New passivity conditions with fewer slack variables for uncertain neural networks with mixed delays

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This paper introduces an effective approach to studying the passivity of neutral-type neural networks with discrete and continuous distributed time-varying delays. By employing a novel Lyapunov–Krasovskii functional based on delay partitioning, several improved delay-dependent passivity conditions are established to guarantee the passivity of uncertain neural networks by applying the Jensen integral inequality. These criteria are expressed in the framework of linear matrix inequalities, which can be verified easily by means of standard Matlab software. One special case of the obtained criteria turns out to be equivalent to some existing result with same reduced conservatism but including fewer slack variables. As the present passivity conditions involve fewer free-weighting matrices, the computational burden is largely reduced. Three examples are provided to demonstrate the advantage of the theoretical results.

1. Introduction

Neural networks have been extensively studied over the past few decades and have found many applications in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, and combinatorial optimization. Although considerable effort has been devoted to analyzing the stability of neural networks without a time delay, in recent years, the stability of delayed neural networks has also received attention since time delay is frequently encountered in neural networks, and it is often a source of instability and oscillation in a system. Generally speaking, the stability criteria for delayed neural networks can be classified into two categories, namely, delay-independent [18,19,21–25] and delay-dependent [8,20,26,28]. Since delay-independent criteria tend to be conservative, especially when the delay is small, much attention has been paid to the delay-dependent type.

The passivity theory has long been a nice tool for analyzing the stability of systems, which has been applied in diverse areas such as stability, complexity, signal processing, chaos control and synchronization, and fuzzy control [12]. Recently, the problem of passivity analysis for delayed neural networks has been addressed in [6,10], where sufficient conditions for passivity were established. Considering that the passivity criteria in both [6] and [10] are delay-independent, several delay-dependent passivity conditions for delayed neural networks were proposed in [11,14,17,27], which are based on linear matrix inequalities (LMIs) techniques and Jensen integral inequality or free-weighting matrix method. For neural networks with discrete and bounded distributed time-varying delays, delay-dependent passivity results were obtained in [2] in terms of LMIs techniques and free-weighting matrix method. However, all these criteria are based on the assumption that the activation functions are monotonic non-decreasing. Furthermore, the conditions of [6,10,11,14] are established with the assumption that the derivatives of time-vary delays are less than 1. On the other hand, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Therefore there will be a distribution of conduction velocities along these pathways and a distribution of propagation be modeled with discrete delays and a more appropriate way is to incorporate continuously distributed delays. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

Motivated by aforementioned discussion, in this paper we will relax the constraint on the monotonicity of the activation function and the assumption that the derivatives of time-vary delays are...
less than 1, and study the passivity of neutral-type neural networks with discrete and continuous distributed time-varying delays. Based on delay partitioning, a new Lyapunov–Krasovskii functional is constructed to obtain several improved delay-dependent passivity conditions which guarantee the passivity of uncertain neural networks by applying the Jensen integral inequality. These criteria are expressed in the framework of linear matrix inequalities, which can be verified easily by means of standard Matlab software. By Finseis Lemma, one special case of the obtained criteria turns out to be equivalent to some existing result with same reduced conservatism but including fewer slack variables [7]. As the present passivity conditions involve fewer free-weighting matrices, the computational burden is largely reduced. Finally three examples are provided to verify the effectiveness of the proposed criteria.

Notation: Throughout this paper, let $W^T$, $W^{-1}$ denote the transpose and the inverse of a square matrix $W$, respectively. Let $W > 0$ ($< 0$) denote a positive (negative) definite symmetric matrix, $I_n$, $O_n$ denote the identity matrix and the zero matrix of $n$-dimension respectively, $0_{m,n}$ denotes the $m \times n$ zero matrix, the symbol $\ast$ denotes a block that is readily inferred by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem description

Considering the following uncertain neutral-type neural networks with discrete and distributed time-varying delays:

$$x'(t) = -AX(t) x(t) + W_1 f(t) x(t) + W_2 f(t) x(t - r(t))$$
$$+ W_3 (t) \tilde{x}(t - \sigma(t)) dt + \tilde{x}(t - m) dt + \eta(t),$$

$$y(t) = g(x(t)),$$

$$x(t) = \phi(t), \ t \in [-max \{\tau, \sigma, m\}, 0] \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is a neural state vector, $\eta(t)$ is an external input vector, and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T \in \mathbb{R}^n$ denotes the neural activation function, continuous function $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T \in \mathbb{R}^n$ is the initial condition. $0 \leq \tau \leq r(t) \leq \sigma$, $0 \leq \sigma(t) \leq \tau, m \geq 0$ are time-varying delays, where $\tau$, $\sigma$, $m$ are constant scalars. $A(t) = A + \Delta A(t)$, $W(t) = W_1 + \Delta W_1(t)$ ($i = 0, 1, 2, 3$). $A = \text{diag}(a_1, a_2, \ldots, a_n)$ is a positive diagonal matrix, $W_i = (w_{ij})_{n \times n}$ is known constant matrix, $\Delta W_i(t)$ is the parametric uncertainty. It is assumed that $\tilde{g}_i(x_i(t))$ ($j = 1, \ldots, n$) is bounded and satisfies the following conditions:

Assumption 1 (Liu et al. [9]). There exist constants $k_i^-, k_i^+$, such that $k_i^- < k_i^+$ and

$$k_i^- \leq \frac{\tilde{g}_i(s_1) - \tilde{g}_i(s_2)}{s_1 - s_2} \leq k_i^+,$$

$$g_i(0) = 0,$$

for any $s_1, s_2 \in \mathbb{R}, s_1 \neq s_2$.

