker} \chi \), so \( \sigma \) is not maximal.

The significance of the maximal lifting homomorphisms is the decomposability. Suppose that \( \sigma : G \to \text{Aut} N \) is a group homomorphism. By definition, \( \sigma \) is a lifting homomorphism of \( \chi \) if and only if \( \chi = \sigma \rho \), that is, \( \chi_{0} = \sigma \rho \), which amounts to saying that \( \sigma \) fits into the following commutative diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\chi_{0}} & G_{0} \\
\sigma \downarrow & & \downarrow \iota \\
\text{Aut} N & \xrightarrow{\rho} & \text{Out} N
\end{array}
\]

(3)
If \( \sigma \) is a maximal lifting homomorphism of \( \chi \), then \( \sigma \rho = \chi_0 \iota \) and \( \text{Ker} \sigma = \text{Ker} \chi = \text{Ker} \chi_0 \) by definition. Notice that \( \chi_0 : G \to G_0 \) is surjective, so there exists a unique group homomorphism \( \lambda : G_0 \to \text{Aut} N \) such that \( \sigma = \chi_0 \lambda \). In this case, we have \( \chi_0 \iota \rho = \chi_0 \iota \) and hence \( \lambda \rho = \iota \). Conversely, suppose that there exists a group homomorphism \( \lambda \) satisfying \( \lambda \rho = \iota \). Set \( \sigma = \chi_0 \lambda \). Then \( \sigma \rho = \chi_0 \lambda \rho = \chi_0 \iota \), which implies that \( \sigma \) is a lifting homomorphism of \( \chi \). Further, from \( \sigma = \chi_0 \lambda \) we obtain \( \text{Ker} \chi = \text{Ker} \chi_0 \leq \text{Ker} \sigma \), which forces \( \text{Ker} \sigma = \text{Ker} \chi \). This shows that \( \sigma \) is exactly a maximal lifting homomorphism of \( \chi \). Thus, we have proved the following result:

**Lemma 1.** Let \( \sigma : G \to \text{Aut} N \) be a group homomorphism. Then \( \sigma \) is a maximal lifting homomorphism of \( \chi \) if and only if there exists a group homomorphism \( \lambda : G_0 \to \text{Aut} N \) such that \( \sigma = \chi_0 \lambda \) and \( \iota = \lambda \rho \). In particular, there is a bijection between the set \( \{ \sigma \} \) of all maximal lifting homomorphisms of \( \chi \) and the set \( \{ \lambda \} \) of all lifting homomorphisms of \( \iota \).

The following theorem is a criterion for the existence of a maximal lifting. For convenience, we write \( X \) for the full inverse image of \( G_0 \) under \( \rho : \text{Aut} N \to \text{Out} N \). So we have the group extension \( \mathcal{E}^* : 1 \to \text{Inn} N \to X \to \text{Im} \chi \to 1 \).

**Theorem 1.** Keep the above notation.

(i) There exists a maximal lifting homomorphism of \( \chi \) if and only if \( \mathcal{E}^* \) splits.

(ii) A lifting homomorphism \( \sigma \) of \( \chi \) is maximal if and only if \( \text{Im} \sigma \cap \text{Inn} N = 1 \).

**Proof.** (i) By [2, Proposition 1], it is easy to verify that the above \( \mathcal{E}^* \) is the associated group extension with \( 1 \to \text{Inn} N \to \text{Aut} N \to \text{Out} N \to 1 \). So \( \iota \) can be lifted to \( \text{Aut} N \) if and only if \( \mathcal{E}^* \) splits. This is equivalent to saying that \( \chi \) has a maximal lifting homomorphism by Lemma 1.

(ii) Let \( \sigma \) be a maximal lifting homomorphism of \( \chi \). By definition, \( \text{Ker} \chi = \text{Ker} \sigma \). For any \( \alpha \in \text{Im} \sigma \cap \text{Inn} N \), we may write \( \alpha = g^\sigma \) for some \( g \in G \). From \( \text{Ker} \rho = \text{Inn} N \) and \( \chi = \sigma \rho \) we have \( 1 = \alpha^\rho = g^{\sigma \rho} = g^\chi \) and hence \( g \in \text{Ker} \chi = \text{Ker} \sigma \). Therefore, \( \alpha = g^\sigma = 1 \). It follows that \( \text{Im} \sigma \cap \text{Inn} N = 1 \).

