Generalized atmospheric turbulence MTF for wave propagating through non-Kolmogorov turbulence

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Abstract: A generalized exponential spectrum model is derived, which considers finite turbulence inner and outer scales and has a general spectral power law value between the range 3 to 5 instead of standard power law value 11/3. Based on this generalized spectrum model, a new generalized long exposure turbulence modulation transfer function (MTF) is obtained for optical plane and spherical wave propagating through horizontal path in weak fluctuation turbulence. When the inner scale and outer scale are set to zero and infinite, respectively, the new generalized MTF is reduced to the classical generalized MTF derived from the non-Kolmogorov spectrum.

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References and links
1. Introduction

Atmospheric turbulence has a significant degrading impact on the quality of imaging systems, and this can be described by the MTF. Traditionally, in imaging systems, the long exposure MTF and the short-exposure MTF are derived with the assumption that the turbulence is Kolmogorov type and the Kolmogorov spectrum is applied for mathematical simplicity [1]. However, the Kolmogorov spectrum is only effective in the inertial subrange. When it is used to estimate the MTF of the imaging system, it is extended to all ranges by assuming the inner scale size is zero and the outer scale is infinite. This assumption may lead to divergent integrals in some cases [2]. So some turbulence spectrums with specific inner and outer scale were used to estimate the MTF [2].

In the past decade, both the experimental data and theoretical investigations [3–6] have exhibited non-Kolmogorov turbulence case in certain portions of the atmosphere. The non-Kolmogorov spectrum with variable power law value between the range 3 to 5 instead of standard power law value 11/3 for Kolmogorov turbulence and a more general amplitude factor instead of constant value 0.033 has been proposed, and applied to obtain the generalized MTF [7,8]. Still it has the same problem as the Kolmogorov spectrum.

In this study, the exponential spectrum model (with both finite inner scale and outer scale) [2] is generalized to apply in non-Kolmogorov atmospheric turbulence. Based on the new generalized exponential spectrum model, a new generalized long exposure turbulence MTF is derived for optical plane and spherical wave propagating through horizontal path in weak fluctuation non-Kolmogorov turbulence. And then the influences of turbulence inner scale, outer scale and spectral power law’s variations on the atmospheric turbulence MTF are analyzed.

2. Generalized exponential spectrum

2.1 Exponential spectrum

Exponential spectrum is a turbulence spectrum model which considers the influence of finite inner and outer scales, and is given by [2]

\[
\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}\exp\left(-\frac{\kappa^2}{\kappa_0^2}\right) \left[1 - \exp\left(-\frac{\kappa^2}{\kappa_0^2}\right)\right] \quad (0 \leq k < \infty) \tag{1}
\]

Where \(C_n^2\) represents the refractive-index structure parameter for Kolmogorov turbulence and has the unit of \(m^{-2/3}\), \(\kappa\) is the spatial wave number, \(\kappa_0 = 5.92/l_0\), \(\kappa_0 = C_0/L_0\), \(l_0\) and \(L_0\) are the turbulence inner and outer scale, \(C_0\) is chosen differently depending on the application, and because the outer scale itself is not well defined, it is difficult to proclaim any particular constant \(C_0\) with the outer scale parameter \(\kappa_0\). In this study, we set \(C_0 = 4\pi\) just as in [2].

In order to include both inner scale and outer scale’s effects, the generalized spectrum recently adopted has the form as [8,9]

\[
\Phi_n(\kappa, \alpha) = A(\alpha) \cdot \tilde{C}_n^2 \cdot F(k_l, l_0, \alpha) \quad (0 \leq \kappa < \infty, \ 3 < \alpha < 5) \tag{2}
\]

Where \(A(\alpha)\) is a constant which maintains consistency between the refractive index structure function and its power spectrum, \(\tilde{C}_n^2\) is the generalized refractive-index structure parameter with unit \(m^{3-\alpha}\), \(F(k_l, l_0, \alpha)\) is the function which includes the influence of finite inner and/or outer scale, and has different forms for different spectrums [8,9].

To generalize the exponential spectrum model to apply in non-Kolmogorov atmospheric turbulence, the exponential spectrum model should take the form as
\[
\Phi_n(\kappa, \alpha, l_0, L_0) = \hat{A}(\alpha) \cdot \hat{C}_n^2 \cdot \kappa^{-\alpha} \cdot f(k, l_0, L_0, \alpha) \quad (0 \leq \kappa < \infty, \quad 3 < \alpha < 5)
\]  
(3)

\[
f(k, l_0, L_0, \alpha) = \exp\left(-\frac{\kappa^2}{\kappa_0^2}\right) \left[1 - \exp\left(-\frac{\kappa^2}{\kappa_0^2}\right)\right]
\]  
(4)

Where \(\hat{A}(\alpha)\) has the same meaning as \(A(\alpha)\), \(\kappa_i = c(\alpha) / l_0\), \(\kappa_0 = C_0 / L_0\), \(c(\alpha)\) is the scaling constant. In the next section, the expression forms of \(\hat{A}(\alpha)\) and \(c(\alpha)\) would be derived.

