New results on delay-dependent stability analysis for neutral stochastic delay systems

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Abstract

This paper is concerned with the problem of stability analysis for neutral stochastic delay systems. Firstly, expectations of stochastic cross terms containing the Itô integral are investigated by the martingale theory. Based on this, an improved delay-dependent stability criterion is derived for neutral stochastic delay systems. In the derivation process, the mathematical development avoids bounding stochastic cross terms, and neither the model transformation method nor free-weighting-matrix method is used. Thus the method leads to a simple criterion and shows less conservatism. Finally, two examples are provided to demonstrate the effectiveness and reduced conservatism of the proposed conditions.
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1. Introduction

Time delays arise in various practical systems and they are often a source of instability and poor performance. Thus, problems of stability and control of time-delay systems have been of great importance and interest. Recently, much attention has been focused on delay dependent conditions for the analysis and control of time-delay systems, because delay-dependent

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conditions are generally less conservative than delay-independent ones. Many effective methods have been proposed such as the discretized Lyapunov–Krasovskii functional method [1], the model transformation method [2–5], the free-weighting-matrix method [6–9], the Finsler projection approach [10], and so on.

On the other hand, some practical systems such as chemical engineering systems, distributed networks containing lossless transmission lines, can be modeled by using the model of neutral-type delay systems [35]. Due to the fact that the neutral delay system involves the delays in both its state and the derivative of the state, stability of neutral delay systems proves to be a more complex issue, and delay-dependent stability problems of neutral delay systems have received considerable attention by using above methods during the past decades [1,3,5,6,11–14]. Recently, since systems in the real world are always perturbed by stochastic noises, Kolmanovskii and Nosov [36,37] introduced the neutral stochastic differential functional equations to represent mathematical models for practical systems such as chemical engineering systems and the theory of aeroelasticity. And neutral stochastic delay systems have been studied over recent years [15–19]. For instance, the delay-dependent stability problems were addressed in [18,19] by the free-weighting-matrix method, while the $H_\infty$ control problem was investigated in [16]. Moreover, for nonlinear stochastic neutral systems, delay-dependent stability results were obtained in [17]. Although these results given are effective, there are still some problems to be pointed out for neutral stochastic delay systems, even these problems exist in stochastic delay systems.

- The Newton–Leibniz formula is still valid in stochastic case?

Recently, some papers such as [20] used the Newton–Leibniz formula to obtain the delay-dependent stability condition for stochastic delay systems. Unfortunately, from the preface of [25], we can know that the Newton–Leibniz formula is not valid in stochastic differential equations. In fact, for the following neutral stochastic functional differential equation:

$$d[x(t)-D(x_t)] = f(t,x_t) dt + g(t,x_t) dw(t)$$

on $t \geq 0$ with the initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C_{\mathbb{F}_0}^b([-h,0]; \mathbb{R}^n)$, it is known in [15,25] that Eq. (1) is a symbolic differential form and is interpreted as meaning the stochastic integral equation

$$x(t)-D(x_t) = x(0)-D(x_0) + \int_0^t f(s,x_s) \, ds + \int_0^t g(s,x_s) \, dw(s).$$

- How to deal with stochastic cross terms containing the Itô integral?

Since the Newton–Leibniz formula is not valid in stochastic case, both of the model transformation method and the free-weighting-matrix method must use the stochastic integral equation (2) to obtain delay-dependent conditions. Then there will be unavoidable to appear the following stochastic cross-terms containing the Itô integral:

$$x(t)^T M \int_{t-h}^t g(s,x_s) \, dw(s), \quad x(t-h)^T N \int_{t-h}^t g(s,x_s) \, dw(s), \quad (\int_{t-h}^t f(s,x_s) \, ds)^T L \int_{t-h}^t g(s,x_s) \, dw(s).$$

It is still very difficult to calculate the expectations of these stochastic cross terms. The existing results in [18,21–23] resorted to bounding techniques to deal with stochastic cross terms, which obviously can bring unavoidable conservatism. For these stochastic cross...
terms of stochastic delay systems, [26–30] considered that the expectations of these stochastic terms are all equal to zero. However, these results were not given by strict mathematical proofs, and we can find examples to illustrate that expectations of some stochastic cross terms are not equal to zero in Remark 1.