Conversely, suppose \( \text{Im} \sigma \cap \text{Inn} N = 1 \). Then we have \( 1 = g^\chi = g^{\sigma \rho} \) for any \( g \in \text{Ker} \chi \), so \( g^\sigma \in \text{Ker} \rho = \text{Inn} N \). But \( g^\sigma \in \text{Im} \sigma \), which forces \( g^\sigma = 1 \) and \( g \in \text{Ker} \sigma \). Therefore, \( \text{Ker} \chi \leq \text{Ker} \sigma \), namely \( \text{Ker} \chi = \text{Ker} \sigma \). Hence, \( \sigma \) is a maximal lifting homomorphism of \( \chi \). \( \square \)

Next we discuss when a given split extension is maximal from pure group theory.

**Theorem 2.** If \( \mathcal{E} : 1 \to N \to E \xrightarrow{\iota} G \to 1 \) is a split group extension, regarding \( N \leq E \), then \( \mathcal{E} \) is maximal if and only if there exists a complement subgroup \( H \) of \( N \) in \( E \) such that \( C_E(N) = Z(N) \times C_H(N) \).

**Proof.** Since \( \mathcal{E} \) splits, there exists a group homomorphism \( \lambda : G \to E \) such that \( \lambda \iota = 1 \). Let \( H = G^\lambda \). Then \( \lambda \) gives rise to a group action \( \sigma : G \xrightarrow{\lambda} H \xrightarrow{\iota} \text{Aut} N \) of \( G \) on \( N \), namely \( \sigma = \lambda \tau \) is a composition homomorphism, where \( \tau \) is the action of \( H \) on \( N \) given by conjugation. By Theorem 1, \( \mathcal{E} \) is a maximal split extension if and only if \( \text{Im} \sigma \cap \text{Inn} N = H^\tau \cap \text{Inn} N = 1 \), that is, whenever \( h \in H \) with \( h^\tau \in \text{Inn} N \) we must have \( h \in \text{Ker} \tau = C_H(N) \). Notice that \( h^\tau \in \text{Inn} N \) if and only if there exists
some \( n \in N \) such that \( a^h = a^n \) for all \( a \in N \), that is, \( hn^{-1} \in C_E(N) \). Therefore, \( \mathcal{E} \) is a maximal split extension if and only if whenever \( hn^{-1} \in C_E(N) \) for \( h \in H \) and \( n \in N \), then \( h \in C_H(N) \) and hence \( n \in Z(N) \). Clearly, \([Z(N), C_H(N)] = 1\) and \( Z(N) \cap C_H(N) = 1\) and the result follows.

Finally, we consider the connection between split \( \chi \)-extensions and split \( \iota \)-extensions. Write \( \text{Ext}_\chi(G, N) \) for the set of equivalence classes of extensions of \( N \) by \( G \) with \( \chi \), and \( \text{Ext}_\iota(G_0, N) \) the set of equivalence classes of extensions of \( N \) by \( G_0 \) with \( \iota \).

If \( \mathcal{E}_0 : 1 \to N \to E_0 \to G_0 \to 1 \) is an \( \iota \)-extension, then by [2, Proposition 1], there exists a unique \( \chi \)-extension (up to equivalence) \( \mathcal{E} : 1 \to N \to E \to G \to 1 \) such that the following diagram

\[
\begin{array}{cccccc}
\mathcal{E} : & 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_0 : & 1 & \longrightarrow & N & \longrightarrow & E_0 & \longrightarrow & G_0 & \longrightarrow & 1 \\
\end{array}
\]

is commutative.

Replacing \( \mathcal{E}_0 \) by an \( \iota \)-extension equivalent to \( \mathcal{E}_0 \), the equivalence class \([\mathcal{E}]\) does not change, where \([\mathcal{E}]\) is the equivalence class containing \( \mathcal{E} \). From this we define a mapping \( \chi^*: \text{Ext}_\iota(G_0, N) \to \text{Ext}_\chi(G, N) \) given by \([\mathcal{E}_0] \mapsto [\mathcal{E}] \). We shall describe the image of the split \( \iota \)-extension under \( \chi^* \). Notice that \( \iota : G_0 \hookrightarrow \text{Out} N \) is injective. So all the lifting homomorphisms of \( \iota \) are maximal, which implies that \( \mathcal{E}_0 \) splits if and only if \( \mathcal{E}_0 \) is a maximal split extension.