2.2 Refractive-index structure function for generalized exponential spectrum model

One important parameter to infer the generalized exponential spectrum is the refractive-index structure function \(D_n(R, \alpha)\), which describes the behavior of the correlations of turbulence refractive index filed fluctuations between two given points separated by a distance \(R\). For Kolmogorov turbulence, the relationship between \(D_n(R)\) and \(\Phi_n(\kappa)\) can be described as [1]

\[
D_n(R) = 8\pi \int_0^\infty \kappa^2 \cdot \Phi_n(k) \left(1 - \frac{\sin \kappa R}{\kappa R}\right) d\kappa
\]  
(5)

For non-Komogorov turbulence, the relationship between \(D_n(R, \alpha)\) and \(\Phi_n(\kappa)\) is

\[
D_n(R, \alpha) = 8\pi \int_0^\infty \kappa^2 \cdot \Phi_n(k, \alpha) \left(1 - \frac{\sin \kappa R}{\kappa R}\right) d\kappa
\]  
(6)

Substituting Eq. (3) into Eq. (6), and set \(L_0\) to infinite (this will be explained later), the following expression is derived:

\[
D_n(R, \alpha) = 8\pi \int_0^\infty \kappa^2 \cdot \hat{A}(\alpha) \cdot \hat{C}_n^2 \cdot \exp\left(-\frac{\kappa^2}{\kappa_0^2}\right) \left(1 - \frac{\sin \kappa R}{\kappa R}\right) d\kappa
\]  
(7)

Expanding \(1 - \frac{\sin \kappa R}{\kappa R}\) with Maclaurin series [10]:

\[
1 - \frac{\sin \kappa R}{\kappa R} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!} \kappa^{2n} R^{2n}
\]  
(8)

and inserting it into Eq. (7), then interchanging the order of series summation and integration, as a result, Eq. (7) becomes:

\[
D_n(R, \alpha) = 8\pi \cdot \hat{A}(\alpha) \cdot \hat{C}_n^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!} R^{2n} \int_0^\infty \kappa^{2-\alpha+2n} \exp\left(-\frac{\kappa^2}{\kappa_0^2}\right) d\kappa
\]  
(9)

In view of the definition of gamma function \(\Gamma(x)\) and the hypergeometric function \(_1F_1(a; b; z)\) [10]:

\[
\Gamma(x) = \int_0^\infty \kappa^{x-1} e^{-\kappa} d\kappa \quad (\kappa > 0, x > 0), \quad _1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n \cdot z^n}{(b)_n \cdot n!}
\]  
(10)

Where \((a)_n\) is the Pochhammer Symbol and has the form
\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)
\] (11)

The integration and series summation in Eq. (9) is solved, as a result, the refractive-index structure function can be expressed as:

\[
D_n(R,\alpha) = 4\pi \hat{A}(\alpha) \hat{\zeta}^{3-\alpha} \cdot \left[ \Gamma\left(-\frac{\alpha}{2} + \frac{3}{2}\right) \left( 1 - {}_1F_1\left(-\frac{\alpha}{2} + \frac{3}{2}; 2; -\frac{R^2\kappa^2_i}{4}\right) \right) \right]
\] (12)

For statistically homogeneous and isotropic fluctuation non-Komogorov atmospheric turbulence, the related refractive-index structure function is given by [2,9]

\[
D_n(R,\alpha) = \begin{cases} 
\hat{C}_n^2 l_0^{n-3} R^2 & 0 \leq R \ll l_0 \\
\hat{C}_n^2 R^{\alpha-3} & l_0 \ll R \ll L_0
\end{cases}
\] (13)

It needs to say that because the random field of index refractive fluctuation is nonisotropic for scale sizes larger than outer scale \(L_0\), no general description of the refractive-index structure function can be predicted for \(R > L_0\). So, in the derivation of refractive-index structure function for generalized modified atmospheric spectrum, \(L_0\) is set to infinite for calculation purpose [2,9].

Using Eq. (12) and Eq. (13), the unknown \(\hat{A}(\alpha)\) and \(c(\alpha)\) in Eq. (3) will be derived next.