- How to give a simple delay-dependent stability condition with less conservatism and lower computational cost?

As is known to all, model transformations may lead to additional dynamics of the original systems [31]. And Xu and Lam [32] pointed out that in some cases free matrix variables may not be useful to the reduction of conservatism and those variables will increase the computational burden. Actually, Chen et al. [24] have noticed these problems, and proposed the generalized Finsler lemma to solve these problems for neutral stochastic delay systems. However, there are some errors in this method (see Remark 4).

Based on above reasons, there is a strong need to establish some improved stability results with less conservatism and lower computational cost for neutral stochastic delay systems. Specifically, bounding techniques should be avoided in the derivation process and the stability criteria should have less decision matrices.

To solve above problems, this paper is concerned with the delay-dependent stability analysis for neutral stochastic delay systems. The main contributions of this paper are summarized as follows: (1) Expectations of stochastic cross terms containing the Itô integral are investigated by Lemma 1. We prove that the expectation of \( x(t-h)^T N \int_{t-h}^t g(s,x_s) \, dw(s) \) is equal to zero by the martingale theory, and expectations of other stochastic cross terms are not. (2) Based on this lemma, a new delay-dependent stability condition for neutral stochastic delay systems is given. In the derivation process, the mathematical development avoids bounding stochastic cross terms, and neither the model transformation method nor free-weighting-matrix method is used. Thus the method leads to a simple criterion and shows less conservatism, and numerical examples are provided to demonstrate the reduced conservatism of the proposed condition.

**Notation.** Throughout the paper, unless otherwise specified, we will employ the following notation. Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a complete probability space with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( \mathcal{E}(\cdot) \) be the expectation operator with respect to the probability measure. If \( A \) is a vector or matrix, its transpose is denoted by \( A^T \). If \( P \) is a square matrix, \( P > 0 \) means that is a symmetric positive definite matrix of appropriate dimensions while \( P \geq 0 \) (\( P \leq 0 \)) is a symmetric positive (negative) semidefinite matrix. The identity matrix of appropriate dimensions is denoted by \( I \). Let \( \lambda_{\text{min}}(\cdot) \) be the minimum eigenvalue of a given matrix. Let \(|\cdot|\) denote the Euclidean norm of a vector and its induced norm of a matrix. Unless explicitly specified, matrices are assumed to have real entries and compatible dimensions. Let \( h > 0 \) and \( C([-h,0]; \mathbb{R}^n) \) denote the family of all continuous \( \mathbb{R}^n \)-valued functions \( \varphi \) on \([-h,0] \) with the norm \( \| \varphi \| = \sup \{|\varphi(\theta)| : -h \leq \theta \leq 0 \} \). Let \( C^b_{\mathcal{F}_0}([-h,0]; \mathbb{R}^n) \) be the family of all \( \mathcal{F}_0 \)-measurable bounded \( C([-h,0]; \mathbb{R}^n) \)-valued random variables, and \( \mathcal{L}^2([a,b]; \mathbb{R}^n) \) is the family of all \( \mathbb{R}^n \)-valued \( \mathcal{F}_t \)-adapted processes \( f(t) \) such that \( \int_a^b |f(t)|^2 \, dt < \infty \) a.s. Let \( \mathcal{M}^2([a,b]; \mathbb{R}^n) \) be the family of processes \( \{f(t)\}_{a \leq t \leq b} \) in \( \mathcal{L}^2([a,b]; \mathbb{R}^n) \) such that \( \mathcal{E}(\int_a^b |f(t)|^2 \, dt) < \infty \).
2. Preliminaries

Consider the following neutral stochastic delay system described by:

\[ d[x(t) - D x(t-h)] = [A x(t) + A_d x(t-h)] \, dt + [H x(t) + H_d x(t-h)] \, dw(t) \]  

on \( t \geq 0 \) with the initial data \( x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C_{b}^{T}([-h, 0], \mathbb{R}^{n}) \), where \( x(t) \in \mathbb{R}^{n} \) is the state vector, \( w(t) \) is a scalar standard Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with a natural filtration \( \{\mathcal{F}_{t}\}_{t \geq 0} \), \( h > 0 \) is the time delay, and \( A, A_d, D, H, H_d \) are known real constant matrices. The spectrum radius of the matrix \( D, \rho(D) \), satisfies \( \rho(D) < 1 \).