**Theorem 3.** In the diagram (4), if \( \mathcal{E}_0 \) splits, then \( \mathcal{E} \) is a maximal split extension. As \([\mathcal{E}_0]\) runs through all equivalence classes of split \( \iota \)-extensions, \([\mathcal{E}]\) can run through those equivalence classes each of which contains a maximal split \( \chi \)-extension.

**Proof.** If \( \mathcal{E}_0 \) splits, then there exists a group action \( \lambda : G_0 \to \text{Aut} N \) of \( G_0 \) on \( N \) such that \( \mathcal{E}_0 \) is equivalent to the semidirect product extension \( 1 \to N \to G_0 \rtimes \lambda N \to G_0 \to 1 \). In this case, \( \lambda \) is obviously a lifting homomorphism of \( \iota : G_0 \hookrightarrow \text{Out} N \), that is, \( \iota = \lambda \rho \). Without loss of generality, we may replace \( E_0 \) by the semidirect product \( G_0 \rtimes \lambda N \) in the diagram (4). Let \( \sigma = \chi \lambda : G \to \text{Aut} N \). Using the notation in the diagram (3), we have \( \sigma \rho = (\chi \lambda) \rho = \chi (\lambda \rho) = \chi \iota \), which implies that \( \sigma \) is a maximal lifting homomorphism of \( \chi \). If \( \mathcal{E}' \) is the corresponding semidirect product extension \( 1 \to N \to G \rtimes \rho N \to G \to 1 \), then \( \mathcal{E}' \) is the maximal split \( \chi \)-extension. Define a mapping \( \gamma : G \rtimes \rho N \to G_0 \rtimes \lambda N \) given by \((g, a) \mapsto (g^\lambda, a)\) for all \( g \in G \) and \( a \in N \). It is straightforward to verify that \( \gamma \) is a homomorphism and makes the diagram (4) commute. By [2, Proposition 1], it follows that \( \mathcal{E}' \) and \( \mathcal{E} \) are equivalent. By definition, \( \mathcal{E} \) is exactly a maximal split \( \chi \)-extension.

Conversely, suppose that \( \mathcal{E} : 1 \to N \to E \to G \to 1 \) is a maximal split \( \chi \)-extension. By definition, we may replace \( \mathcal{E} \) by the semidirect product extension \( \mathcal{E}_\sigma : 1 \to N \to G \rtimes \sigma N \to G \to 1 \), where \( \sigma \) is a maximal lifting homomorphism of \( \chi \). So we may assume that \( \mathcal{E} = \mathcal{E}_\sigma \). By the diagram (3) again, there is a group homomorphism \( \lambda : G_0 \to \text{Aut} N \) such that \( \sigma = \chi \lambda \) and \( \lambda \rho = \iota \), which implies that
λ is a lifting homomorphism of \( \iota : G_0 \hookrightarrow \text{Out} N \). Now we have the corresponding semidirect product extension \( E_0 : 1 \to N \to G_0 \rtimes_\lambda N \to G_0 \to 1 \).

By the construction of the above homomorphism \( \gamma \), it follows that \( E \) and \( E_0 \) fit into the diagram (4), which implies that the image of \( [E_0] \) under \( \chi^* \) is exactly \( [E] \). This completes the proof.

It should be pointed out that if \( \sigma \) and \( \sigma' \) are lifting homomorphisms of \( \chi \) such that \( \sigma \sim \sigma' \) and that \( \sigma \) is a maximal lifting of \( \chi \), then \( \sigma' \) is clearly not a maximal lifting of \( \chi \). For instance, take \( \chi \) and \( \sigma \) to be trivial homomorphisms. It is obvious that \( \sigma \) is a maximal lifting homomorphism of \( \chi \). By Proposition 1, \( \sigma' \sim \sigma \) if and only if \( \text{Im} \sigma' \subseteq \text{Inn} N \). If we choose groups \( G \) and \( N \) such that \( \sigma' : G \to \text{Inn} N \) is non-trivial, then \( \sigma \sim \sigma' \) and \( \text{Ker} \sigma' \neq \text{Ker} \chi = G \), which implies that \( \sigma' \) is not maximal.

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