2.2.1 Expression derivation of \(\hat{A}(\alpha)\)

When \(l_0 \ll R \ll L_0\) then \(\frac{R^2\kappa^2_i}{4} = \frac{R^2 c^2(\alpha)}{4l_0^2} \gg 1\), the \( {}_1F_1(a;b;-x) \) in the Eq. (12) can be approximately expanded with big arguments and is given by [10]

\[
{}_1F_1(a;b;-x) = \frac{\Gamma(b)}{\Gamma(b-a)} x^{-a} \quad (x \gg 1)
\] (14)

Substituting Eq. (14) into Eq. (12), then \(D_n(R,\alpha)\) (see Eq. (12)) becomes

\[
D_n(R,\alpha) \approx -4\pi \hat{A}(\alpha) \hat{\zeta}^{3-\alpha} \cdot \left[ \Gamma\left(-\frac{\alpha}{2} + \frac{3}{2}\right) \left( 1 - \frac{3}{2} \right) \right] R^{\alpha-3} \quad \left( l_0 \ll R \ll L_0 \right)
\] (15)

Using Eq. (15) and Eq. (13), and considering the properties of gamma function [10]

\[
\Gamma(\alpha+1) = \alpha \Gamma(\alpha) \quad \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\pi \alpha)} \quad \Gamma(\alpha)\Gamma\left(\alpha + \frac{1}{2}\right) = 2^{1-\alpha} \sqrt{\pi} \Gamma(2\alpha)
\] (16)

As a result, \(\hat{A}(\alpha)\) can be expressed with the form

\[
\hat{A}(\alpha) = \frac{\Gamma(\alpha-1)}{4\pi^2} \sin\left(\frac{(\alpha-3)\pi}{2}\right)
\] (17)
It has the same expression as \( A(\alpha) \) in non-Kolmogorov spectrum model [7,8].

### 2.2.2 Expression derivation of \( c(\alpha) \)

When \( 0 \leq R \ll l_0 \) then \( \frac{R^2 k_j^2}{4} = \frac{R^2 c^2(\alpha)}{4 l_0^2} \ll 1 \), the \( \hat{F}_i(a;b;x) \) in the Eq. (12) can be approximately expanded with small arguments and is given by [10]

\[
\hat{F}_i(a;b;x) \approx \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} = 1 + \frac{a}{b} x \quad (x \ll 1)
\] (18)

Substituting Eq. (18) into Eq. (12), then \( D_n(R,\alpha) \) becomes

\[
D_n(R,\alpha) \approx \pi \hat{A}(\alpha) \hat{c}^2 k_j^{2-\alpha} R^2 \left[ \Gamma \left( \frac{\alpha}{2} + \frac{3}{2} \right) \left( \frac{3-\alpha}{3} \right) \right]^{\frac{1}{\alpha-3}} (0 \leq R \ll l_0)
\] (19)

Using Eq. (19) and Eq. (13), the expression of \( c(\alpha) \) can be derived:

\[
c(\alpha) = \left\{ \pi \hat{A}(\alpha) \left[ \Gamma \left( \frac{\alpha}{2} + \frac{3}{2} \right) \left( \frac{3-\alpha}{3} \right) \right]^{\frac{1}{\alpha-3}} \right\}^{-1}
\] (20)

### 2.3 Generalized exponential spectrum model

Substituting Eq. (17) and Eq. (20) into Eq. (3), we can obtain the expression of the generalized exponential spectrum model. The curves of the \( \hat{A}(\alpha) \) and \( c(\alpha) \) as the function of \( \alpha \) are plotted and shown in Fig. 1(a) and Fig. 1(b). When \( \alpha = 11/3 \), \( \hat{A}(11/3) \approx 0.033 \) and \( c(11/3) \approx 5.92 \), Eq. (3) is reduced to the exponential spectrum.

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![Fig. 1. \( A(\alpha) \) and \( c(\alpha) \) as a function of \( \alpha \). (a): \( A(\alpha) \), (b): \( c(\alpha) \)](image-url)

When inner scale is set to zero and outer scale is set to infinite, the generalized exponential spectrum becomes

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\[ \Phi_n(\kappa, \alpha) = \hat{A}(\alpha) \cdot \hat{C}_n^2 \cdot \kappa^{-\alpha} \quad (0 \leq \kappa < \infty, \quad 3 < \alpha < 5) \]  

(21)

Where \( \hat{A}(\alpha) \) has the same expression as \( A(\alpha) \) in non-Kolmogorov spectrum model, and that means the generalized exponential spectrum can be reduced to the classical non-Kolmogorov spectrum with the particular case of zero inner scale and infinite outer scale.

3. Generalized atmospheric turbulence MTF based on the generalized exponential spectrum

In this section, a new generalized atmospheric turbulence MTF model is derived with the generalized exponential spectrum for plane and spherical wave propagating through horizontal path in weak fluctuation non-Kolmogorov turbulence.