The objective of this paper is to establish a new delay-dependent sufficient condition for exponential stability of the system (3).

For simplicity, let us introduce some notations as follows:

\[ f(t) = f(t, x_t) = A x(t) + A_d x(t-h), \]
\[ g(t) = g(t, x_t) = H x(t) + H_d x(t-h), \]

for all \( t \geq 0 \).

By Eq. (3), one can observe that both \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) satisfy the local Lipschitz condition and the linear growth condition. It is easy to verify by [15] that the neutral stochastic delay differential equation (3) has a unique continuous solution denoted by \( \{x(t)\}_{t \geq -h} \), that is adapted to \( \{\mathcal{F}_{t}\}_{t \geq -h} \) (as usual, let \( \mathcal{F}_{t} = \mathcal{F}_{0} \) for \(-h \leq t \leq 0\)).

At the end of this section, we provide the following definitions and propositions that are useful for the development of our results.

**Definition 1 (Kuo [25]).** Let \( \{\mathcal{F}_{t}\}_{t \in T} \) be an increasing family of \( \sigma \)-algebras of subset of \( \Omega \). A stochastic process \( \{X_{t}\}_{t \in T} \) is said to be adapted to \( \{\mathcal{F}_{t}\}_{t \in T} \) if for each \( t \), the random variable \( X_{t} \) is \( \mathcal{F}_{t} \)-measurable.

**Definition 2 (Kuo [25]).** Let \( \{X_{t}\}_{t \in T} \) be a stochastic process adapted to a filtration \( \{\mathcal{F}_{t}\}_{t \in T} \) and \( \mathbb{E}(|X_{t}|) < \infty \) for all \( t \in T \). Then \( \{X_{t}\}_{t \in T} \) is called a martingale with respect to \( \{\mathcal{F}_{t}\}_{t \in T} \) if for any \( s \leq t \) in \( T \)

\[ \mathbb{E}(X_{t} | \mathcal{F}_{s}) = X_{s} \quad \text{a.s. (almost surely).} \]

**Definition 3 (Mao [15]).** System (3) is said to be exponentially stable in mean square if there exist a scalar \( \lambda > 0 \) such that

\[ \lim_{t \to \infty} \sup_{t} \frac{1}{t} \log \mathbb{E}(|x(t)|^2) \leq -\lambda. \]

**Proposition 1 (Øksendal [33]).** Suppose \( (\Omega, \mathcal{F}, \{\mathcal{F}_{t}\}_{t \geq 0}, \mathbb{P}) \) be a complete probability space and \( X, Y \) are two \( n \)-dimensions random variables such that \( \mathbb{E}(|X|) < \infty \) and \( \mathbb{E}(|Y|) < \infty \). Let \( \mathcal{H} \subset \mathcal{F} \) be a \( \sigma \)-algebra. Then

(a) \( \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})) \).
(b) \( \mathbb{E}(Y^T X|\mathcal{H}) = Y^T \mathbb{E}(X|\mathcal{H}) \) if \( Y \) is \( \mathcal{H} \)-measurable.

**Proposition 2 (Øksendal [33] and Friedman [34]).** Let \( \{h(t)\}_{t_{0} \leq t \leq T} \) is a stochastic process and belongs to \( \mathcal{M}^{2}([t_{0}, T]; \mathbb{R}^{n}) \), then
(a) \( \{\int_{t_0}^{t} h(s) \, dw(s) : t_0 \leq t \leq T \} \) is a martingale with respect to \( \{\mathcal{F}_t\}_{t_0 \leq t \leq T} \).