For notational simplicity, we denote

$$K_1 = \text{diag}(k_1^-, k_1^+, k_2^-, \ldots, k_n^-, k_n^+),$$

$$K_2 = \text{diag}(k_1^-, k_1^+, k_2^-, k_2^+, \ldots, k_n^-, k_n^+),$$

$$K_3 = \text{diag}(k_1^-, k_2^-, \ldots, k_n^+).$$

Remark 1. As pointed out by Liu et al. [9], the constants $k_i^-$, $k_i^+$ in (2) are allowed to be positive, negative, or zero. Hence, the resulting activation functions may be non-monotonic, and more general than the usual sigmoid functions in [2,6,14,17,27].

Suppose that the time-varying uncertain matrices $\Delta A(t)$, $\Delta W_i(t)$ ($i = 0, 1, 2, 3$) are linear fractional norm-bounded, which are in the form of

$$[\Delta A(t), \Delta W_i(t)] = HA(t) [G G],$$

where $H, G, G_i$ ($i = 0, 1, 2, 3$) are known real constant matrices with appropriate dimensions. The uncertainty $\Delta(t)$ is defined as

$$\Delta(t) = (I_n - F(t)\tilde{f})^{-1} F(t),$$

where $f$ is also a known real constant matrix satisfying $f^T f < I_n$ and $F(t)$ is an unknown time-varying matrix satisfying

$$F^T(t) F(t) \leq I_n.$$  

Remark 2. The aforementioned structured linear fractional form includes the norm-bounded uncertainty as a special case when $J = 0$. Note also that condition $f^T f < I_n$ and (5) guarantee that $I - F(t)\tilde{f}$ is invertible.

We now introduce the following definition of passivity.

Definition 1 (Li and Liao [6]). The system in (1) is said to be passive if there exists a scalar $\gamma > 0$ such that for all $t \geq 0$

$$2 \int_0^t y^T(s) u(s) ds \geq -\gamma \int_0^t u^T(s) u(s) ds,$$

under the zero initial condition.

In order to obtain the results, we need the following lemmas.

Lemma 1 (See Gu [4]). For any positive symmetric constant matrix $M \in \mathbb{R}^{n \times n}$, scalars $r_1 < r_2$ and vector function $\omega : [r_1, r_2] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$\left( \int_{r_1}^{r_2} \omega(s) ds \right)^T M \left( \int_{r_1}^{r_2} \omega(s) ds \right) \leq (r_2 - r_1) \int_{r_1}^{r_2} \omega^T(s) M \omega(s) ds.$$

Lemma 2 (See Li et al. [8]). Suppose that $\Delta(t)$ is given by (4). Given matrices $Q = Q^T, H$ and $G$ with compatible dimensions, the matrix inequality

$$Q + \Delta(t) N^T + N \Delta(t)^T S < 0$$

holds for any $\tilde{f}(t)$ satisfying $\tilde{f}(t) F(t) \leq I_n$ if and only if for any positive scalar $\epsilon$, the following matrix inequality holds:

$$\left[ \begin{array}{cc} Q & S \\ -\epsilon I_n & \epsilon I_n \end{array} \right] < 0.$$  

Lemma 3 (See de Oliveira [13]). Let $y \in \mathbb{R}^n$, $L \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$ such that $rank(B) < n$ and $L = L^T$. Then the following statements are equivalent:

(i) $y^T L y < 0$ holds for any $y \neq 0$. By $y = 0$;

(ii) $(B^T)^T L B^T < 0$;

(iii) There exists a matrix $X \in \mathbb{R}^{n \times m}$ such that $L + B X + B^T X^T < 0$;

where $B^T$ is a matrix whose columns form the bases of the right null space of $B$.

3. Main results

In the sequel, we will establish several passivity results by employing the so-called delay-partitioning approach introduced in Zhang et al. [26]. In order to estimate the upper bound of the delay for passivity, we partition delay interval $[\tau, \sigma]$ into several components, that is, $0 \leq \tau = r_0 < r_1 < \cdots < r_T = \sigma$, where $r$ is a positive integer.