Following the analysis of Hufnagel [11] and Fried [12], the long-exposure weak fluctuation atmospheric turbulence MTF can be deduced from the formulation of the mutual coherence function (MCF) in the receiver aperture plane, and it is the product of the MCF evaluated at \( \rho = \lambda F \nu \). For the case of plane/spherical wave, the long exposure weak fluctuation turbulence MTF in the focal plane of the receiver is given by [12]

\[ MTF_{\text{long}}(\nu) = \exp \left[ -\frac{1}{2} D_{\text{ps}}(\lambda F \nu) \right] \]  

(22)

Where \( \rho \) is the separation between points in the image plane transverse to the direction, \( \nu \) is the spatial frequency measured in cycles per unit length, \( F \) is focal length. \( D_{\text{ps}}(\lambda F \nu) \) represents the wave structure function for plane/spherical wave, and it is the sum of the log-amplitude structure function and the phase structure function.

For weak, homogeneous, and isotropic turbulence, the method of small perturbations can be used to solve the wave propagation problem. The solution yields the wave structure function for plane and spherical waves from point object within the atmospheric turbulence [1]

\[ D_{\text{ps}}(\rho) = 8\pi^2 k^2 \int_0^\infty dz \int_0^\infty \left[ 1 - J_0(\kappa \rho) \right] \Phi_n(\kappa, z) \kappa d\kappa \]  

(23)

\[ D_{\text{ps}}(\rho) = 8\pi^2 k^2 \int_0^\infty dz \int_0^\infty \left[ 1 - J_0(\kappa \rho z / L) \right] \Phi_n(\kappa, z) \kappa d\kappa \]  

(24)

Where \( J_0 \) is Bessel function of the first kind and zero order. \( L \) is the optical path. \( D_{\text{ps}}(\rho) \) and \( D_{\text{ps}}(\rho) \) represents plane and spherical wave structure function, respectively.

It is noted that in the derivation of Eq. (22), Eq. (23) and Eq. (24), no particular assumption of Kolmogorov turbulence is needed, so the relationships between \( MTF(\cdot) \), \( D_{\text{ps}}(\cdot) \) and \( \Phi_n(\cdot) \) can still be applied in the non-Kolmogorov turbulence case.

In the next section, the expression forms of plane and spherical wave structure functions for non-Kolmogorov turbulence case will be derived, and subsequently the generalized MTF can be obtained. It is noted that in the following calculation, the range of \( \alpha \) is further restricted to the range 3 to 4 just as [7,8,13].

3.1 Plane wave structure function for non-Kolmogorov turbulence

For generalized exponential spectrum, Eq. (23) can be written in the form as
Substituting Eq. (3) into Eq. (25), and expanding \( J_0 \) in the form of Maclaurin series \[10\]

\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+1)} \left( \frac{x}{2} \right)^{2n}
\]

(26)

Where, \( \Gamma(n+1) \) in the above equation can be expressed with \( \Gamma(n+1) = n! = (1)_n \).

The wave structure function for plane wave becomes

\[
D_{\omega\rho}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \tilde{A}(\alpha) \tilde{C}_n^2 \cdot L \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \left( \frac{\rho^2}{4} \right)^{2n} \int_0^\infty \kappa^{2n-\alpha+1} f(\kappa, l_0, L_0, \alpha) d\kappa
\]

(27)

Interchanging the order of series summation and integration, and making integration with respect to \( \kappa \), as a result, Eq. (27) is expressed as

\[
D_{\omega\rho}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \tilde{A}(\alpha) \tilde{C}_n^2 L \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \left( \frac{\rho^2}{4} \right)^{2n} \int_0^\infty \kappa^{2n-\alpha+1} f(\kappa, l_0, L_0, \alpha) d\kappa
\]

(28)

To calculate for simplicity, \( f(\kappa, l_0, L_0, \alpha) \) is divided into two parts

\[
f_1(\kappa, l_0, L_0, \alpha) = \exp \left( -\frac{\kappa^2}{\kappa_i^2} \right), \quad f_2(\kappa, l_0, L_0, \alpha) = -\exp \left[ -\kappa^2 \left( \frac{1}{\kappa_0^2} + \frac{1}{\kappa_i^2} \right) \right]
\]

(29)

Then using the Eq. (10) for the calculation of integration and series summation, the plane wave structure function for non-Kolmogorov turbulence is finally derived

\[
D_{\omega\rho}(\rho, \alpha, l_0, L_0) = D_{\omega\rho_1}(\rho, \alpha, l_0, L_0) + D_{\omega\rho_2}(\rho, \alpha, l_0, L_0)
\]