(b) \( \int_{t_0}^{t} h(s) \, dw(s) \) is \( \mathcal{F}_t \)-measurable, \( t_0 \leq t \leq T \).

(c) \( \mathcal{E}(\int_{t_0}^{T} g(t) \, dw(t))^2) = \mathcal{E}(\int_{t_0}^{T} |g(t)|^2 \, dt) \), where \( \{g(t)\}_{t \leq T} \in \mathcal{M}^2([t_0, T]; \mathbb{R}^n) \).

3. Delay-dependent exponential stability

In this section, we will give sufficient conditions for exponential stability of systems (3), that is

\[
d[x(t) - D x(t-h)] = f(t) \, dt + g(t) \, dw(t)
\]

with the initial data \( x_0 = \xi \in C^b_{\mathcal{F}_0}([-h,0], \mathbb{R}^n) \), where \( f(t) \) and \( g(t) \) are given in Eqs. (4) and (5). Firstly, we give the following lemma which will be used in the proof of our main results.

**Lemma 1.** If \( x(t) \) is the solution of system (7), \( N \) is a any compatible dimension matrix, then

\[
\mathcal{E}(x(t-h)^T N \left[ \int_{t-h}^{t} g(s) \, dw(s) \right]) = 0, \quad t \geq h.
\]

Specially, when \( D = 0 \) in (7), that is

\[
dx(t) = f(t) \, dt + g(t) \, dw(t).
\]

Eq. (9) is a common stochastic delay system and Eq. (8) is also tenable for this case.

**Proof.** By Proposition 1(a), we have

\[
\mathcal{E}(x(t-h)^T N \left[ \int_{t-h}^{t} g(s) \, dw(s) \right]) = \mathcal{E}\left( \mathcal{E}\left( x(t-h)^T N \left[ \int_{t-h}^{t} g(s) \, dw(s) \right] \big| \mathcal{F}_{t-h} \right) \right).
\]

It should be noted that the solution \( \{x(t)\}_{t \geq h} \) of systems (7) is adapted to \( \{\mathcal{F}_t\}_{t \geq -h} \), then \( x(t) \) is \( \mathcal{F}_t \)-measurable and \( x(t-h) \) is \( \mathcal{F}_{t-h} \)-measurable for \( t \geq h \). Considering this and by Proposition 1(b), we can have

\[
\mathcal{E}\left( x(t-h)^T N \left[ \int_{t-h}^{t} g(s) \, dw(s) \right] \big| \mathcal{F}_{t-h} \right) = x(t-h)^T N \mathcal{E}\left( \int_{t-h}^{t} g(s) \, dw(s) \big| \mathcal{F}_{t-h} \right).
\]

From Proposition 2(a), we know that \( \{\int_{0}^{z} g(s) \, dw(s) : 0 \leq z \leq t \} \) is a martingale with respect to \( \{\mathcal{F}_z\}_{0 \leq z \leq t} \). Then, by Definition 2, we can obtain

\[
\mathcal{E}\left( \int_{0}^{t} g(s) \, dw(s) \big| \mathcal{F}_{t-h} \right) = \int_{0}^{t-h} g(s) \, dw(s), \quad t \geq h.
\]

From Eq. (12) and the fact that \( \int_{0}^{t-h} g(s) \, dw(s) \) is \( \mathcal{F}_{t-h} \)-measurable by Proposition 2(b), it is easy to obtain

\[
\mathcal{E}\left( \int_{t-h}^{t} g(s) \, dw(s) \big| \mathcal{F}_{t-h} \right) = 0, \quad t \geq h.
\]

Then, by Eqs. (10), (11) and (13), it is easy to obtain Eq. (8). If \( D = 0 \) in Eq. (7), that is a common stochastic delay system, we can also easily prove that Eq. (8) is tenable for this case. □
**Remark 1.** For neutral stochastic delay systems

\[ d[x(t) - Dx(t-h)] = f(t) \, dt + g(t) \, dw(t). \]

Lemma 1 has proved

\[ \mathcal{E}\left( (x(t-h))^T N \left[ \int_{t-h}^t g(s) \, dw(s) \right] \right) = 0, \quad t \geq h. \]