Before introducing the main results for system (1) with $r > 0$, following notations are defined for simplicity:

$$\Omega_{11} = -(1 - \eta) Q_1 - 2T T_1 K_1,$$

$$\Omega_{1, r+7} = -(1 - \eta) Q_2 + T_1 K_2.$$
\[ \Omega_{22} = Q_1 + R_0 + U_1 + (\tau - \tau)^2 U_2 - 2T K_1 - \frac{1}{\delta_0} Z_0, \]
\[ \Omega_{23} = P_2 - \frac{1}{\tau^3} Z_0, \quad \Omega_{2, r+1} = -P_2, \]
\[ \Omega_{2, r+4} = P_1 - K_3 A, \quad \Omega_{2, r+6} = Q_2 + Y_0 + T K_2, \]
\[ \Omega_{33} = R_1 - R_0 - U_2 - 2 T_0 K_1 - \frac{1}{\delta_0} Z_0 - \frac{1}{\delta_1} Z_1, \]
\[ \Omega_{2, r+5} = P_3, \quad \Omega_{3, r+8} = Y_1 - Y_0 + T_0 K_2, \]
\[ \Omega_{2, r+2} = r - R_1 - 2 T K_1 - \frac{1}{\delta_0} Z_0 - \frac{1}{\delta_1} Z_1, \]
\[ \Omega_{2, r+3 + i} = Y_{r-1} + T K_2, \]
\[ \Omega_{r+4, r+3} = -Y_{r-1}, \quad \Omega_{r+4, r+4} = \delta Z_1 + U_4, \]
\[ \Omega_{r+4, r+5} = P_2, \quad \Omega_{r+4, r+6} = \lambda, \quad \Omega_{r+5, r+5} = -U_2, \]
\[ \Omega_{r+6, r+6} = Q_3 + \sigma^2 U_3 - 2 T_2, \quad \Omega_{r+6, r+11} = -u_7, \]
\[ \Omega_{r+7, r+7} = -(1-\eta) Q_3 - 2 T_2, \quad \Omega_{r+8, r+8} = S_0 - 2 T_0, \]
\[ \Omega_{r+8, r+11} = -g_7, \quad \Omega_{r+9, r+9} = -U_3, \]
\[ \Omega_{r+10, r+11} = -u_4, \quad \Omega_{r+11, r+11} = -g_7, \]
where \( \tau = 0, \delta = \tau - r, i = 0, 1, \ldots; r = 2, \ldots, r. \)

Now we present a delay-dependent criterion for the passivity of system (1) with \( \tau > 0. \)

**Theorem 1** (See Appendix for a proof). Under Assumption 1, for given scalars \( \tau > 0, \sigma \geq 0, \) and \( \eta, \) system (1) is passive for \( \xi \leq \tau(t) \leq \tau, \) \( 0 \leq \sigma(t) \leq \sigma, \) \( i(t) \leq \eta \) if there exist constant scalars \( \epsilon_1 > 0, \gamma > 0, \) positive definite symmetric matrices \( P_1, P_2, Q_1, Q_2, R_1, S_1, Z_1, U_1, \) positive diagonal matrices \( \Lambda, T, T_2, T_3, \) real matrices \( P_2, Q_2, Y_1, M_1 \) (i = 0, 1, ..., r; j = 1, ..., 4; k = 1, ..., 9) with compatible dimensions such that (6)–(8) and the following LMIs hold ( \( \kappa = 1, \ldots, r; \) )

\[
\begin{bmatrix}
\hat{\Omega} + \hat{\Omega}^T + \hat{\Lambda} \hat{\Lambda}^T + \hat{\Lambda} \hat{\Lambda}^T M \hat{H} \tilde{e}^T \\
* & -\tilde{e}_n^T & \tilde{e}_r^T \\
* & * & -\tilde{e}_n^T
\end{bmatrix} < 0,
\]

where

\[
\hat{\Omega} = [\hat{\Omega}_{ij}]_{(2r+1)\times(2r+1)}, \quad \hat{\Omega}_{ij} = [\Omega_{ij}]_{(2r+1)\times(2r+1)},
\]

\[
\hat{\Lambda} = [\Lambda_{ij}]_{(2r+1)\times(2r+1)}, \quad \hat{\Lambda}_{ij} = [\Lambda_{ij}]_{(2r+1)\times(2r+1)},
\]

\[
\tilde{e}^T = [e_1, \ldots, e_r, \tilde{e}_n, \ldots, \tilde{e}_n],
\]

\[
\tilde{e}_r^T = [e_1, \ldots, e_r, \tilde{e}_n],
\]

\[
\tilde{e}^r = [e_1, \ldots, e_r, \tilde{e}_n],
\]

other parameters \( \hat{\Omega}(i < j) \) are all equal to zero's.

**Remark 3.** Note that the conditions in Theorem 1 and Corollary 1 depend not only on the size of the delay but also on the rate of the delay. Hence, when \( i(t) \) is unknown or not differentiable, Theorem 1 and Corollary 1 are no longer applicable. In this case, however, delay-dependent and rate-independent passivity conditions can be readily obtained from Theorem 1 and Corollary 1 by setting \( Q_i = 0 \) ( \( i = 1, 2, 3. \) )

In order to show the equivalence with the existing criterion, we consider the following uncertain neural networks with discrete and distributed time-varying delays:

\[
x'(t) = -Ax(t) + W_0(tg(x(t)) + W_1(tg(x(t-r(t)))) + W_2(t) \int_{t-r(t)}^{t} g(s) \, ds \, u(t).
\]

\[
y(t) = g(x(t)).
\]