(30)

Where \( D_{\omega\rho_1}(\rho, \alpha, l_0, L_0) \) and \( D_{\omega\rho_2}(\rho, \alpha, l_0, L_0) \) are obtained from \( f_1(\kappa, l_0, L_0, \alpha) \) and \( f_2(\kappa, l_0, L_0, \alpha) \) separately, and expressed with the forms as

\[
D_{\omega\rho_1}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \tilde{A}(\alpha) \tilde{C}_n^2 L \cdot \kappa_i^{2-\alpha} \left[ \frac{1}{2} \Gamma \left( -\frac{\alpha}{2} + 1 \right) \left( \frac{\alpha}{2} + 1 ; -\frac{\rho^2 \kappa_i^2}{4} \right) \right]
\]

(31)

\[
D_{\omega\rho_2}(\rho, \alpha, l_0, L_0) = -8\pi^2 k^2 \tilde{A}(\alpha) \tilde{C}_n^2 L \cdot \frac{1}{2} \kappa_i^{-\alpha-1} \Gamma \left( -\frac{\alpha}{2} + 1 \right) \left( \frac{\alpha}{2} + 1 ; -\frac{\rho^2 \kappa_i^2}{4} \right)
\]

(32)

Where, \( \kappa_i = \frac{1}{\kappa_0^2} + \frac{1}{\kappa_i^2} \).

Hitherto, the wave structure function for plane wave propagating through horizontal path has been obtained, and it contains finite inner and outer scale. To compare with the result derived from the non-Kolmogorov spectrum in which the inner scale is set to zero and outer scale is set to infinite for calculation purpose, we discuss the particular case of \( L_0 \to \infty \) and \( l_0 \to 0 \).
When \( L_0 \to \infty \) and \( l_0 \to 0 \), then \( \kappa_0 = \frac{C_0}{L_0} \to 0 \) and \( \kappa_1 = \frac{c(\alpha)}{l_0} \to \infty \), so \( \kappa_2 = \frac{1}{\kappa_0^2} + \frac{1}{\kappa_1^2} \to \infty \).

\[
\rho^2 \to 0, \quad \frac{\rho^2 \kappa_2}{4} \to \infty.
\]

The \( F_1(a;b;z) \) in Eq. (31) and Eq. (32) can be approximately expanded with big arguments (see Eq. (14)) and small arguments (see Eq. (18)), respectively. So the wave structure function can be approximately expressed as

\[
D_{\text{np}}(\rho, \alpha, l_0, L_0) \approx -2^{4-\alpha} \pi^2 k^2 \hat{A}(\alpha) C_0^2 L - \frac{\Gamma(-\alpha/2+1)}{\Gamma(\alpha/2)} \cdot \rho^{\alpha-2} \quad (3 < \alpha < 4) \tag{33}
\]

At this time, the \( D_{\text{np}}(\rho, \alpha, l_0, L_0) \) can be expressed with Eq. (33) which has the same form as the case derived from the non-Kolmogorov spectrum [13], and this means the plane wave structure function derived in this study can be reduced to the classical result under the special condition of zero inner scale and infinite outer scale.

### 3.2 Spherical wave structure function for non-Kolmogorov turbulence

For generalized exponential spectrum, Eq. (24) can be expressed as

\[
D_{\text{sn}}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \int_0^L dz \left[ 1 - J_0(\kappa \rho z / L) \right] \Phi_n(\kappa, \alpha, l_0, L_0) \kappa d\kappa \quad (3 < \alpha < 4) \tag{35}
\]

Substituting Eq. (3) into Eq. (35), and expanding \( J_0 \) in the form of Maclaurin series just as the Eq. (26), the wave structure function for spherical wave becomes

\[
D_{\text{sn}}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \hat{A}(\alpha) C_0^2 L \sum_{n=1}^\infty \frac{(-1)^n}{n!(1)_n} \left( \frac{\rho^2}{2L} \right)^{2n} \int_0^\infty \kappa^{2n-\alpha+1} f(\kappa, l_0, L_0, \alpha) d\kappa \tag{36}
\]

Interchanging the order of series summation and integration, and making the integrating with respect to \( z \), Eq. (36) becomes

\[
D_{\text{sn}}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \hat{A}(\alpha) C_0^2 L \sum_{n=1}^\infty \frac{(-1)^n}{n!(1)_n} \frac{1}{2n+1} \left( \frac{\rho^2}{2} \right)^{2n} \int_0^\infty \kappa^{2n-\alpha+1} f(\kappa, l_0, L_0, \alpha) d\kappa \tag{37}
\]

Where, \( \frac{1}{2n+1} \) in the above equation can be expressed with the form \( \frac{1}{2n+1} = \frac{(1/2)_n}{(3/2)_n} \).