However, for any compatible dimension matrix \( N \), it cannot prove

\[ \mathcal{E}\left( x(t)^T N \left[ \int_{t-h}^t g(s) \, dw(s) \right] \right) = 0, \quad t \geq h. \]

For example, considering the following stochastic equation:

\[ d(x(t) - 0x(t-h)) = dw(t), \quad (14) \]

which has a one solution \( x(t) = w(t) \). However, we can easily verify that

\[ \mathcal{E}\left( x(t)^T N \left[ \int_{t-h}^t g(s) \, dw(s) \right] \right) = \mathcal{E}\left( w(t)N \left[ \int_{t-h}^t dw(s) \right] \right) = Nh \neq 0, \quad \forall N \neq 0. \quad (15) \]

Then, we can also consider the following one-dimension Langevin equation [25], that can be regarded as a special class of neutral stochastic delay systems

\[ d(x(t) - 0x(t-h)) = f(t) \, dt + g(t) \, dw(t), \quad x(0) = \xi, \]

where \( f(t) = -x(t) \) and \( g(t) = 1 \). From the pages 104–105 of [25], this equation has a solution

\[ x(t) = e^{-(t-u)} x(u) + \int_u^t e^{-(t-s)} \, dw(s), \quad u \leq t \quad (16) \]

or

\[ x(t) = e^{-t} x(0) + \int_0^t e^{-(t-s)} \, dw(s). \quad (17) \]

Then, we can see that by Eq. (16)

\[ \mathcal{E}\left( x(t)N \left[ \int_{t-h}^t g(s) \, dw(s) \right] \right) \]

\[ = \mathcal{E}\left( e^{-h} x(t-h) + \int_{t-h}^t e^{-(t-s)} \, dw(s) \right) N \left[ \int_{t-h}^t dw(s) \right] \]

\[ = e^{-h} \mathcal{E}\left( x(t-h)N \left[ \int_{t-h}^t dw(s) \right] \right) + \mathcal{E}\left( \int_{t-h}^t e^{-(t-s)} \, dw(s)N \int_{t-h}^t dw(s) \right) \]

\[ = 0 + N \int_{t-h}^t \mathcal{E}(e^{-(t-s)}) \, ds \]

\[ = N - Ne^{-h} \neq 0, \quad \forall N \neq 0 \quad (18) \]
and
\[\mathbb{E}\left( \int_{t-h}^{t} f(s) \, ds N\left[ \int_{t-h}^{t} g(s) \, dw(s) \right] \right)\]
\[= \mathbb{E}\left( x(t)-x(t-h) - \int_{t-h}^{t} g(s) \, dw(s) \right) N\left[ \int_{t-h}^{t} g(s) \, dw(s) \right] \]
\[= \mathbb{E}\left( x(t)N \int_{t-h}^{t} g(s) \, dw(s) \right) - \mathbb{E}\left( x(t-h)N \int_{t-h}^{t} g(s) \, dw(s) \right) ,\]
\[-\mathbb{E}\left( \int_{t-h}^{t} g(s) \, dw(s)N \int_{t-h}^{t} g(s) \, dw(s) \right) \]
\[= N(1-e^{-h})-0 - \int_{t-h}^{t} ds \]
\[= N(1-e^{-h})-0 \quad \forall N \neq 0. \quad (19)\]

Recently, some papers such as [26–30] considered that the expectations of these stochastic terms are all equal to zero. However, this is not the case. From above examples and Lemma 1, we can see that \(x(t-h)^{T}N \int_{t-h}^{t} g(s) \, dw(s)\) is the only one, whose expectation is equal to zero.

Now, the sufficient condition for the exponential stability of stochastic time-delay systems in (3) is proposed as follows.