We propose the following corollary, which is straightforward from Corollary 1 by setting \( J = Q_2 = Y_1 = S_1 = U_1 = 0, \) \( r = 1, \) \( k_i^* = 0 \) ( \( i = 1, \ldots, n. \) )

**Corollary 2.** Under Assumption 1, for given scalars \( \tau > 0, \sigma \geq 0, \) and \( \eta, \) system (11) is passive for \( 0 \leq \tau(t) \leq \tau, \) \( 0 \leq \sigma(t) \leq \sigma, \) \( i(t) \leq \eta \) if there exist constant scalars \( \epsilon_1 > 0, \gamma > 0, \) positive definite symmetric matrices
$P_i, P_2, Q_1, Q_3, R_1, Z_1, U_2, U_3$, positive diagonal matrices $\Lambda T_1 T_2$ real matrices $P_2, M_2 (k = 1, \ldots, 9)$ with compatible dimensions such that (6) and the following LMI hold:

$$\begin{bmatrix} \mathcal{F} + \mathcal{S}^T & \mathcal{M}^T \mathcal{H} + \mathcal{S}^T & \mathcal{S} & \mathcal{S} \\ \mathcal{S}^T & * & -I_n & 0 \\ * & * & 0 & -I_n \\ * & * & * & 0 \end{bmatrix} < 0,$$

(12)

where

$$\mathcal{S} = [\mathcal{S}_j]_{j=3}, \quad \mathcal{S} = \mathcal{M} \mathcal{T}, \quad \mathcal{M}^T = [M_1^T, M_2^T, M_3^T, M_4^T, M_5^T, M_6^T, M_7^T, M_8^T, M_9^T],$$

$$\mathcal{T} = [0_n, -A, 0_n, -I_n, 0_n, W_0, W_1, W_2, I_n],$$

$$\mathcal{S}^T = [0_n, -G_0, G_0, G_1, G_2, 0],$$

$$\mathcal{T}_1 = -(1-q)I_n, -I_n,$$

$$\mathcal{T}_2 = \frac{1}{7} I_n, \quad \mathcal{T}_3 = \frac{1}{7} I_n, \quad \mathcal{T}_4 = T_1 K_2,$$

$$\mathcal{S}_2 = P_2 + P_2^T + Q_1 + R_1 \frac{1}{7} Z_1 + \tau^2 U_2,$$

$$\mathcal{S}_3 = -P_2, \quad \mathcal{S}_4 = P_1, \quad \mathcal{S}_5 = P_3, \quad \mathcal{S}_6 = T_1 K_2,$$

$$\mathcal{S}_3 = -R_1 \frac{1}{7} I_n, \quad \mathcal{S}_5 = -P_3, \quad \mathcal{S}_4 = \mathcal{T}_1,$$

$$\mathcal{S}_5 = P_2, \quad \mathcal{S}_6 = \mathcal{S}_7 = \mathcal{S}_8 = \mathcal{S}_9 = \mathcal{T}_3, \quad \mathcal{S}_9 = \mathcal{T}_3, \quad \mathcal{S}_9 = \mathcal{T}_3, \quad \mathcal{S}_9 = \mathcal{T}_3.$$

other parameters $\mathcal{T}_i (i < j)$ are all equal to zeros.

**Remark 4.** Recently, a less conservative passivity condition for neural networks with mixed delays is proposed in [2] via free-weighting matrix method. However, it is found that this approach needs to introduce too many free-weighting matrices in obtaining LMI conditions and thus leads to a significant increase in the computational demand. In order to avoid increasing computational burden, Theorem 1, Corollaries 1 and 2 involve fewer slack matrices. However, they can still provide less conservative results than the results in [2,6,11,14,17,27].

**4. Comparison with the existing result**

In this section, we will establish theoretically that Corollary 2 is equivalent to Theorem 1 in [2] while $\eta < 1$. To compare clearly, applying the Schur complement to Theorem 1 in [2] gives the following lemma:

**Lemma 4** (Chen et al. [2]). Under Assumption 1, for given scalars $\tau > 0, \tau > 0, \eta < 1$, system (11) is passive for $0 \leq \tau (t) \leq \tau, \eta \leq \eta$, if there exist constant scalar $\epsilon > 0, \gamma > 0$, definite positive symmetric matrices $P_1, P_2, Q_1, Q_3, R_1, Z_1, U_2, U_3$, positive diagonal matrices $\Lambda T_1 T_2$, real matrices $P_2, M_2, N_1, S_1 (k = 1, \ldots, 9)$ with compatible dimensions such that (6) and the following LMI hold:

$$\begin{bmatrix} \mathcal{F} + \mathcal{S}^T & \mathcal{M}^T \mathcal{H} + \mathcal{S}^T & \mathcal{S} & \mathcal{S} \\ \mathcal{S}^T & * & -I_n & 0 \\ * & * & 0 & -I_n \\ * & * & * & 0 \end{bmatrix} < 0,$$

(13)

where

$$\mathcal{F} = [\mathcal{F}_j]_{j=3,9}, \quad \mathcal{S} = \mathcal{S}^T,$$

$$\mathcal{S} = [S-N, N, -S, 0_n, 0_n],$$

$$\mathcal{M} = [N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9],$$

$$\mathcal{S}_j = [S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9],$$

$$\Pi_{11} = -(1-\eta)Q_1, \quad \Pi_{12} = \Pi_{13} = \Pi_{14} = 0,$$

$$\Pi_{22} = P_2 + P_2^T + Q_1 + R_1 + \tau^2 U_2, \quad \Pi_{33} = -R_1,$$

and other parameters are defined as $\Pi_{ij} = \Pi_{ij} (i < j)$.