To calculate for simplicity, \( f(\kappa, l_0, L_0, \alpha) \) is expressed with the sum of two parts just as Eq. (29), then considering the definition of \( \Gamma(x) \) function (see Eq. (10)) and the generalized hypergeometric function \( \text{}_{2}F_1(a,b;c;d;z) \) which is defined as [10]:

\[
\text{}_{2}F_1(a,b;c;d;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n (d)_n} \frac{z^n}{n!}
\]

As a result, the spherical wave structure function for non-Kolmogorov turbulence can be expressed as
\[ D_{\text{sw}}(\rho, \alpha, l_0, L_0) = D_{\text{sw}}(\rho, \alpha, l_0, L_0) + D_{\text{sw}}(\rho, \alpha, l_0, L_0) \]  
(39)

Where \( D_{\text{sw}}(\rho, \alpha, l_0, L_0) \) and \( D_{\text{sw}}(\rho, \alpha, l_0, L_0) \) are obtained from the part of \( f_1(\kappa, l_0, L_0, \alpha) \) and \( f_2(\kappa, l_0, L_0, \alpha) \) separately, and expressed with the forms as

\[
D_{\text{sw}}(\rho, \alpha, l_0, L_0) = 8\pi^2 k^2 \hat{A}(\alpha) \tilde{C}_\alpha^2 L \cdot \kappa^{-2a} \left\{ \frac{1}{2} \Gamma \left( -\frac{\alpha}{2} + 1 \right) \left[ 1 - F_2 \left( \frac{-\alpha}{2} + 1, \frac{3}{2}; \frac{\rho^2}{4} \right) \right] \right\} 
\]
(40)

\[
D_{\text{sw}}(\rho, \alpha, l_0, L_0) = -8\pi^2 k^2 \hat{A}(\alpha) \tilde{C}_\alpha^2 L \cdot \left\{ \frac{1}{2} \kappa^{-2} \Gamma \left( -\frac{\alpha}{2} + 1 \right) \left[ 1 - F_2 \left( \frac{-\alpha}{2} + 1, \frac{3}{2}; \frac{\rho^2}{4} \right) \right] \right\} 
\]
(41)

Hitherto, the wave structure function for spherical wave propagating through horizontal path has been obtained, and it contains finite inner and outer scale. To compare with the result derived from the non-Kolmogorov spectrum in which the inner scale is set to zero and outer scale is set to infinite for calculation purpose, we discuss the particular case of \( L_0 \to \infty \) and \( l_0 \to 0 \).

Similar to the discussion in section 3.2, here it needs to consider the approximate expansions of \( F_2(a, b; c; d; z) \) with big arguments and small arguments, and they are given by [10]:

\[
F_2(a, b; c; d; -x) \approx \frac{\Gamma(c) \Gamma(d) (b - a)}{\Gamma(b) \Gamma(c - a) \Gamma(d - a)} x^a + \frac{\Gamma(c) \Gamma(d) (a - b)}{\Gamma(a) \Gamma(c - b) \Gamma(d - b)} x^b \quad (x \gg 1) \quad (42)
\]

\[
F_2(a, b; c; d; z) \approx \sum_{n=0}^{1} \frac{(a)_n (c)_n}{(b)_n (d)_n} \cdot z^n = 1 + \frac{ab}{cd} x \quad (x \ll 1) \quad (43)
\]

Substituting Eq. (42) and Eq. (43) into the Eq. (40) and Eq. (41), the spherical wave structure function for non-Kolmogorov turbulence can be approximately expressed as

\[
D_{\text{sw}}(\rho, \alpha, l_0, L_0) \approx \frac{1}{\alpha - 1} \left\{-2^{1-a} \pi^2 k^2 \hat{A}(\alpha) \tilde{C}_\alpha^2 L \cdot \frac{\Gamma(\alpha/2 + 1)}{\Gamma(\alpha/2)} \cdot \rho^{a-2} \right\} \quad (3 < \alpha < 4) \quad (44)
\]

\[
D_{\text{sw}}(\rho, \alpha, l_0, L_0) \approx 0 \quad (3 < \alpha < 4) \quad (45)
\]

At this time, the \( D_{\text{sw}}(\rho, \alpha, l_0, L_0) \) can be expressed with Eq. (44) which has the same form as the case derived from the non-Kolmogorov spectrum [13], and this means the spherical wave structure function derived in this study can also be reduced to the classical result under the special condition of zero inner scale and infinite outer scale.