**Theorem 1.** The neutral stochastic delay systems in (3) is mean-square exponentially stable, if there exist matrices \(P>0, Q_{1}>0, Q_{2}>0, Z>0\) such that the following LMI holds:

\[
\begin{bmatrix}
\Gamma_{1} & -A^{T}PD & 0 & 0 & H^{T}P & hA^{T}Z \\
* & \Gamma_{2} & -A_{d}^{T}PD & hA_{d}^{T}P & H_{d}^{T}P & hA_{d}^{T}Z \\
* & * & -Q_{2} & 0 & 0 & 0 \\
* & * & * & -hZ & 0 & 0 \\
* & * & * & * & -P & 0 \\
* & * & * & * & * & -hZ
\end{bmatrix}
< 0, \quad (20)
\]

where \(\Gamma_{1} = PA + A^{T}P + Q_{1} + Q_{2}\) and \(\Gamma_{2} = -Q_{1} + A_{d}^{T}P + PA_{d}\).

**Proof.** Firstly, as is pointed out in the introduction, Eq. (3) is a symbolic differential form, and it should be interpreted as its integral form

\[x(t)-Dx(t-h) = x(0)-Dx(-h) + \int_{0}^{t} f(s) \, ds + \int_{0}^{t} g(s) \, dw(s), \quad t \geq 0.\]

Then, from this equation, it is easy to obtain

\[\begin{aligned}
\left[ x(t)-Dx(t-h) \right] - \left[ x(t-h)-Dx(t-2h) \right] \\
= \int_{t-h}^{t} f(s) \, ds + \int_{t-h}^{t} g(s) \, dw(s), \quad t \geq h.
\end{aligned}\]  

(21)
Choose the following Lyapunov–Krasovskii functional for system (3) as
\[ V(x, t) = [x(t) - Dx(t-h)]^T P [x(t) - Dx(t-h)] + \int_{t-h}^{t} x(s)^T Q_1 x(s) \, ds \]
\[ + \int_{t-2h}^{t} x(s)^T Q_2 x(s) \, ds + \int_{-h}^{0} \int_{-h}^{t} f(s)^T Z f(s) \, ds \, d\theta \]
for all \( t \geq h \). Then, by Itô’s formula, the stochastic differential \( dV(x, t) \) can be obtained
\[ dV(x, t) = \mathcal{L} V(x, t) \, dt + 2[x(t) - Dx(t-h)]^T P g(t) \, dw(t), \]
where
\[ \mathcal{L} V(x, t) = 2[x(t) - Dx(t-h)]^T P f(t) + g(t)^T P g(t) + x(t)^T Q_1 x(t) \]
\[ - x(t-h)^T Q_1 x(t-h) + x(t)^T Q_2 x(t) - x(t-2h)^T Q_2 x(t-2h) \]
\[ + hf(t)^T Z f(t) - \int_{t-h}^{t} [f(s)^T Z f(s)] \, ds. \]

By Eq. (21), we have
\[ 2[x(t) - Dx(t-h)]^T P f(t) = 2[x(t) - Dx(t-h)]^T PA x(t) + 2[x(t) - Dx(t-h)]^T P A_d x(t-h) \]
\[ = 2[x(t) - Dx(t-h)]^T P A x(t) + 2[x(t-h) - Dx(t-2h)]^T P A_d x(t-h) \]
\[ + \int_{t-h}^{t} f(s) \, ds + \int_{t-h}^{t} g(s) \, dw(s) \]
\[ = \mathcal{P} A_d x(t-h). \] (25)

By Lemma 1, we have
\[ \mathcal{E} \left( \int_{t-h}^{t} g(s) \, dw(s) \right)^T P A_d x(t-h) = 0. \] (26)

Thus, it is easy, from Eqs. (25) and (26), to obtain
\[ \mathcal{E}(2[x(t) - Dx(t-h)]^T P f(t)) = \mathcal{E}(2[x(t) - Dx(t-h)]^T P A x(t)) \]
\[ + 2[x(t-h) - Dx(t-2h)] + \int_{t-h}^{t} f(s) \, ds \]
\[ = \mathcal{E}(2[x(t) - Dx(t-h)]^T P A x(t)) \] (27)