Although Corollary 2 and Lemma 4 are obtained via different methods, they turned out to be equivalent. To show this, we give the following theorem.

**Theorem 2.** LMI (12) in Corollary 2 is feasible if and only if LMI (13) in Lemma 4 is feasible.

**Proof.** By applying the Schur complement, the LMI (12) in Corollary 2 is equivalent to the following matrix inequality:

$$\mathcal{S} + \mathcal{S}^T < 0,$$

(14)

where

$$\mathcal{S} = [S-N, N, -S, 0_n, 0_n],$$

$$\mathcal{S}^T = [S-N, N, -S, 0_n, 0_n].$$

On the other hand, the LMI (13) in Lemma 4 can be expressed as

$$\begin{bmatrix} \mathcal{F} + \mathcal{S}^T & \mathcal{M}^T \mathcal{H} + \mathcal{S}^T & \mathcal{S} & \mathcal{S} \\ \mathcal{S}^T & * & -I_n & 0 \\ * & * & 0 & -I_n \\ * & * & * & 0 \end{bmatrix} < 0.$$

where

$$\mathcal{S} = [S-N, N, -S, 0_n, 0_n],$$

$$\mathcal{S} = [S-N, N, -S, 0_n, 0_n].$$

Note that $W_2(t) = 0$, where

$$\chi(t) = \left[ \chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t), \chi_5(t), \chi_6(t), \chi_7(t), \chi_8(t), \chi_9(t) \right],$$

$$g(t) = \left[ g_1(t), g_2(t), g_3(t), g_4(t), g_5(t), g_6(t), g_7(t), g_8(t), g_9(t) \right],$$

$$u(t) = \left[ u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t), u_7(t), u_8(t), u_9(t) \right].$$

According to Finsler's Lemma (see Lemma 3), it is readily seen that $\dot{\mathcal{F}} < 0$ if and only if $(\mathcal{W}^*)^T \mathcal{F} \mathcal{W}^* < 0$ hold. Since

$$\mathcal{W}^* = \begin{bmatrix} I_n & 0_n & 0_n \end{bmatrix},$$

it is easy to see that $(\mathcal{W}^*)^T \mathcal{F} \mathcal{W}^* = \mathcal{S} + \mathcal{S}^T < 0$. From (14), this completes the proof.

**Remark 5.** From the proof of Theorem 2, it is easy to see that Corollary 2 is equivalent to Theorem 1 of [2]. This means that the free weighting matrices $N_1, S_1 (i = 1, \ldots, 9)$ can be removed while maintaining the effectiveness of the passivity condition.

**Remark 6.** The presented criteria involve fewer slack variables which largely reduce the computational burden and conservatism, the following examples will show this. For system (11) with $W_2(t) = 0$, Table 1 provides a comparison of the number of the variables involved in Corollaries 1 and 2 and in some other existing results. From Table 1, it is easy to see that the number of the decision variables involved in this paper is smaller than those in [2,27].
5. Illustrative examples

In this section, we will provide three numerical examples to demonstrate the effectiveness and less conservativeness of our delay-dependent stability criteria over some recent results in the literature.

Example 1. Consider system (1) with

\[ A = \text{diag}(1.3, 1.5), \quad W_0 = \begin{bmatrix} -1.198 & 0.1 \\ 0.1 & -1.198 \end{bmatrix}, \]

\[ W_1 = \begin{bmatrix} 0.1 & 0.16 \\ 0.05 & 0.1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.2 \end{bmatrix}, \]

\[ W_3 = \begin{bmatrix} 0.3 & -0.15 \\ 0.5 & -0.2 \end{bmatrix}, \quad J = 0, H = 0.2, G = 0.2, \]

\[ G_0 = 0.25, \quad G_1 = 0.15, \quad G_2 = 0.20, \quad G_3 = 0.25, \]

\[ g(x) = 0.5(x + 1) - |x - 1|, \quad i = 1, 2. \]

Obviously, Assumption 1 is satisfied with \( K_1 = K_2 = 0, \ K_2 = 1. \)

We are aiming at searching the upper bounds of admissible delays \( \tau, \sigma \) and \( m \) with fixed \( \tau \) which ensure the passivity of system (1). It is easy to see that the results of this paper are delay-independent of \( m \). The maximal upper bounds of times delays \( \tau = \sigma \) with \( r = 1, 2, 3 \) for fixed \( \tau = 0 \) and various \( \eta \)'s are listed in Table 2.