3.3 Generalized atmospheric turbulence MTF for plane wave and spherical wave

Taking into account Eq. (30) and Eq. (39) for plane and spherical wave structure function for wave propagating through horizontal path in weak fluctuation non-Kolmogorov turbulence, the generalized form of the atmospheric turbulence MTF can be expressed as

\[
\text{MTF}_{\text{atm}(p)}(u, \alpha, l_0, L_0) = \exp \left[ -\frac{1}{2} D_{\text{sw}}(uD, \alpha, l_0, L_0) \right] \quad (3 < \alpha < 4) \quad (46)
\]
\[ MTF_{\text{turb}(p)}(u, \alpha, l_0, L_0) = \exp\left[ -\frac{1}{2} D_{\text{eq}}(uD, \alpha, l_0, L_0) \right] \quad (3 < \alpha < 4) \] (47)

Where \( MTF_{\text{turb}(p)}(u, \alpha, l_0, L_0) \) and \( MTF_{\text{turb}(s)}(u, \alpha, l_0, L_0) \) represent the MTF for plane wave and spherical wave propagating through horizontal path in weak non-Kolmogorov atmospheric turbulence, respectively. \( u = \frac{\lambda F \nu}{D} \) denotes the normalized spatial frequency, and it is between 0 and 1. \( D \) is the receiver aperture diameter.

In practice, if we further consider the influence of receiver, atmospheric particulates (aerosol, dust, etc.), the total long-exposure MTF of propagation path up to detector of imaging system can be described as the product [7]

\[ MTF_{\text{total}}(u) = \frac{2}{\pi} \left[ \cos^{-1} u - u \sqrt{1 - u^2} \right] \times MTF_{\text{turb}}(u) \times MTF_{\text{aerosol}}(u) \] (48)

Where \( \frac{2}{\pi} \left[ \cos^{-1} u - u \sqrt{1 - u^2} \right] \) is the MTF of the receiver optics, \( MTF_{\text{aerosol}}(u) \) represents the MTF caused by atmospheric particulates (aerosol, dust, etc.) scattering.

In this study, we focus on the influence of atmospheric turbulence on the imaging system, that is \( MTF_{\text{turb}}(u) \).

### 4. Numerical results

In this section, simulations are conducted to analyze the generalized atmospheric turbulence MTF as a function of normalized spatial frequency. Because the MTF contains three changeable parameters: \( \alpha \), \( l_0 \) (turbulence inner scale) and \( L_0 \) (turbulence outer scale), in order to avoid the mutual interferences between parameters, in the following simulations, we fix two of the three parameters at a time and analyze only one parameter’s influence on the MTF. All the experiments are conducted for plane wave and spherical wave propagating in horizontal path with the settings:

\[ \tilde{C}_a^2 = 1.6 \times 10^{-14} \text{m}^{3/4} \text{a} \land \lambda = 1.55\mu \text{m} \land L = 1000\text{m} \land D = 0.1\text{m} \]

#### 4.1 Effect of outer scale’s variation on the MTF

In the first simulation experiment, we fix \( \alpha \) and \( l_0 \) to constant values, and choose different turbulence outer scale sizes to analyze the influence on MTF. To well compare with the classical MTF derived from the non-Kolmogorov spectrum in which \( l_0 \) equals zero and \( L_0 \) is infinite for calculation purpose, \( l_0 \) is set to a very small value of \( 0.01 \times (\lambda L)^{3/2} \) and \( \alpha \) is set to a constant value of 11/3. Then outer scale \( L_0 \) is set to 5m, 50m and 500m, respectively, and the corresponding atmospheric turbulence MTFs are plotted in Fig. 2. To further analyze the impact of \( L_0 \)'s variation on MTF, we also exhibit the classical generalized MTF derived from non-Kolmogorov spectrum.

As shown in Fig. 2, with the increase of the turbulent outer scale, the value of MTF decreases (it means the quality of imaging system is degraded more severely by the turbulence), and especially when \( L_0 \) is set to a very big value (here it equals 500m), at this time, the result is very close to the classical non-Kolmogorov case (with infinite outer scale). This can be explained directly from the function \( f(\kappa, \alpha, l_0, L_0) \) in Eq. (4). When \( L_0 \) increases, \( f(\kappa, \alpha, l_0, L_0) \) raises, that makes the plane and spherical wave structure function in Eq. (25) and Eq. (35) increases, so the final MTF decreases.
This can also be interpreted from another point: big turbulent cells \( (> (\lambda L)^{1/2}) \) mainly bring the variation of wave phase [14], and it can be expressed with phase structure function which is the dominant component in the wave structure function [15]. So when the \( L_0 \) is assumed high value the wave meets a major number of large-scale turbulent cells along its propagation length and these cells leads to higher phase wave structure function with respect to the case of low outer scale value (where more large scale cells are cut out) [9], and this makes the plane and spherical wave structure function increases, subsequently the final MTF decreases.