It then follows from Eqs. (24) and (27) that
\[ \mathcal{E}(\mathcal{L} V(t)) = \mathcal{E}(2[x(t)]^T P A x(t) - 2x(t-h)^T D^T P A x(t) + 2x(t-h)^T P A_d x(t-h) \]
\[ - 2x(t-2h)^T D^T P A_d x(t-h) + 2 \int_{t-h}^{t} f(s) \, ds \]
\[ + x(t)^T Q_1 x(t) - x(t-h)^T Q_1 x(t-h) + x(t)^T Q_2 x(t) - x(t-2h)^T Q_2 x(t-2h) \]
\[ + hf(t)^T Z f(t) - \left( \int_{t-h}^{t} f(s)^T Z f(s) \, ds \right) = \mathcal{E}(\frac{1}{h} \int_{t-h}^{t} \zeta(t, s) \Theta \zeta(t, s) \, ds), \] (28)
where
\[
\Theta = \begin{pmatrix}
\Gamma_1 & -A^T P D & 0 & 0 \\
\ast & \Gamma_2 & -A_d^T P D & h A_d^T P \\
\ast & \ast & -Q_2 & 0 \\
\ast & \ast & \ast & -hZ
\end{pmatrix} + \begin{pmatrix}
H^T \\
H_d^T \\
0 \\
0
\end{pmatrix}^T \begin{pmatrix}
H^T \\
H_d^T \\
0 \\
0
\end{pmatrix}^T A^T \begin{pmatrix}
A^T \\
A_d^T \\
Z \\
0
\end{pmatrix}^T,
\]
\[ \zeta(t, s) = [x(t)^T x(t-h)^T x(t-2h)^T f(s)^T]^T. \]
From Eq. (20) and by Schur complement, we can obtain \( \Theta < 0 \). Then, by Eq. (28), it is easy to prove that there exists a scalar \( c > 0 \) such that

\[
\mathcal{E}(L V(t)) \leq -c \mathcal{E}|x(t)|^2.
\]

(29)

Finally, using a method similar to that used in [15,19], we can prove that system (3) is exponentially stable. This completes the proof. □

**Remark 2.** It should point out that bounding techniques including Jensen inequality were adopted in [18,19] to obtain delay-dependent stability conditions for neutral stochastic delay systems. These methods will increase the conservatism. In the derivation of Theorem 1, any bounding technique and Jensen inequality are not used. Therefore, our method will show less conservatism, which will be illustrated by numerical examples in Section 4.

**Remark 3.** It is obvious that in deriving Theorem 1, none of model transformations and free-weighting matrices method have been employed. However, the free-weighting matrix approach was applied in [18,19], which will result in high computational costs. In fact, 20 and 11 decision matrices were introduced in [18,19], respectively. Theorem 1 only has four decision matrices. Therefore, our method is more appealing since it involves fewer decision matrices.

**Remark 4.** Let us give an example to illustrate that \((T_2) \nRightarrow (T_1)\) in the generalized Finsler lemma method of [24]. Let

\[
B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix},
\]

where \( \bar{\xi} \) is a Gaussian random variable with mean 0 and variance \( \sigma > 0 \). Then \( B^\perp = (0 \ 1)^T \), \( \xi \neq 0 \) a.s. And we can easily obtain that

\[
(B^\perp)^T \Theta B^\perp = -1 < 0, \quad \int_{t_0}^{t} \mathcal{E}(B\theta) \, ds = \int_{t_0}^{t} \mathcal{E}(\xi) \, ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

However, we can verify that

\[
\mathcal{E}(\theta^T \Theta \theta) = \mathcal{E}\left(\begin{pmatrix} \bar{\xi} \\ \xi \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{\xi} \\ \xi \end{pmatrix}\right) = \mathcal{E}(\xi^2) = \sigma > 0.
\]

Therefore, it can be seen that \((T_2) \nRightarrow (T_1)\), and the generalized Finsler lemma in [24] is not tenable.