By resorting to the Matlab LMI Control Toolbox to solve the LMIs in Corollary 1 with \( r = 3, \ \tau = 0, m > 0, \ \tau = \sigma = 2.6 \) and unknown \( \eta \), we obtain

\[ P_1 = \begin{bmatrix} 12.9615 & -3.0609 \\ -3.0609 & 8.1988 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -0.0009 & 0.0073 \\ 0.0020 & 0.0192 \end{bmatrix}, \]

\[ P_3 = \begin{bmatrix} 0.2010 & -0.2067 \\ -0.2067 & 0.2355 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 5.3036 & -4.2276 \\ -4.2276 & 6.7787 \end{bmatrix}, \]

\[ R_2 = \begin{bmatrix} 3.6945 & -2.9595 \\ -2.9595 & 4.7391 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 2.0190 & -1.6096 \\ -1.6096 & 2.6117 \end{bmatrix}, \]

\[ Z_1 = \begin{bmatrix} 0.4131 & 0.3515 \\ 0.3515 & 0.8367 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.3699 & 0.2869 \\ 0.2869 & 0.7365 \end{bmatrix}, \]

\[ Z_3 = \begin{bmatrix} 0.4120 & 0.3504 \\ 0.3504 & 0.8276 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.2989 & -0.3231 \\ -0.3231 & 0.3709 \end{bmatrix}. \]

\[ U_3 = \begin{bmatrix} 3.6125 & -0.6972 \\ -0.6972 & 1.4079 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 3.5827 & -1.3905 \\ -1.3905 & 0.9732 \end{bmatrix}. \]

\[ \Lambda = \text{diag}(13.7221, 13.7221), \quad T_+ = \text{diag}(0.6546, 1.2168), \]

\[ T_* = \text{diag}(24.7375, 12.2392), \]

\[ M_1 = \begin{bmatrix} 0.0271 & 0.0310 \\ -0.0006 & 0.0633 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 12.4142 & -1.9116 \\ -1.5407 & 9.5258 \end{bmatrix}, \]

\[ M_3 = \begin{bmatrix} 0.0796 & 0.0379 \\ 0.0348 & 0.1582 \end{bmatrix}, \quad M_4 = \begin{bmatrix} -0.0946 & -0.0449 \\ -0.0426 & -0.1486 \end{bmatrix}, \]

\[ M_5 = \begin{bmatrix} -0.0201 & 0.0119 \\ 0.0070 & -0.0403 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 6.2652 & -0.3826 \\ -0.2941 & 4.9340 \end{bmatrix}, \]

\[ M_7 = \begin{bmatrix} 0.0464 & -0.0524 \\ -0.0467 & 0.0590 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 5.6623 & -0.7121 \\ -0.8912 & 4.4115 \end{bmatrix}, \]

\[ M_9 = \begin{bmatrix} -0.2617 & -0.1765 \\ -0.3829 & -0.2916 \end{bmatrix}, \quad M_{10} = \begin{bmatrix} 30.6319 & -0.1510 \\ -0.2539 & 27.9905 \end{bmatrix}, \]

\[ M_{11} = \begin{bmatrix} -0.8937 & -0.8186 \\ 0.2956 & -0.3958 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0.3879 & 0.4366 \end{bmatrix}. \]

\[ \gamma = 205.0075, \quad \epsilon = 2.5017. \]

Example 2. Consider system (11) with

\[ A = \text{diag}(2.3, 2.5), \quad W_0 = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \]

\[ W_1 = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.4 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.5 & -0.3 \\ 0.2 & 1.2 \end{bmatrix}, \]

\[ H = 0.2, \quad G = G_0 = G_1 = G_2 = 1, \quad J = 0, \]

\[ g(x) = 0.5(x + 1) - |x - 1|, \quad i = 1, 2. \]

Obviously, Assumption 1 is satisfied with \( K_1 = K_2 = 0, \ K_2 = 1. \)

This model was investigated in [2]. If we set \( \tau = \sigma \), the maximal upper bounds of time delays with \( \tau = 0 \) for various \( \eta \)'s from Corollaries 1, 2 or Remark 3 in this paper and those in [2] are listed in Table 3.

Remark 7. From Table 3 it is seen that Corollary 2 of this paper gives the same maximal upper bounds of time delays with Theorem 1 in [2], which coincides with the theoretical results which are stated in Remark 5 of Theorem 2. In addition, Corollary 1 of this paper with \( r = 1, 2, 3 \) gives larger maximal upper bounds of time delays than Theorem 1 in [2]. This shows that for this example, the results of our paper are less conservative than those in [2].

Example 3. Consider system (11) with

\[ A = \text{diag}(2.2, 1.5), \quad W_2(t) = 0, J = 0, \]

\[ W_0 = \begin{bmatrix} 1 & 0.6 \\ 0.1 & 0.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \]

\[ H = 0.1, \quad G = 0.1, \quad G_0 = 0.2, \quad G_1 = 0.3, \]

\[ g(x) = 0.5(x + 1) - |x - 1|, \quad i = 1, 2. \]

Obviously, Assumption 1 is satisfied with \( K_1 = K_3 = 0, \ L_2 = I. \)

This model was investigated in [2]. It is proved that the condition given in [6] fails to conclude whether this model is passive or not for \( \eta = 0 \). For fixed \( \tau = 0 \) and different \( \eta \)'s, Table 4 gives the comparison results on the maximum \( \tau \) allowed via the methods in [2,14,17,27] and Corollaries 1, 2 or Remark 3 in our paper, where the results of [14] (corrected) are the corrected ones of [14] based on [17].