4.2 Effect of inner scale’s variation on MTF

To analyze turbulence inner scale’s influence on MTF, \( \alpha \) and \( L_0 \) are fixed to constant values, and different inner scale sizes are chosen. In order to alleviate the other two parameters’ interference, \( \alpha \) is chosen to be 11/3 and \( L_0 \) is fixed to a very big value of 5000 m. Then the inner scale \( l_0 \) is set to \( 0.1 \times (\lambda L)^{1/2}, (\lambda L)^{1/2} \) and \( 10 \times (\lambda L)^{1/2} \), respectively, and the corresponding MTFs as a function of normalized spatial frequency are plotted in Fig. 3. To
further analyze this simulation, we also exhibit the classical generalized MTF derived from
conventional non-Kolmogorov spectrum.

From Fig. 3, we can infer that when the inner scale is much smaller than the first Fresnel
zone, that is $l_0 \ll (\lambda L)^{1/2}$, its influence on the final form of the MTF can be ignored compared
with the classical result. That is because the log-amplitude structure function has a correlation
distance of the order of $(\lambda L)^{1/2}$, and it is primarily affected by the turbulence cells with the
size of $(\lambda L)^{1/2}$ [14]. When $l_0 \ll (\lambda L)^{1/2}$, the number of turbulent cells with the size of
$(\lambda L)^{1/2}$ almost keeps parallel with the classical case which assumes zero inner scale,
therefore, log-amplitude structure function will not be changed. Meanwhile, the outer scale
$L_0$ is fixed to a very big value in this simulation, which makes the phase structure function
nearly the same as the classical case which takes infinite outer scale. All these make the wave
structure function and the MTF unchanged compared with the result derived from non-
Kolmogorov spectrum.

However, when the finite inner scale is much larger than the Fresnel zone, that is
$l_0 \gg (\lambda L)^{1/2}$, the log-amplitude structure function is nearly ignorable and the number of big
turbulent cells $(\gg (\lambda L)^{1/2})$ that leads to high phase structure function becomes less. This
makes the wave structure function decreases and the MTF increases just as shown in Fig. 3.
At this time, the influence of finite inner scale on the MTF should be considered theoretically.
However, in practice, the inner scale is commonly in the unit of millimeter and does not
satisfy $l_0 \gg (\lambda L)^{1/2}$.
4.3 Effect of $\alpha$’s variation on MTF

To analyze $\alpha$’s influence on MTF, $l_0$ is fixed to a constant value of $0.1 \times (\lambda L)^{1/2}$ and $L_0$ is set to 5000m. So, we can only focus on the influence of $\alpha$’s changes on MTF. In this simulations, $\alpha$ is set to 10/3, 11/3 and 3.9 respectively, and the corresponding MTFs as a function of normalized spatial frequency are plotted in Fig. 4, from which we can infer that various $\alpha$ brings different impact on the form of MTF just as in [7,8].

As shown in Fig. 4, for helical turbulence, $\alpha = 10/3$, compared with the Kolmogorov case, the turbulence brings more degrading on the imaging in the low frequency and less degrading in the high frequency, and this is consistent with [7], although the latter describes the condition of slant path propagation. While, for the power law value higher than the Kolmogorov case, in this simulation, $\alpha = 3.9 > 11/3$, the opposite trend appears just as [7].
5. Conclusions

In this study, a theoretical generalized exponential spectrum to apply in non-Kolmogorov atmospheric turbulence is derived, which considers finite inner scale, finite outer scale and general power law values. Based on this atmospheric spectrum model, a new generalized atmospheric turbulence long exposure MTF model is developed for plane wave and spherical wave propagating through non-Kolmogorov atmospheric turbulence medium for a horizontal path. When the inner scale and outer scale are set to zero and infinite, respectively, the new generalized MTF is reduced to the classical generalized MTF derived from the non-Kolmogorov spectrum.

Simulation results show that finite outer and inner scale does introduce influences on the MTF for plane wave and spherical wave. The former alleviates the influence of turbulence on the imaging system especially in high spatial frequency than the case derived from non-Kolmogorov spectrum, and the inner scale with very small value (here it is set to $l_i \ll (\lambda L)^{1/2}$) has ignorable influence on the MTF compared with the classical result, but as it continues to increase, the effect on the propagating needs to be considered. Different power law values also produces different impacts on imaging system, and this conclusion is consist with [7].
The results will help to better study the effects of turbulence on the optical plane wave and spherical wave propagating through horizontal path.

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