**4. Numerical examples**

As is pointed out in the introduction, many practical systems can be modeled by neutral-type delay systems [35]. On the other hand, systems in the real world are always perturbed by stochastic noises [15]. Therefore, Kolmanovskii and Nosov [36,37] introduced the neutral stochastic differential functional equations to represent mathematical models of practical systems. For example, a distributed network (see the pages 5 and 6 in [35]) can be
modeled by the following neutral delay equation:

\[
\frac{d}{dt}(u(t) - Ku(t-h)) = f(u(t), u(t-h)),
\]

where \(h = \frac{2}{\sqrt{LC}}, C\) is the capacity and \(L\) is the inductance. Then, considering that stochastic perturbations are inevitable in practical engineering, the distributed networks model in practice should be modeled by

\[
d[u(t) - Ku(t-h)] = f(u(t), u(t-h)) \, dt + g(u(t), u(t-h)) \, dw(t).
\]

(31)

Let \(f(u(t), u(t-h)) = Au(t) + A_1 u(t-h), g(u(t), u(t-h)) = Bu(t) + B_1 u(t-h)\), Eq. (31) is the model (3) in our paper.

Then, numerical examples are provided to demonstrate the reduced conservatism of the proposed condition of our paper.

**Example 1.** Let us consider the stochastic delay system (3) with

\[
D = \begin{bmatrix}
-0.3 & 0 \\
0 & -0.3
\end{bmatrix}, \quad A = \begin{bmatrix}
-2 & 0 \\
0 & -1.9
\end{bmatrix}, \quad A_d = \begin{bmatrix}
-c & 0 \\
0 & -c
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
0.2 & 0 \\
0 & 0.2
\end{bmatrix}, \quad H_d = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}.
\]

Employing the most recent results of [18,19], and Theorem 1 in this paper, we give the comparisons of the maximum allowed delay \(h\) for various parameter \(c\) in Table 1.

It is obvious that Theorem 1 is less conservative than those in the literature [18,19]. It should be noted that the numbers of decision variables to be determined in [18,19] and

<table>
<thead>
<tr>
<th>(c)</th>
<th>[18]</th>
<th>[19]</th>
<th>Theorem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.1435</td>
<td>–</td>
<td>0.1858</td>
</tr>
<tr>
<td>1.8</td>
<td>0.1459</td>
<td>–</td>
<td>0.1966</td>
</tr>
<tr>
<td>1.6</td>
<td>0.1479</td>
<td>–</td>
<td>0.2089</td>
</tr>
<tr>
<td>1.4</td>
<td>0.1493</td>
<td>–</td>
<td>0.2219</td>
</tr>
<tr>
<td>1.2</td>
<td>0.1497</td>
<td>–</td>
<td>0.2392</td>
</tr>
<tr>
<td>1</td>
<td>0.1491</td>
<td>–</td>
<td>0.2586</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>[18]</th>
<th>[19]</th>
<th>Theorem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers of decision variables</td>
<td>(\frac{37n^2}{2} + 7n)</td>
<td>(\frac{17n^2}{2} + 5n)</td>
<td>(\frac{4n^2}{2} + 4n)</td>
</tr>
</tbody>
</table>
Theorem 1 are given in Table 2. Thus, our result is more computationally efficient than [18,19].

Example 2. Let us consider the stochastic delay system (3) with

$$D = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.2 \end{bmatrix}, \quad A = \begin{bmatrix} -1.4 & 0 \\ 0 & -1.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$H = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad H_d = \begin{bmatrix} -1.9 & 0 \\ 0 & -1.9 \end{bmatrix}. $$

The maximal allowable delays for exponential stability of this system by [18,19] and Theorem 1, are listed in Table 3. It is obvious that Theorem 1 is less conservative than those in the literature [18,19].

5. Conclusions

This paper investigates the problem of stability analysis for neutral stochastic delay systems. An improved delay-dependent stability criterion is derived for neutral stochastic delay systems. In the derivation process, the mathematical development avoids bounding stochastic cross terms containing the Itô integral by the martingale theory; and neither the model transformation method nor free-weighting-matrix method is used. Thus the method leads to a simple criterion and shows less conservatism. Finally, examples are provided to demonstrate the reduced conservatism of the proposed conditions.

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References