From Table 4, one can see that Corollaries 1 and 2 provide larger upper bounds than those criteria in [2,14,17,27]. In particular, when \( \tau = 0 \) and \( \eta = 0.1 \), the achieved maximal upper bound \( \tau \) by Corollary 1 with \( r = 3 \) is 1012.9%, 660.8%, 303.2%, 127.2% and 20.2% larger than those in [14,17,27,22].
Remark 8. From Table 4 it is seen that Corollary 2 of this paper gives the same maximal upper bounds of time delays with Theorem 1 in [2], which coincides with the theoretical results which are stated in Remark 5 of Theorem 2. Furthermore, Corollary 1 of this paper with r = 1, 2, 3 gives larger upper bounds of time delays than those in [2,11,14,17,27]. This demonstrates that for this example, the results of our paper are much effective and less conservative than those in [2,6,11,14,27].

6. Conclusion

This paper has established improved passivity criteria for uncertain neutral-type neural networks with discrete and continuous distributed time-varying delays. The approach is based on the delay-partitioning method, a simple integral inequality and free-weighting matrices method. One special case of the proposed results is shown to be equivalent to some existing result but including fewer variables in the LMI conditions. Thus, the present method could largely reduce the computational burden in solving LMs. Both theoretical and numerical comparisons have been provided to demonstrate the effectiveness and efficiency of the present method. The results could be easily used for analysis of other kinds of neural networks, such as the dynamics analysis for complex networks [3,15,16], and this will be our future research direction.

Appendix A. Proof of Theorem 1

Consider the following Lyapunov–Krasovskii functional:

\[ V(t, x_t) = \sum_{i=1}^{n} V_i(t, x_t) \]  

(15)

with

\[ V_i(t, x_t) = \Phi^T(t) P_i \Phi(t) + 2 \sum_{i=1}^{n} \left[ \lambda_i \left[ g_i(x(t))-k_i^2 x(t) \right] \right] ds, \]

\[ V_2(t, x_t) = \int_{t-	au}^{t} \psi^T(s) \psi(s) ds, \]

\[ V_3(t, x_t) = \sum_{i=0}^{r} \int_{t-	au_i}^{t} \psi^T(s) R_i \psi(s) ds, \]

\[ V_4(t, x_t) = \sum_{i=0}^{r} \int_{t-	au_i}^{t} x^T(s) Z_i x(s) ds, \]

\[ V_3(t, x_t) = \int_{t-	au}^{t} x^T(s) U_1 x(s) ds + (\tau-t) \int_{0}^{\tau} x^T(s) U_2 x(s) ds, \]

\[ V_0(t, x_t) = \tau \int_{0}^{\tau} g^T(x(t)) U_3 g(x(t)) ds, \]

where

\[ \phi^T(t) = \left[ x^T(t), \int_{t-\tau}^{t} x^T(s) ds \right]. \]

\[ \Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \quad \psi^T(s) = [x^T(s), g^T(x(s))]. \]

Remark 9. From (6)–(8), Assumption 1 and the definition of positive definite symmetric matrices \( Z_i (i = 0, 1, \ldots, r) \), \( U_j (j = 1, \ldots, 4) \), it is easy to see that \( V(t, x_t) > 0 \) for any \( t > 0 \). This means that \( V(t, x_t) \) can be chosen as a Lyapunov–Krasovskii functional. As \( V(t, x_t) \) includes the upper bounds \( \tau, \sigma \) of time-varying delays \( \tau(t) \) and \( \sigma(t) \), the established passivity conditions are delay-dependent which cannot be applied for very large time delays.

For convenience, we denote \( x_t = x(t-\tau(t)) \). The time derivative of functional (15) along the trajectories of system (1) is obtained as follows:

\[ \dot{V}_1(t, x_t) = 2 \phi^T(t) P \phi(t) + 2 \sum_{i=1}^{r} \lambda_i \left[ g_i(x(t))-k_i^2 x_t \right] \]

\[ = 2x^T(t) P_1 \dot{x}_t + 2x^T(t) P_2 [x(t-\tau)-x(t-\sigma)] \]

\[ + 2\dot{x}_t^T P_2 \int_{t-\tau}^{t} x(s) ds, \]

\[ + 2g^T(x(t)) x(t) - x(t) x(t) ] P_3 \int_{t-\tau}^{t} x(s) ds \]

\[ + 2g^T(x(t)) x(t) x(t) ] K \lambda x(t), \]

\[ \dot{V}_2(t, x_t) = \psi^T(t) \psi(t) - (1-\eta) \psi^T(t-\tau(t)) \psi(t-(t-\tau(t))) \]

\[ \leq \psi^T(t) \psi(t) - (1-\eta) \psi^T(t-\tau(t)) \psi(t-(t-\tau(t))) \]

\[ = x^T(t) Q_1 x(t) + 2x^T(t) Q_2 g(x(t)) \]

\[ + g^T(x(t)) Q_3 g(x(t)) - (1-\eta) [2x^T(t) Q_2 g(x(t)) + g^T(x(t)) Q_3 g(x(t))], \]

\[ \dot{V}_3(t, x_t) = \sum_{i=0}^{r} \psi^T(t-\tau_i) R_i \psi(t-\tau_i) \]

\[ \leq \sum_{i=0}^{r} \psi^T(t-\tau_i) R_i \psi(t-\tau_i) \]

\[ + 2x^T(t-\tau_i) Y_i g(x(t-\tau_i)) \]

\[ + g^T(x(t-\tau_i)) S_i g(x(t-\tau_i)) \]

\[ - x^T(t-\tau_i) R_i x(t-\tau_i) \]

\[ - 2x^T(t-\tau_i) Y_i g(x(t-\tau_i)) \]

\[ - g^T(x(t-\tau_i)) S_i g(x(t-\tau_i)), \]

\[ \dot{V}_4(t, x_t) = \sum_{i=0}^{r} \left[ \delta_i \dot{x}_t^T(t) Z_i \dot{x}_t - \int_{t-\tau_i}^{t} x^T(s) Z_i x(s) ds \right]. \]
\[ V_s(t,x_t) = x^T(t)U_jx(t) - x^T(t-\tau)U_jx(t-\tau) \]
\[ + \sigma(t) \int_{t-\tau}^{t} g^T(s)U_jg(s) \, ds, \]
\[ \dot{V}_i(t,x_t) = \sigma^2g^T(t)U_jg(t) \]
\[ - \pi \int_{t-\tau}^{t} g^T(s)U_jg(s) \, ds \]
\[ + x^T(t)U_jx(t) - x^T(t-\tau)U_jx(t-\tau). \]

Now, we suppose that \( \tau_{i-1} \leq t \leq \tau_i \) (1 \( \leq i \leq r \)). Obviously the following inequality holds:

\[ - \int_{t-\tau_{i-1}}^{t-\tau_i} x^T(s)Z_i^*x(s) \, ds \]
\[ = - \int_{t-\tau_{i-1}}^{t-\tau_i} x^T(s)Z_i^*x(s) \, ds - \int_{t-\tau_i}^{t} x^T(s)Z_i^*x(s) \, ds. \]

From Lemma 1 and Leibniz–Newton formula, the following inequalities hold for any positive \( \tau_i \) or \( \tau_j \), \( 0 \leq i \leq r \):

\[ - \delta_\tau \int_{t-\tau}^{t-\tau_{i-1}} x^T(s)Z_i^*x(s) \, ds \]
\[ \leq - (t - \tau - \tau_{i-1}) \int_{t-\tau}^{t-\tau_{i-1}} x^T(s)Z_i^*x(s) \, ds \]
\[ \leq \left( \int_{t-\tau}^{t-\tau_{i-1}} x^T(s)Z_i^*x(s) \, ds \right) \int_{t-\tau}^{t-\tau_{i-1}} x^T(s)Z_i^*x(s) \, ds \]
\[ \leq \begin{bmatrix} x(t) & x(t-\tau_{i-1}) \end{bmatrix} \begin{bmatrix} -Z_i & Z_i \\ * & -Z_i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_{i-1}) \end{bmatrix}. \]

To get our passivity criterion, we introduce the following equality for any real matrix \( M \) with compatible dimension:

\[ 0 = 2\dot{C}_{1}(t)M \{-\dot{x}(t) - A(t)x(t) + W_0(t)g(x(t)) \}
\[ + W_1(t)g(x(t)) + W_2(t) \int_{t-\tau}^{t} g(x(s)) \, ds \]
\[ + W_3(t)u(t) + u(t), \]

where

\[ \dot{C}_{1}(t) = \begin{bmatrix} x^T(t), x^T(t-\tau_{i-1}), x^T(t-\tau_{i-2}), \ldots, x^T(t-\tau_{i}), x^T(t-\tau), x^T(t) \end{bmatrix}, \]

\[ g^T(t) = \begin{bmatrix} g^T(x(t)), g^T(x(t-\tau_{i-1})), \ldots, g^T(x(t-\tau)), g^T(x(t-\tau_{i-1})) \end{bmatrix}. \]

By utilizing the well-known Schur complement (see [1] and Lemma 2, we can prove that

\[ \dot{V}(x_i) - \dot{V}(u)(t-2) \leq C_{1}(t)Q + \Omega + M \dot{A}_i + \dot{M}A_i^T + \dot{M}A_i^T + \dot{M}A_i^T + \dot{M}A_i^T + \dot{M}A_i^T + \dot{M}A_i^T. \]

is equivalent to (9). Thus if LMIls (6)–(9) hold, from (26) we have

\[ \dot{V}(x_i) - \dot{V}(u)(t-2) \leq 0. \]
Thus $\dot{V}(t) \leq 0$ holds for any $\tau > 0$ while LMIs (6)–(9) are true. From the definition of passivity, the proof of Theorem 1 is completed.

References


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